6
Cotangent Bundles

In many mechanics problems, the phase space is the cotangent bundle T^*Q of a configuration space Q. There is an "intrinsic" symplectic structure on T^*Q that can be described in various equivalent ways. Assume first that Q is n-dimensional, and pick local coordinates $(q^1, \ldots, q^n)$ on Q. Since $(dq^1, \ldots, dq^n)$ is a basis of T_q^*Q, we can write any $\alpha \in T_q^*Q$ as $\alpha = p_i \, dq^i$. This procedure defines induced local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on T^*Q. Define the ***canonical symplectic form*** on T^*Q by

$$\Omega = dq^i \wedge dp_i.$$

This defines a two-form Ω, which is clearly closed, and in addition, it can be checked to be independent of the choice of coordinates $(q^1, \ldots, q^n)$. Furthermore, observe that Ω is locally constant, that is, the coefficient multiplying the basis forms $dq^i \wedge dp_i$, namely the number 1, does not explicitly depend on the coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ of phase space points. In this section we show how to construct Ω intrinsically, and then we will study this canonical symplectic structure in some detail.

6.1 The Linear Case

To motivate a coordinate-independent definition of Ω, consider the case in which Q is a vector space W (which could be infinite-dimensional), so that $T^*Q = W \times W^*$. We have already described the canonical two-form on

$W \times W^*$:

$$\Omega_{(w,\alpha)}((u,\beta),(v,\gamma)) = \langle \gamma, u \rangle - \langle \beta, v \rangle\,, \tag{6.1.1}$$

where $(w,\alpha) \in W \times W^*$ is the base point, $u, v \in W$, and $\beta, \gamma \in W^*$. This canonical two-form will be constructed from the ***canonical one-form*** Θ, defined as follows:

$$\Theta_{(w,\alpha)}(u,\beta) = \langle \alpha, u \rangle\,. \tag{6.1.2}$$

The next proposition shows that the canonical two-form (6.1.1) is exact:

$$\Omega = -\mathbf{d}\Theta. \tag{6.1.3}$$

We begin with a computation that reconciles these formulas with their coordinate expressions.

Proposition 6.1.1. *In the finite-dimensional case the symplectic form* Ω *defined by* (6.1.1) *can be written* $\Omega = dq^i \wedge dp_i$ *in coordinates* $q^1, \ldots, q^n$ *on* W *and corresponding dual coordinates* $p_1, \ldots, p_n$ *on* W^*. *The associated canonical one-form is given by* $\Theta = p_i\, dq^i$, *and* (6.1.3) *holds.*

Proof. If $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are coordinates on T^*W, then

$$\left(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \right)$$

denotes the induced basis for $T_{(w,\alpha)}(T^*W)$, and $(dq^1, \ldots, dq^n, dp_1, \ldots, dp_n)$ denotes the associated dual basis of $T^*_{(w,\alpha)}(T^*W)$. Write

$$(u,\beta) = \left(u^j \frac{\partial}{\partial q^j},\ \beta_j \frac{\partial}{\partial p_j} \right)$$

and similarly for (v,γ). Hence

$$\begin{aligned} (dq^i \wedge dp_i)_{(w,\alpha)}((u,\beta),(v,\gamma)) &= (dq^i \otimes dp_i - dp_i \otimes dq^i)((u,\beta),(v,\gamma)) \\ &= dq^i(u,\beta) dp_i(v,\gamma) - dp_i(u,\beta) dq^i(v,\gamma) \\ &= u^i \gamma_i - \beta_i v^i. \end{aligned}$$

Also, $\Omega_{(w,\alpha)}((u,\beta),(v,\gamma)) = \gamma(u) - \beta(v) = \gamma_i u^i - \beta_i v^i$. Thus,

$$\Omega = dq^i \wedge dp_i.$$

Similarly,

$$(p_i\, dq^i)_{(w,\alpha)}(u,\beta) = \alpha_i\, dq^i(u,\beta) = \alpha_i u^i,$$

and

$$\Theta_{(w,\alpha)}(u,\beta) = \alpha(u) = \alpha_i u^i.$$

Comparing, we get $\Theta = p_i\, dq^i$. Therefore,

$$-\mathbf{d}\Theta = -\mathbf{d}(p_i\, dq^i) = dq^i \wedge dp_i = \Omega. \qquad \blacksquare$$

To verify (6.1.3) for the infinite-dimensional case, use (6.1.2) and the second identity in item 6 of the table at the end of §4.4 to give

$$\begin{aligned}\mathbf{d}\Theta_{(w,\alpha)}((u_1,\beta_1),(u_2,\beta_2)) &= [\mathbf{D}\Theta_{(w,\alpha)}\cdot(u_1,\beta_1)]\cdot(u_2,\beta_2)\\ &\quad - [\mathbf{D}\Theta_{(w,\alpha)}\cdot(u_2,\beta_2)]\cdot(u_1,\beta_1)\\ &= \langle\beta_1,u_2\rangle - \langle\beta_2,u_1\rangle\,,\end{aligned}$$

since $\mathbf{D}\Theta_{(w,\alpha)}\cdot(u,\beta) = \langle\beta,\cdot\rangle$. But this equals $-\Omega_{(w,\alpha)}((u_1,\beta_1),(u_2,\beta_2))$.

To give an intrinsic interpretation to Θ, let us prove that

$$\Theta_{(w,\alpha)}\cdot(u,\beta) = \left\langle \alpha, T_{(w,\alpha)}\pi_W(u,\beta)\right\rangle, \tag{6.1.4}$$

where $\pi_W : W \times W^* \to W$ is the projection. Indeed, (6.1.4) coincides with (6.1.2), since $T_{(w,\alpha)}\pi_W : W\times W^* \to W$ is the projection on the first factor.

Exercises

◇ **6.1-1** (Jacobi–Haretu Coordinates)**.** Consider the three-particle configuration space $Q = \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$ with elements denoted by $\mathbf{r}_1,\mathbf{r}_2$, and $\mathbf{r}_3$. Call the conjugate momenta $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ and equip the phase space T^*Q with the canonical symplectic structure Ω. Let $\mathbf{j} = \mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$. Let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and let $\mathbf{s} = \mathbf{r}_3 - \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. Show that the form Ω pulled back to the level sets of $\mathbf{j}$ has the form $\Omega = d\mathbf{r}\wedge d\pi + d\mathbf{s}\wedge d\sigma$, where the variables π and σ are defined by $\pi = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$ and $\sigma = \mathbf{p}_3$.

6.2 The Nonlinear Case

Definition 6.2.1. *Let Q be a manifold. We define $\Omega = -\mathbf{d}\Theta$, where Θ is the one-form on T^*Q defined analogous to* (6.1.4)*, namely*

$$\Theta_\beta(v) = \langle\beta, T\pi_Q\cdot v\rangle\,, \tag{6.2.1}$$

where $\beta\in T^*Q$, $v\in T_\beta(T^*Q)$, $\pi_Q : T^*Q\to Q$ is the projection, and $T\pi_Q : T(T^*Q)\to TQ$ is the tangent map of π_Q.

The computations in Proposition 6.1.1 show that $(T^*Q, \Omega = -\mathbf{d}\Theta)$ is a symplectic manifold; indeed, in local coordinates with $(w,\alpha)\in U\times W^*$, where U is open in W, and where $(u,\beta),(v,\gamma)\in W\times W^*$, the two-form $\Omega = -\mathbf{d}\Theta$ is given by

$$\Omega_{(w,\alpha)}((u,\beta),(v,\gamma)) = \gamma(u) - \beta(v). \tag{6.2.2}$$

Darboux' theorem and its corollary can be interpreted as asserting that any (strong) symplectic manifold locally looks like $W\times W^*$ in suitable local coordinates.

Hamiltonian Vector Fields. For a function $H : T^*Q \to \mathbb{R}$, the Hamiltonian vector field X_H on the cotangent bundle T^*Q is given in canonical cotangent bundle charts $U \times W^*$, where U is open in W, by

$$X_H(w, \alpha) = \left(\frac{\delta H}{\delta \alpha}, -\frac{\delta H}{\delta w} \right). \tag{6.2.3}$$

Indeed, setting $X_H(w, \alpha) = (w, \alpha, v, \gamma)$, for any $(u, \beta) \in W \times W^*$ we have

$$\begin{aligned} \mathbf{d}H_{(w,\alpha)} \cdot (u, \beta) &= \mathbf{D}_w H_{(w,\alpha)} \cdot u + \mathbf{D}_\alpha H_{(w,\alpha)} \cdot \beta \\ &= \left\langle \frac{\delta H}{\delta w}, u \right\rangle + \left\langle \beta, \frac{\delta H}{\delta \alpha} \right\rangle, \end{aligned} \tag{6.2.4}$$

which, by definition and (6.2.2), equals

$$\Omega_{(w,\alpha)}(X_H(w, \alpha), (u, \beta)) = \langle \beta, v \rangle - \langle \gamma, u \rangle . \tag{6.2.5}$$

Comparing (6.2.4) and (6.2.5) gives (6.2.3). In finite dimensions, (6.2.3) is the familiar right-hand side of Hamilton's equations.

Poisson Brackets. Formula (6.2.3) and the definition of the Poisson bracket show that in canonical cotangent bundle charts,

$$\{f, g\}(w, \alpha) = \left\langle \frac{\delta f}{\delta w}, \frac{\delta g}{\delta \alpha} \right\rangle - \left\langle \frac{\delta g}{\delta w}, \frac{\delta f}{\delta \alpha} \right\rangle, \tag{6.2.6}$$

which in finite dimensions becomes

$$\{f, g\}(q^i, p_i) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \tag{6.2.7}$$

Pull-Back Characterization. Another characterization of the canonical one-form that is sometimes useful is the following:

Proposition 6.2.2. *Θ is the unique one-form on T^*Q such that*

$$\alpha^* \Theta = \alpha \tag{6.2.8}$$

for any local one-form α on Q, where on the left-hand side, α is regarded as a map (of some open subset of) Q to T^*Q.

Proof. In finite dimensions, if $\alpha = \alpha_i(q^j)\, dq^i$ and $\Theta = p_i\, dq^i$, then to calculate $\alpha^*\Theta$ means that we substitute $p_i = \alpha_i(q^j)$ into Θ, a process that clearly gives back α, so $\alpha^*\Theta = \alpha$. The general argument is as follows. If Θ is the canonical one-form on T^*Q, and $v \in T_qQ$, then

$$\begin{aligned} (\alpha^*\Theta)_q \cdot v &= \Theta_{\alpha(q)} \cdot T_q\alpha(v) = \left\langle \alpha(q), T_{\alpha(q)}\pi_Q(T_q\alpha(v)) \right\rangle \\ &= \langle \alpha(q), T_q(\pi_Q \circ \alpha)(v) \rangle = \alpha(q) \cdot v, \end{aligned}$$

since $\pi_Q \circ \alpha =$ identity on Q.

For the converse, assume that Θ is a one-form on T^*Q satisfying (6.2.8). We will show that it must then be the canonical one-form (6.2.1). In finite dimensions this is straightforward: If $\Theta = A_i \, dq^i + B^i \, dp_i$ for A_i, B^i functions of (q^j, p_j), then

$$\alpha^*\Theta = (A_i \circ \alpha) \, dq^i + (B^i \circ \alpha) \, d\alpha_i = \left(A_j \circ \alpha + (B^i \circ \alpha) \frac{\partial \alpha_i}{\partial q^j} \right) dq^j,$$

which equals $\alpha = \alpha_i \, dq^i$ if and only if

$$A_j \circ \alpha + (B^i \circ \alpha) \frac{\partial \alpha_i}{\partial q^j} = \alpha_j.$$

Since this must hold for all α_j, putting $\alpha_1, \ldots, \alpha_n$ constant, it follows that $A_j \circ \alpha = \alpha_j$, that is, $A_j = p_j$. Therefore, the remaining equation is

$$(B^i \circ \alpha) \frac{\partial \alpha_i}{\partial q^j} = 0$$

for any α_i; choosing $\alpha_i(q^1, \ldots, q^n) = q_0^i + (q^i - q_0^i)p_i^0$ (no sum) implies $0 = (B^j \circ \alpha)(q_0^1, \ldots, q_0^n)p_j^0$ for all (q_0^j, p_j^0); therefore, $B^j = 0$ and thus $\Theta = p_i \, dq^i$.[1] ■

Exercises

◇ **6.2-1.** Let N be a submanifold of M and denote by Θ_N and Θ_M the canonical one-forms on the cotangent bundles $\pi_N : T^*N \to N$ and $\pi_M : T^*M \to M$, respectively. Let $\pi : (T^*M)|N \to T^*N$ be the projection defined by $\pi(\alpha_n) = \alpha_n | T_nN$, where $n \in N$ and $\alpha_n \in T_n^*M$. Show that $\pi^*\Theta_N = i^*\Theta_M$, where $i : (T^*M)|N \to T^*M$ is the inclusion.

◇ **6.2-2.** Let $f : Q \to \mathbb{R}$ and $X \in \mathfrak{X}(T^*Q)$. Show that

$$\Theta(X) \circ \mathbf{d}f = X[f \circ \pi_Q] \circ \mathbf{d}f.$$

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[1] In infinite dimensions, the proof is slightly different. We will show that if (6.2.8) holds, then Θ is locally given by (6.1.4), and thus it is the canonical one-form. If $U \subset E$ is the chart domain in the Banach space E modeling Q, then for any $v \in E$ we have

$$(\alpha^*\Theta)_u \cdot (u, v) = \Theta(u, \alpha(u)) \cdot (v, \mathbf{D}\alpha(u) \cdot v),$$

where α is given locally by $u \mapsto (u, \alpha(u))$ for $\alpha : U \to E^*$. Thus (6.2.8) is equivalent to

$$\Theta_{(u,\alpha(u))} \cdot (v, \mathbf{D}\alpha(u) \cdot v) = \langle \alpha(u), v \rangle ,$$

which would imply (6.1.4) and hence Θ being the canonical one-form, provided that we can show that for prescribed γ, $\delta \in E^*$, $u \in U$, and $v \in E$, there is an $\alpha : U \to E^*$ such that $\alpha(u) = \gamma$, and $\mathbf{D}\alpha(u) \cdot v = \delta$. Such a mapping is constructed in the following way. For $v = 0$ choose $\alpha(u)$ to equal γ for all u. For $v \neq 0$, by the Hahn–Banach theorem one can find a $\varphi \in E^*$ such that $\varphi(v) = 1$. Now set $\alpha(x) = \gamma - \varphi(u)\delta + \varphi(x)\delta$.

◇ **6.2-3.** Let Q be a given configuration manifold and let the ***extended phase space*** be defined by $(T^*Q) \times \mathbb{R}$. Given a time-dependent vector field X on T^*Q, extend it to a vector field $\tilde{X}$ on $(T^*Q) \times \mathbb{R}$ by $\tilde{X} = (X, 1)$.

Let H be a (possibly time-dependent) function on $(T^*Q) \times \mathbb{R}$ and set

$$\Omega_H = \Omega + dH \wedge dt,$$

where Ω is the canonical two-form. Show that X is the Hamiltonian vector field for H with respect to Ω if and only if

$$\mathbf{i}_{\tilde{X}} \Omega_H = 0.$$

◇ **6.2-4.** Give an example of a symplectic manifold (P, Ω), where Ω is exact but P is *not* a cotangent bundle.

6.3 Cotangent Lifts

We now describe an important way to create symplectic transformations on cotangent bundles.

Definition 6.3.1. *Given two manifolds Q and S and a diffeomorphism $f : Q \to S$, the* ***cotangent lift*** *$T^*f : T^*S \to T^*Q$ of f is defined by*

$$\langle T^*f(\alpha_s), v \rangle = \langle \alpha_s, (Tf \cdot v) \rangle, \tag{6.3.1}$$

where

$$\alpha_s \in T_s^*S, \quad v \in T_qQ, \quad \textit{and} \quad s = f(q).$$

The importance of this construction is that T^*f is guaranteed to be symplectic; it is often called a "point transformation" because it arises from a diffeomorphism on points in configuration space. Notice that while Tf covers f, T^*f covers f^{-1}. Denote by $\pi_Q : T^*Q \to Q$ and $\pi_S : T^*S \to S$ the canonical cotangent bundle projections.

Proposition 6.3.2. *A diffeomorphism $\varphi : T^*S \to T^*Q$ preserves the canonical one-forms Θ_Q and Θ_S on T^*Q and T^*S, respectively, if and only if φ is the cotangent lift T^*f of some diffeomorphism $f : Q \to S$.*

Proof. First assume that $f : Q \to S$ is a diffeomorphism. Then for arbitrary $\beta \in T^*S$ and $v \in T_\beta(T^*S)$, we have

$$\begin{aligned}((T^*f)^*\Theta_Q)_\beta \cdot v &= (\Theta_Q)_{T^*f(\beta)} \cdot TT^*f(v)\\ &= \langle T^*f(\beta), (T\pi_Q \circ TT^*f) \cdot v \rangle\\ &= \langle \beta, T(f \circ \pi_Q \circ T^*f) \cdot v \rangle\\ &= \langle \beta, T\pi_S \cdot v \rangle = \Theta_{S\beta} \cdot v,\end{aligned}$$

since $f \circ \pi_Q \circ T^* f = \pi_S$.

Conversely, assume that $\varphi^* \Theta_Q = \Theta_S$, that is,

$$\langle \varphi(\beta), T(\pi_Q \circ \varphi)(v) \rangle = \langle \beta, T\pi_S(v) \rangle \tag{6.3.2}$$

for all $\beta \in T^*S$ and $v \in T_\beta(T^*S)$. Since φ is a diffeomorphism, the range of $T_\beta(\pi_Q \circ \varphi)$ is $T_{\pi_Q(\varphi(\beta))}Q$, so that letting $\beta = 0$ in (6.3.2) implies that $\varphi(0) = 0$. Arguing similarly for φ^{-1} instead of φ, we conclude that φ restricted to the zero section S of T^*S is a diffeomorphism onto the zero section Q of T^*Q. Define $f : Q \to S$ by $f = \varphi^{-1}|Q$. We will show below that φ is fiber-preserving, or, equivalently, that $f \circ \pi_Q = \pi_S \circ \varphi^{-1}$. For this we use the following:

Lemma 6.3.3. *Define the flow F_t^Q on T^*Q by $F_t^Q(\alpha) = e^t \alpha$ and let V_Q be the vector field it generates. Then*

$$\langle \Theta_Q, V_Q \rangle = 0, \quad \pounds_{V_Q} \Theta_Q = \Theta_Q, \quad \textit{and} \quad \mathbf{i}_{V_Q} \Omega_Q = -\Theta_Q. \tag{6.3.3}$$

Proof. Since F_t^Q is fiber-preserving, V_Q will be tangent to the fibers, and hence $T\pi_Q \circ V_Q = 0$. This implies by (6.2.1) that $\langle \Theta_Q, V_Q \rangle = 0$. To prove the second formula, note that $\pi_Q \circ F_t^Q = \pi_Q$. Let $\alpha \in T_q^*Q$, $v \in T_\alpha(T^*Q)$, and Θ_α denote Θ_Q evaluated at α. We have

$$\begin{aligned} ((F_t^Q)^* \Theta)_\alpha \cdot v &= \Theta_{F_t^Q(\alpha)} \cdot TF_t^Q(v) \\ &= \left\langle F_t^Q(\alpha), (T\pi_Q \circ TF_t^Q)(v) \right\rangle \\ &= \left\langle e^t \alpha, T(\pi_Q \circ F_t^Q)(v) \right\rangle \\ &= e^t \langle \alpha, T\pi_Q(v) \rangle = e^t \Theta_\alpha \cdot v, \end{aligned}$$

that is,

$$(F_t^Q)^* \Theta_Q = e^t \Theta_Q.$$

Taking the derivative relative to t at $t = 0$ yields the second formula. Finally, the first two formulas imply

$$\mathbf{i}_{V_Q} \Omega_Q = -\mathbf{i}_{V_Q} \mathbf{d} \Theta_Q = -\pounds_{V_Q} \Theta_Q + \mathbf{d} \mathbf{i}_{V_Q} \Theta_Q = -\Theta_Q. \qquad \blacktriangledown$$

Continuing the proof of the proposition, note that by (6.3.3) we have

$$\begin{aligned} \mathbf{i}_{\varphi^* V_Q} \Omega_S &= \mathbf{i}_{\varphi^* V_Q} \varphi^* \Omega_Q = \varphi^* (\mathbf{i}_{V_Q} \Omega_Q) \\ &= -\varphi^* \Theta_Q = -\Theta_S = \mathbf{i}_{V_S} \Omega_S, \end{aligned}$$

so that weak nondegeneracy of Ω_S implies $\varphi^* V_Q = V_S$. Thus φ commutes with the flows F_t^Q and F_t^S, that is, for any $\beta \in T^*S$ we have $\varphi(e^t \beta) =$

$e^t\varphi(\beta)$. Letting $t \to -\infty$ in this equality implies $(\varphi \circ \pi_S)(\beta) = (\pi_Q \circ \varphi)(\beta)$, since $e^t\beta \to \pi_S(\beta)$ and $e^t\varphi(\beta) \to (\pi_Q \circ \varphi)(\beta)$ for $t \to -\infty$. Thus

$$\pi_Q \circ \varphi = \varphi \circ \pi_S, \quad \text{or} \quad f \circ \pi_Q = \pi_S \circ \varphi^{-1}.$$

Finally, we show that $T^*f = \varphi$. For $\beta \in T^*S$, $v \in T_\beta(T^*S)$, (6.3.2) gives

$$\begin{aligned}\langle T^*f(\beta), T(\pi_Q \circ \varphi)(v)\rangle &= \langle \beta, T(f \circ \pi_Q \circ \varphi)(v)\rangle \\ &= \langle \beta, T\pi_S(v)\rangle = (\Theta_S)_\beta \cdot v \\ &= (\varphi^*\Theta_Q)_\beta \cdot v = (\Theta_Q)_{\varphi(\beta)} \cdot T_\beta\varphi(v) \\ &= \langle \varphi(\beta), T_\beta(\pi_Q \circ \varphi)(v)\rangle\,,\end{aligned}$$

which shows that $T^*f = \varphi$, since the range of $T_\beta(\pi_Q \circ \varphi)$ is the whole tangent space at $(\pi_Q \circ \varphi)(\beta)$ to Q. ■

In finite dimensions, the first part of this proposition can be seen in coordinates as follows. Write $(s^1, \ldots, s^n) = f(q^1, \ldots, q^n)$ and define

$$p_j = \frac{\partial s^i}{\partial q^j} r_i, \tag{6.3.4}$$

where $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are cotangent bundle coordinates on T^*Q and $(s^1, \ldots, s^n, r_1, \ldots, r_n)$ on T^*S. Since f is a diffeomorphism, it determines the q^i in terms of the s^j, say $q^i = q^i(s^1, \ldots, s^n)$, so both q^i and p_j are functions of $(s^1, \ldots, s^n, r_1, \ldots, r_n)$. The map T^*f is given by

$$(s^1, \ldots, s^n, r_1, \ldots, r_n) \mapsto (q^1, \ldots, q^n, p_1, \ldots, p_n). \tag{6.3.5}$$

To see that (6.3.5) preserves the canonical one-form, use the chain rule and (6.3.4):

$$r_i\, ds^i = r_i \frac{\partial s^i}{\partial q^k}\, dq^k = p_k\, dq^k. \tag{6.3.6}$$

Note that if f and g are diffeomorphisms of Q, then

$$T^*(f \circ g) = T^*g \circ T^*f, \tag{6.3.7}$$

that is, the cotangent lift switches the order of composition; in fact, *it is useful to think of T^*f as the* ***adjoint*** *of Tf*; this is because in coordinates the matrix of T^*f is the *transpose* of the matrix of the derivative of f.

Exercises

◇ **6.3-1.** The ***Lorentz group*** $\mathcal{L}$ is the group of invertible linear transformations of $\mathbb{R}^4$ to itself that preserve the quadratic form $x^2 + y^2 + z^2 - c^2t^2$, where c is a constant, the speed of light. Describe all elements of this group. Let Λ_0 denote one of these transformations. Map $\mathcal{L}$ to itself by $\Lambda \mapsto \Lambda_0\Lambda$. Calculate the cotangent lift of this map.

◇ **6.3-2.** We have shown that a transformation of T^*Q is the cotangent lift of a diffeomorphism of configuration space if and only if it preserves the canonical one-form. Find this result in Whittaker's book.

6.4 Lifts of Actions

A ***left action*** of a group G on a manifold M associates to each group element $g \in G$ a diffeomorphism Φ_g of M such that $\Phi_{gh} = \Phi_g \circ \Phi_h$. Thus, the collection of Φ_g's is a *group of transformations of* M. If we replace the condition $\Phi_{gh} = \Phi_g \circ \Phi_h$ by $\Psi_{gh} = \Psi_h \circ \Psi_g$, we speak of a ***right action***. We often write $\Phi_g(m) = g \cdot m$ and $\Psi_g(m) = m \cdot g$ for $m \in M$.

Definition 6.4.1. *Let Φ be an action of a group G on a manifold Q. The **right lift** Φ^* of the action Φ to the symplectic manifold T^*Q is the right action defined by the rule*

$$\Phi_g^*(\alpha) = (T^*_{g^{-1}\cdot q}\Phi_g)(\alpha), \tag{6.4.1}$$

where $g \in G$, $\alpha \in T_q^*Q$, and $T^*\Phi_g$ is the cotangent lift of the diffeomorphism $\Phi_g : Q \to Q$.

By (6.3.7), we see that

$$\Phi_{gh}^* = T^*\Phi_{gh} = T^*(\Phi_g \circ \Phi_h) = T^*\Phi_h \circ T^*\Phi_g = \Phi_h^* \circ \Phi_g^*, \tag{6.4.2}$$

so Φ^* is a right action. To get a ***left action***, denoted by Φ_* and called the ***left lift*** of Φ, one sets

$$(\Phi_*)_g = T^*_{g\cdot q}(\Phi_{g^{-1}}). \tag{6.4.3}$$

In either case, these lifted actions are actions by canonical transformations because of Proposition 6.3.2. We shall return to the study of actions of groups after we study Lie groups in Chapter 9.

Examples

(a) For a system of N particles in $\mathbb{R}^3$, we choose the configuration space $Q = \mathbb{R}^{3N}$. We write $(\mathbf{q}_j)$ for an N-tuple of vectors labeled by $j = 1, \ldots, N$. Similarly, elements of the momentum phase space $P = T^*\mathbb{R}^{3N} \cong \mathbb{R}^{6N} \cong \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ are denoted by $(\mathbf{q}_j, \mathbf{p}^j)$. Let the additive group $G = \mathbb{R}^3$ of translations act on Q according to

$$\Phi_{\mathbf{x}}(\mathbf{q}_j) = \mathbf{q}_j + \mathbf{x}, \quad \text{where } \mathbf{x} \in \mathbb{R}^3. \tag{6.4.4}$$

Each of the N position vectors $\mathbf{q}_j$ is translated by the same vector $\mathbf{x}$.

Lifting the diffeomorphism $\Phi_{\mathbf{x}} : Q \to Q$, we obtain an action Φ^* of G on P. We assert that

$$\Phi^*_{\mathbf{x}}(\mathbf{q}_j, \mathbf{p}^j) = (\mathbf{q}_j - \mathbf{x}, \mathbf{p}^j). \tag{6.4.5}$$

To verify (6.4.5), observe that $T\Phi_{\mathbf{x}} : TQ \to TQ$ is given by

$$(\mathbf{q}_i, \dot{\mathbf{q}}_j) \mapsto (\mathbf{q}_i + \mathbf{x}, \dot{\mathbf{q}}_j), \tag{6.4.6}$$

so its dual is $(\mathbf{q}_i, \mathbf{p}^j) \mapsto (\mathbf{q}_i - \mathbf{x}, \mathbf{p}^j)$. ◆

(b) Consider the action of $\mathrm{GL}(n, \mathbb{R})$, the group of $n \times n$ invertible matrices, or, more properly, the group of invertible linear transformations of $\mathbb{R}^n$ to itself, on $\mathbb{R}^n$ given by

$$\Phi_A(\mathbf{q}) = A\mathbf{q}. \tag{6.4.7}$$

The group of induced canonical transformations of $T^*\mathbb{R}^n$ to itself is given by

$$\Phi^*_A(\mathbf{q}, \mathbf{p}) = (A^{-1}\mathbf{q}, A^T\mathbf{p}), \tag{6.4.8}$$

which is readily verified. Notice that this reduces to the same transformation of $\mathbf{q}$ and $\mathbf{p}$ when A is orthogonal. ◆

Exercises

◇ **6.4-1.** Let the multiplicative group $\mathbb{R}\backslash\{0\}$ act on $\mathbb{R}^n$ by $\Phi_\lambda(\mathbf{q}) = \lambda\mathbf{q}$. Calculate the cotangent lift of this action.

6.5 Generating Functions

Consider a symplectic diffeomorphism $\varphi : T^*Q_1 \to T^*Q_2$ described by functions

$$p_i = p_i(q^j, s^j), \quad r_i = r_i(q^j, s^j), \tag{6.5.1}$$

where (q^i, p_i) and (s^j, r_j) are cotangent coordinates on T^*Q_1 and on T^*Q_2, respectively. In other words, assume that we have a map

$$\Gamma : Q_1 \times Q_2 \to T^*Q_1 \times T^*Q_2 \tag{6.5.2}$$

whose image is the graph of φ. Let Θ_1 be the canonical one-form on T^*Q_1 and Θ_2 be that on T^*Q_2. By definition,

$$\mathbf{d}(\Theta_1 - \varphi^*\Theta_2) = 0. \tag{6.5.3}$$

This implies, in view of (6.5.1), that

$$p_i\, dq^i - r_i\, ds^i \tag{6.5.4}$$

is closed. Restated, $\Gamma^*(\Theta_1 - \Theta_2)$ is closed. This condition holds if $\Gamma^*(\Theta_1 - \Theta_2)$ is exact, namely,

$$\Gamma^*(\Theta_1 - \Theta_2) = \mathbf{d}S \tag{6.5.5}$$

for a function $S(q, s)$. In coordinates, (6.5.5) reads

$$p_i\, dq^i - r_i\, ds^i = \frac{\partial S}{\partial q^i}\, dq^i + \frac{\partial S}{\partial s^i}\, ds^i, \tag{6.5.6}$$

which is equivalent to

$$p_i = \frac{\partial S}{\partial q^i}, \quad r_i = -\frac{\partial S}{\partial s^i}. \tag{6.5.7}$$

One calls S a ***generating function*** for the canonical transformation. With generating functions of this sort, one may run into singularities even with the identity map! See Exercise 6.5-1.

Presupposed relations other than (6.5.1) lead to conclusions other than (6.5.7). Point transformations are generated in this sense; if $S(q^i, r_j) = s^j(q) r_j$, then

$$s^i = \frac{\partial S}{\partial r_i} \quad \text{and} \quad p_i = \frac{\partial S}{\partial q^i}. \tag{6.5.8}$$

(Here one writes $p_i\, dq^i + s^i\, dr_i = \mathbf{d}S$.)

In general, consider a diffeomorphism $\varphi : P_1 \to P_2$ of one symplectic manifold (P_1, Ω_1) to another (P_2, Ω_2) and denote the graph of φ by

$$\Gamma(\varphi) \subset P_1 \times P_2.$$

Let $i_\varphi : \Gamma(\varphi) \to P_1 \times P_2$ be the inclusion and let $\Omega = \pi_1^* \Omega_1 - \pi_2^* \Omega_2$, where $\pi_i : P_1 \times P_2 \to P_i$ is the projection. One verifies that φ *is symplectic if and only if* $i_\varphi^* \Omega = 0$. Indeed, since $\pi_1 \circ i_\varphi$ is the projection restricted to $\Gamma(\varphi)$ and $\pi_2 \circ i_\varphi = \varphi \circ \pi_1$ on $\Gamma(\varphi)$, it follows that

$$i_\varphi^* \Omega = (\pi_1 | \Gamma(\varphi))^* (\Omega_1 - \varphi^* \Omega_2),$$

and hence $i_\varphi^* \Omega = 0$ if and only if φ is symplectic, because $(\pi_1 | \Gamma(\varphi))^*$ is injective. In this case, one says that $\Gamma(\varphi)$ is an ***isotropic*** submanifold of $P_1 \times P_2$ (equipped with the symplectic form Ω); in fact, since $\Gamma(\varphi)$ has half the dimension of $P_1 \times P_2$, it is *maximally* isotropic, or a ***Lagrangian manifold***.

Now suppose one *chooses* a form Θ such that $\Omega = -\mathbf{d}\Theta$. Then $i_\varphi^*\Omega = -\mathbf{d}i_\varphi^*\Theta = 0$, so *locally* on $\Gamma(\varphi)$ there is a function $S : \Gamma(\varphi) \to \mathbb{R}$ such that

$$i_\varphi^*\Theta = \mathbf{d}S. \tag{6.5.9}$$

This defines the ***generating function*** of the canonical transformation φ. Since $\Gamma(\varphi)$ is diffeomorphic to P_1 and also to P_2, we can regard S as a function on P_1 or P_2. If $P_1 = T^*Q_1$ and $P_2 = T^*Q_2$, we can equally well regard (at least locally) S as defined on $Q_1 \times Q_2$. In this way, the general construction of generating functions reduces to the case in equations (6.5.7) and (6.5.8) above. By making other choices of Q, the reader can construct other generating functions and reproduce formulas in, for instance, Goldstein [1980] or Whittaker [1927]. The approach here is based on Sniatycki and Tulczyjew [1971].

Generating functions play an important role in Hamilton–Jacobi theory, in the classical–quantum-mechanical relationship (where S plays the role of the quantum-mechanical phase), and in numerical integration schemes for Hamiltonian systems. We shall see a few of these aspects later on.

Exercises

◇ **6.5-1.** Show that

$$S(q^i, s^j, t) = \frac{1}{2t}\|\mathbf{q} - \mathbf{s}\|^2$$

generates a canonical transformation that is the identity at $t = 0$.

◇ **6.5-2** (A first-order symplectic integrator)**.** Given H, let

$$S(q^i, r_j, t) = r_k q^k - tH(q^i, r_j).$$

Show that S generates a canonical transformation that is a first-order approximation to the flow of X_H for small t.

6.6 Fiber Translations and Magnetic Terms

Momentum Shifts. We saw above that cotangent lifts provide a basic construction of canonical transformations. Fiber translations provide a second.

Proposition 6.6.1 (Momentum Shifting Lemma)**.** *Let A be a one-form on Q and let $t_A : T^*Q \to T^*Q$ be defined by $\alpha_q \mapsto \alpha_q + A(q)$, where $\alpha_q \in T_q^*Q$. Let Θ be the canonical one-form on T^*Q. Then*

$$t_A^*\Theta = \Theta + \pi_Q^*A, \tag{6.6.1}$$

where $\pi_Q : T^*Q \to Q$ *is the projection. Hence*

$$t_A^*\Omega = \Omega - \pi_Q^* \mathbf{d}A, \tag{6.6.2}$$

where $\Omega = -\mathbf{d}\Theta$ *is the canonical symplectic form. Thus,* t_A *is a canonical transformation if and only if* $\mathbf{d}A = 0$*.*

Proof. We prove this using a finite-dimensional coordinate computation. The reader is asked to supply the coordinate-free and infinite-dimensional proofs as an exercise. In coordinates, t_A is the map

$$t_A(q^i, p_j) = (q^i, p_j + A_j). \tag{6.6.3}$$

Thus,

$$t_A^*\Theta = t_A^*(p_i \mathbf{d}q^i) = (p_i + A_i)\mathbf{d}q^i = p_i \mathbf{d}q^i + A_i \mathbf{d}q^i, \tag{6.6.4}$$

which is the coordinate expression for $\Theta + \pi_Q^* A$. The remaining assertions follow directly from this. ■

In particular, fiber translation by the differential of a function $A = \mathbf{d}f$ is a canonical transformation; in fact, f induces, in the sense of the preceding section, a generating function (see Exercise 6.6-2). The two basic classes of canonical transformations, lifts, and fiber translations play an important part in mechanics.

Magnetic Terms. A symplectic form on T^*Q different from the canonical one is obtained in the following way. Let B be a closed two-form on Q. Then $\Omega - \pi_Q^* B$ is a closed two-form on T^*Q, where Ω is the canonical two-form. To see that $\Omega - \pi_Q^* B$ is (weakly) nondegenerate, use the fact that in a local chart this form is given at the point (w, α) by

$$((u, \beta), (v, \gamma)) \mapsto \langle \gamma, u \rangle - \langle \beta, v \rangle - B(w)(u, v). \tag{6.6.5}$$

Proposition 6.6.2.

(i) *Let* Ω *be the canonical two-form on* T^*Q *and let* $\pi_Q : T^*Q \to Q$ *be the projection. If* B *is a closed two-form on* Q*, then*

$$\Omega_B = \Omega - \pi_Q^* B \tag{6.6.6}$$

is a (weak) symplectic form on T^*Q*.*

(ii) *Let* B *and* B' *be closed two-forms on* Q *and assume that* $B - B' = \mathbf{d}A$*. Then the mapping* t_A *(fiber translation by* A*) is a symplectic diffeomorphism of* $(T^*Q,\ \Omega_B)$ *with* $(T^*Q,\ \Omega_{B'})$*.*

Proof. Part (i) follows by an argument similar to that in the momentum shifting lemma. For (ii), use formula (6.6.2) to get

$$t_A^*\Omega = \Omega - \pi_Q^* \mathbf{d}A = \Omega - \pi_Q^* B + \pi_Q^* B', \tag{6.6.7}$$

so that

$$t_A^*(\Omega - \pi_Q^* B') = \Omega - \pi_Q^* B,$$

since $\pi_Q \circ t_A = \pi_Q$. ■

Symplectic forms of the type Ω_B arise in the reduction process.[2] In the following section, we explain why the extra term $\pi_Q^* B$ is called a ***magnetic term***.

Exercises

◇ **6.6-1.** Provide the intrinsic proof of Proposition 6.6.1.

◇ **6.6-2.** If $A = \mathbf{d}f$, use a coordinate calculation to check that $S(q^i, r_i) = r_i q^i - f(q^i)$ is a generating function for t_A.

6.7 A Particle in a Magnetic Field

Let B be a closed two-form on $\mathbb{R}^3$ and let $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$ be the associated divergence-free vector field, that is,

$$\mathbf{i}_{\mathbf{B}}(dx \wedge dy \wedge dz) = B,$$

so that

$$B = B_x\, dy \wedge dz - B_y\, dx \wedge dz + B_z\, dx \wedge dy.$$

Thinking of $\mathbf{B}$ as a magnetic field, the equations of motion for a particle with charge e and mass m are given by the ***Lorentz force law***

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B}, \tag{6.7.1}$$

where $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$. On $\mathbb{R}^3 \times \mathbb{R}^3$, that is, $(\mathbf{x}, \mathbf{v})$-space, consider the symplectic form

$$\Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c}B, \tag{6.7.2}$$

[2] Magnetic terms come up in what is called the **cotangent bundle reduction theorem**; see Smale [1970], Abraham and Marsden [1978], Kummer [1981], Nill [1983], Montgomery, Marsden, and Ratiu [1984], Gozzi and Thacker [1987], and Marsden [1992].

that is, (6.6.6). As Hamiltonian, take the kinetic energy

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{6.7.3}$$

Writing $X_H(u, v, w) = (u, v, w, \dot{u}, \dot{v}, \dot{w})$, the condition

$$\mathbf{d}H = \mathbf{i}_{X_H}\Omega_B \tag{6.7.4}$$

is the same as

$$\begin{aligned} &m(\dot{x}\, d\dot{x} + \dot{y}\, d\dot{y} + \dot{z}\, d\dot{z}) \\ &\qquad = m(u\, d\dot{x} - \dot{u}\, dx + v\, d\dot{y} - \dot{v}\, dy + w\, d\dot{z} - \dot{w}\, dz) \\ &\qquad\quad - \frac{e}{c}[B_x v\, dz - B_x w\, dy - B_y u\, dz + B_y w\, dx + B_z u\, dy - B_z v\, dx], \end{aligned}$$

which is equivalent to $u = \dot{x}$, $v = \dot{y}$, and $w = \dot{z}$, together with the equations

$$\begin{aligned} m\dot{u} &= \frac{e}{c}(B_z v - B_y w), \\ m\dot{v} &= \frac{e}{c}(B_x w - B_z u), \\ m\dot{w} &= \frac{e}{c}(B_y u - B_x v), \end{aligned}$$

that is, to

$$\begin{aligned} m\ddot{x} &= \frac{e}{c}(B_z \dot{y} - B_y \dot{z}), \\ m\ddot{y} &= \frac{e}{c}(B_x \dot{z} - B_z \dot{x}), \\ m\ddot{z} &= \frac{e}{c}(B_y \dot{x} - B_x \dot{y}), \end{aligned} \tag{6.7.5}$$

which is the same as (6.7.1). Thus *the equations of motion for a particle in a magnetic field are Hamiltonian, with energy equal to the kinetic energy and with the symplectic form* Ω_B.

If $B = \mathbf{d}A$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}^\flat = A$, then the map $t_A : (\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{x}, \mathbf{p})$, where $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$, pulls back the canonical form to Ω_B by the momentum shifting lemma. Thus, equations (6.7.1) are also Hamiltonian relative to the canonical bracket on $(\mathbf{x}, \mathbf{p})$-space with the Hamiltonian

$$H_A = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2. \tag{6.7.6}$$

Remarks.

1. Not every magnetic field can be written as $\mathbf{B} = \nabla \times \mathbf{A}$ on Euclidean space. For example, the ***field of a magnetic monopole of strength*** $g \neq 0$, namely

$$\mathbf{B}(\mathbf{r}) = g \frac{\mathbf{r}}{\|\mathbf{r}\|^3}, \tag{6.7.7}$$

cannot be written this way, since the flux of $\mathbf{B}$ through the unit sphere is $4\pi g$, yet Stokes' theorem applied to the two-sphere would give zero; see Exercise 4.4-3. Thus, one might think that the Hamiltonian formulation involving only $\mathbf{B}$ (that is, using Ω_B and H) is preferable. However, there is a way to recover the magnetic potential $\mathbf{A}$ by regarding it as a connection on a *nontrivial bundle* over $\mathbb{R}^3 \setminus \{0\}$. (This bundle over the sphere S^2 is the ***Hopf fibration*** $S^3 \to S^2$.) For a readable account of some aspects of this situation, see Yang [1985].

2. When one studies the motion of a particle in a Yang–Mills field, one finds a beautiful generalization of this construction and related ideas using the theory of principal bundles; see Sternberg [1977], Weinstein [1978a], and Montgomery [1984].

3. In Chapter 8 we study centrifugal and Coriolis forces and will see some structures analogous to those here. ♦

Exercises

◇ **6.7-1.** Show that particles in constant magnetic fields move in helixes.

◇ **6.7-2.** Verify "by hand" that $\frac{1}{2}m\|\mathbf{v}\|^2$ is conserved for a particle moving in a magnetic field.

◇ **6.7-3.** Verify "by hand" that Hamilton's equations for H_A are the Lorentz force law equations (6.7.1).