5

Hamiltonian Systems on Symplectic Manifolds

Now we are ready to geometrize Hamiltonian mechanics to the context of manifolds. First we make phase spaces nonlinear, and then we study Hamiltonian systems in this context.

5.1 Symplectic Manifolds

Definition 5.1.1. *A* ***symplectic manifold*** *is a pair* (P, Ω) *where* P *is a manifold and* Ω *is a closed (weakly) nondegenerate two-form on* P*. If* Ω *is strongly nondegenerate, we speak of a* ***strong symplectic manifold****.*

As in the linear case, strong nondegeneracy of the two-form Ω means that at each $z \in P$, the bilinear form $\Omega_z : T_zP \times T_zP \to \mathbb{R}$ is nondegenerate, that is, Ω_z defines an isomorphism

$$\Omega_z^\flat : T_zP \to T_z^*P.$$

For a (weak) symplectic form, the induced map $\Omega^\flat : \mathfrak{X}(P) \to \mathfrak{X}^*(P)$ between vector fields and one-forms is one-to-one, but in general is not surjective. We will see later that Ω is required to be closed, that is, $\mathbf{d}\Omega = 0$, where $\mathbf{d}$ is the exterior derivative, so that the induced Poisson bracket satisfies the Jacobi identity and so that the flows of Hamiltonian vector fields will consist of canonical transformations. In coordinates z^I on P in the finite-dimensional case, if $\Omega = \Omega_{IJ}\, dz^I \wedge dz^J$ (sum over all $I < J$), then

$\mathbf{d}\Omega = 0$ becomes the condition

$$\frac{\partial \Omega_{IJ}}{\partial z^K} + \frac{\partial \Omega_{KI}}{\partial z^J} + \frac{\partial \Omega_{JK}}{\partial z^I} = 0. \tag{5.1.1}$$

Examples

(a) Symplectic Vector Spaces. If (Z, Ω) is a symplectic vector space, then it is also a symplectic manifold. The requirement $\mathbf{d}\Omega = 0$ is satisfied automatically, since Ω is a *constant* form (that is, $\Omega(z)$ is independent of $z \in Z$). ♦

(b) The cylinder $S^1 \times \mathbb{R}$ with coordinates (θ, p) is a symplectic manifold with $\Omega = d\theta \wedge dp$. ♦

(c) The torus $\mathbb{T}^2$ with periodic coordinates (θ, φ) is a symplectic manifold with $\Omega = d\theta \wedge d\varphi$. ♦

(d) The two-sphere S^2 of radius r is symplectic with Ω the standard *area element* $\Omega = r^2 \sin\theta \, d\theta \wedge d\varphi$ on the sphere as the symplectic form. ♦

Given a manifold Q, we will show in Chapter 6 that the cotangent bundle T^*Q has a natural symplectic structure. When Q is the ***configuration space*** of a mechanical system, T^*Q is called the ***momentum phase space***. This important example generalizes the linear examples with phase spaces of the form $W \times W^*$ that we studied in Chapter 2.

Darboux' Theorem. The next result says that, in principle, every strong symplectic manifold is, in suitable local coordinates, a symplectic vector space. (By contrast, a corresponding result for Riemannian manifolds is not true unless they have zero curvature; that is, are flat.)

Theorem 5.1.2 (Darboux' Theorem)**.** *Let (P, Ω) be a strong symplectic manifold. Then in a neighborhood of each $z \in P$, there is a local coordinate chart in which Ω is constant.*

Proof. We can assume $P = E$ and $z = 0 \in E$, where E is a Banach space. Let Ω_1 be the constant form equaling $\Omega(0)$. Let $\Omega' = \Omega_1 - \Omega$ and $\Omega_t = \Omega + t\Omega'$, for $0 \le t \le 1$. For each t, the bilinear form $\Omega_t(0) = \Omega(0)$ is nondegenerate. Hence by openness of the set of linear isomorphisms of E to E^* and compactness of $[0, 1]$, there is a neighborhood of 0 on which Ω_t is strongly nondegenerate for all $0 \le t \le 1$. We can assume that this neighborhood is a ball. Thus by the Poincaré lemma, $\Omega' = \mathbf{d}\alpha$ for some one-form α. Replacing α by $\alpha - \alpha(0)$, we can suppose $\alpha(0) = 0$. Define a smooth time-dependent vector field X_t by

$$\mathbf{i}_{X_t} \Omega_t = -\alpha,$$

which is possible, since Ω_t is strongly nondegenerate. Since $\alpha(0) = 0$, we get $X_t(0) = 0$, and so from the local existence theory for ordinary differential equations, there is a ball on which the integral curves of X_t are defined for a time at least one; see Abraham, Marsden, and Ratiu [1988, Section 4.1], for the technical theorem. Let F_t be the flow of X_t starting at F_0 = identity. By the Lie derivative formula for *time-dependent* vector fields, we have

$$\begin{aligned}\frac{d}{dt}(F_t^*\Omega_t) &= F_t^*(\pounds_{X_t}\Omega_t) + F_t^*\frac{d}{dt}\Omega_t \\ &= F_t^*\mathbf{d}\mathbf{i}_{X_t}\Omega_t + F_t^*\Omega' = F_t^*(\mathbf{d}(-\alpha) + \Omega') = 0.\end{aligned}$$

Thus, $F_1^*\Omega_1 = F_0^*\Omega_0 = \Omega$, so F_1 provides a chart transforming Ω to the constant form Ω_1. ■

This proof is due to Moser [1965]. As was noted by Weinstein [1971], this proof generalizes to the infinite-dimensional *strong* symplectic case. Unfortunately, many interesting infinite-dimensional symplectic manifolds are *not* strong. In fact, the analogue of Darboux's theorem is not valid for weak symplectic forms. For an example, see Exercise 5.1-3, and for conditions under which it is valid, see Marsden [1981], Olver [1988], Bambusi [1999], and references therein. For an equivariant Darboux theorem and references, see Dellnitz and Melbourne [1993] and the discussion in Chapter 9.

Corollary 5.1.3. *If (P, Ω) is a finite-dimensional symplectic manifold, then P is even dimensional, and in a neighborhood of $z \in P$ there are local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ (where $\dim P = 2n$) such that*

$$\Omega = \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{5.1.2}$$

This follows from Darboux' theorem and the canonical form for linear symplectic forms. As in the vector space case, coordinates in which Ω takes the above form are called ***canonical coordinates***.

Corollary 5.1.4. *If (P, Ω) is a $2n$-dimensional symplectic manifold, then P is oriented by the **Liouville volume** form, defined as*

$$\Lambda = \frac{(-1)^{n(n-1)/2}}{n!}\,\Omega \wedge \cdots \wedge \Omega \quad (n \text{ times}). \tag{5.1.3}$$

In canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, Λ has the expression

$$\Lambda = dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n. \tag{5.1.4}$$

Thus, if (P, Ω) is a $2n$-dimensional symplectic manifold, then (P, Λ) is a ***volume manifold*** (that is, a manifold with a volume element). The measure associated to Λ is called the ***Liouville measure***. The factor $(-1)^{n(n-1)/2}/n!$ is chosen so that in canonical coordinates, Λ has the expression (5.1.4).

Exercises

◇ **5.1-1.** Show how to construct (explicitly) canonical coordinates for the symplectic form $\Omega = f\mu$ on S^2, where μ is the standard area element and where $f : S^2 \to \mathbb{R}$ is a positive function.

◇ **5.1-2** (Moser [1965]). Let μ_0 and μ_1 be two volume elements (nowhere-vanishing n-forms) on the compact boundaryless n-manifold M giving M the same orientation. Assume that $\int_M \mu_0 = \int_M \mu_1$. Show that there is a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\mu_1 = \mu_0$.

◇ **5.1-3.** (Requires some functional analysis.) Prove that Darboux' theorem fails for the following weak symplectic form. Let H be a real Hilbert space and $S : H \to H$ a compact, self-adjoint, and positive operator whose range is dense in H but not equal to H. Let $A_x = S + \|x\|^2 I$ and

$$g_x(e, f) = \langle A_x e, f\rangle.$$

Let Ω be the weak symplectic form on $H \times H$ associated to g. Show that there is no coordinate chart about $(0, 0) \in H \times H$ on which Ω is constant.

◇ **5.1-4.** Use the method of proof of the Darboux Theorem to show the following. Assume that Ω_0 and Ω_1 are two symplectic forms on the compact manifold P such that $[\Omega_0] = [\Omega_1]$, where $[\Omega_0]$, $[\Omega_1]$ are the cohomology classes of Ω_0 and Ω_1 respectively in $H^2(P; \mathbb{R})$. If for every $t \in [0, 1]$, the form $\Omega_t := (1-t)\Omega_0 + t\Omega_1$ is non-degenerate, show that there is a diffeomorphism $\varphi : P \longrightarrow P$ such that $\varphi^*\Omega_1 = \Omega_0$.

◇ **5.1-5.** Prove the following **relative Darboux theorem.** Let S be a submanifold of P and assume that Ω_0 and Ω_1 are two strong symplectic forms on P such that $\Omega_0|S = \Omega_1|S$. Then there is an open neighborhood V of S in P and a diffeomorphism $\varphi : V \longrightarrow \varphi(V) \subset P$ such that $\varphi|S =$ identity on S and $\varphi^*\Omega_1 = \Omega_0$. (Hint: Use Exercise 4.2-6.)

5.2 Symplectic Transformations

Definition 5.2.1. *Let (P_1, Ω_1) and (P_2, Ω_2) be symplectic manifolds. A C^∞-mapping $\varphi : P_1 \to P_2$ is called **symplectic** or **canonical** if*

$$\varphi^*\Omega_2 = \Omega_1. \tag{5.2.1}$$

Recall that $\Omega_1 = \varphi^*\Omega_2$ means that for each $z \in P_1$, and all $v, w \in T_zP_1$, we have the following identity:

$$\Omega_{1z}(v, w) = \Omega_{2\varphi(z)}(T_z\varphi \cdot v, T_z\varphi \cdot w),$$

where Ω_{1z} means Ω_1 evaluated at the point z and where $T_z\varphi$ is the tangent (derivative) of φ at z.

If $\varphi : (P_1, \Omega_1) \to (P_2, \Omega_2)$ is canonical, the property $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$ implies that $\varphi^*\Lambda = \Lambda$; that is, φ also preserves the Liouville measure. Thus we get the following:

Proposition 5.2.2. *A smooth canonical transformation between symplectic manifolds of the same dimension is volume preserving and is a local diffeomorphism.*

The last statement comes from the inverse function theorem: If φ is volume preserving, its Jacobian determinant is 1, so φ is locally invertible. It is clear that the set of canonical diffeomorphisms of P form a subgroup of $\mathrm{Diff}(P)$, the group of all diffeomorphisms of P. This group, denoted by $\mathrm{Diff}_{\mathrm{can}}(P)$, plays a key role in the study of plasma dynamics.

If Ω_1 and Ω_2 are exact, say $\Omega_1 = -\mathbf{d}\Theta_1$ and $\Omega_2 = -\mathbf{d}\Theta_2$, then (5.2.1) is equivalent to

$$\mathbf{d}(\varphi^*\Theta_2 - \Theta_1) = 0. \tag{5.2.2}$$

Let $M \subset P_1$ be an oriented two-manifold with boundary ∂M. Then if (5.2.2) holds, we get

$$0 = \int_M \mathbf{d}(\varphi^*\Theta_2 - \Theta_1) = \int_{\partial M} (\varphi^*\Theta_2 - \Theta_1),$$

that is,

$$\int_{\partial M} \varphi^*\Theta_2 = \int_{\partial M} \Theta_1. \tag{5.2.3}$$

Proposition 5.2.3. *The map $\varphi : P_1 \to P_2$ is canonical if and only if (5.2.3) holds for every oriented two-manifold $M \subset P_1$ with boundary ∂M.*

The converse is proved by choosing M to be a small disk in P_1 and using the fact that if the integral of a two-form over any small disk vanishes, then the form is zero. The latter assertion is proved by contradiction, constructing a two-form on a two-disk whose coefficient is a bump function. Equation (5.2.3) is an example of an ***integral invariant***. For more information, see Arnold [1989] and Abraham and Marsden [1978].

Exercises

◇ **5.2-1.** Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a map of the form $\varphi(q,p) = (q, p + \alpha(q))$. Use the canonical one-form $p_i dq^i$ to determine when φ is symplectic.

◇ **5.2-2.** Let $\mathbb{T}^6$ be the six-torus with symplectic form

$$\Omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4 + d\theta_5 \wedge d\theta_6.$$

Show that if $\varphi : \mathbb{T}^6 \to \mathbb{T}^6$ is symplectic and $M \subset \mathbb{T}^6$ is a compact oriented four-manifold with boundary, then

$$\int_{\partial M} \varphi^*(\Omega \wedge \Theta) = \int_{\partial M} \Omega \wedge \Theta,$$

where $\Theta = \theta_1\, d\theta_2 + \theta_3\, d\theta_4 + \theta_5\, d\theta_6$.

◇ **5.2-3.** Show that any canonical map between finite-dimensional symplectic manifolds is an immersion.

5.3 Complex Structures and Kähler Manifolds

This section develops the relation between complex and symplectic geometry a little further. It may be omitted on a first reading.

Complex Structures. We begin with the case of vector spaces. By a ***complex structure*** on a real vector space Z, we mean a linear map $\mathbb{J} : Z \to Z$ such that $\mathbb{J}^2 = -\text{Identity}$. Setting $iz = \mathbb{J}(z)$ gives Z the structure of a complex vector space.

Note that if Z is finite-dimensional, the hypothesis on $\mathbb{J}$ implies that $(\det \mathbb{J})^2 = (-1)^{\dim Z}$, so $\dim Z$ must be an even number, since $\det \mathbb{J} \in \mathbb{R}$. The complex dimension of Z is half the real dimension. Conversely, if Z is a complex vector space, it is also a real vector space by restricting scalar multiplication to the real numbers. In this case, $\mathbb{J}z = iz$ is the complex structure on Z. As before, the real dimension of Z is twice the complex dimension, since the vectors z and iz are linearly independent.

We have already seen that the imaginary part of a complex inner product is a symplectic form. Conversely, *if $\mathcal{H}$ is a real Hilbert space and Ω is a skew-symmetric weakly nondegenerate bilinear form on $\mathcal{H}$, then there is a complex structure $\mathbb{J}$ on $\mathcal{H}$ and a real inner product s such that*

$$s(z, w) = -\Omega(\mathbb{J}z, w). \tag{5.3.1}$$

The expression

$$h(z, w) = s(z, w) - i\Omega(z, w) \tag{5.3.2}$$

defines a Hermitian inner product, and h or s is complete on $\mathcal{H}$ if and only if Ω is strongly nondegenerate. (See Abraham and Marsden [1978, p. 173] for the proof.) Moreover, given any two of $(s, \mathbb{J}, \Omega)$, there is at most one third structure such that (5.3.1) holds.

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and write

$$z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) = ((x_1, y_1), \ldots, (x_n, y_n)),$$

then

$$
\begin{aligned}
-\operatorname{Im}\langle(z_1,\dots,z_n),(z_1',\dots,z_n')\rangle &= -\operatorname{Im}(z_1\overline{z}_1'+\cdots+z_n\overline{z}_n')\\
&= -(x_1'y_1-x_1y_1'+\cdots+x_n'y_n-x_ny_n').
\end{aligned}
$$

Thus, the canonical symplectic form on $\mathbb{R}^{2n}$ may be written

$$
\Omega(z,z') = -\operatorname{Im}\langle z,z'\rangle = \operatorname{Re}\langle iz,z'\rangle\,, \tag{5.3.3}
$$

which, by (5.3.1), agrees with the convention that $\mathbb{J}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is multiplication by i.

An ***almost complex structure*** $\mathbb{J}$ on a manifold M is a smooth tangent bundle isomorphism $\mathbb{J}:TM\to TM$ covering the identity map on M such that for each point $z\in M$, $\mathbb{J}_z=\mathbb{J}(z):T_zM\to T_zM$ is a complex structure on the vector space T_zM. A manifold with an almost complex structure is called an ***almost complex manifold***.

A manifold M is called a ***complex manifold*** if it admits an atlas $\{(U_\alpha,\varphi_\alpha)\}$ whose charts $\varphi_\alpha : U_\alpha\subset M\to E$ map to a complex Banach space E and the transition functions $\varphi_\beta\circ\varphi_\alpha^{-1}:\varphi_\alpha(U_\alpha\cap U_\beta)\to\varphi_\beta(U_\alpha\cap U_\beta)$ are holomorphic maps. The complex structure on E (multiplication by i) induces via the chart maps φ_α an almost complex structure on each chart domain U_α. Since the transition functions are biholomorphic diffeomorphisms, the almost complex structures on $U_\alpha\cap U_\beta$ induced by φ_α and φ_β coincide. This shows that a complex manifold is also almost complex. The converse is not true.

If M is an almost complex manifold, T_zM is endowed with the structure of a complex vector space. A ***Hermitian metric*** on M is a smooth assignment of a (possibly weak) complex inner product on T_zM for each $z\in M$. As in the case of vector spaces, the imaginary part of the Hermitian metric defines a nondegenerate (real) two-form on M. The real part of a Hermitian metric is a Riemannian metric on M. If the complex inner product on each tangent space is strongly nondegenerate, the metric is *strong*; in this case both the real and imaginary parts of the Hermitian metric are strongly nondegenerate over $\mathbb{R}$.

Kähler Manifolds. An almost complex manifold M with a Hermitian metric $\langle\,,\rangle$ is called a ***Kähler manifold*** if M is a complex manifold and the two-form $-\operatorname{Im}\langle\,,\rangle$ is a closed two-form on M. There is an equivalent definition that is often useful: A Kähler manifold is a smooth manifold with a Riemannian metric g and an almost complex structure $\mathbb{J}$ such that $\mathbb{J}_z$ is g-skew for each $z\in M$ and such that $\mathbb{J}$ is covariantly constant with respect to g. (One requires some Riemannian geometry to understand this definition—it will not be required in what follows.) The important fact used later on is the following:

> *Any Kähler manifold is also symplectic, with symplectic form given by*

$$\Omega_z(v_z, w_z) = \langle \mathbb{J}_z v_z, w_z \rangle . \tag{5.3.4}$$

In this second definition of Kähler manifolds, the condition $\mathbf{d}\Omega = 0$ follows from $\mathbb{J}$ being covariantly constant. A ***strong Kähler manifold*** is a Kähler manifold whose Hermitian inner product is strong.

Projective Spaces. Any complex Hilbert space $\mathcal{H}$ is a strong Kähler manifold. As an example of a more interesting Kähler manifold, we shall consider the projectivization $\mathbb{P}\mathcal{H}$ of a complex Hilbert space $\mathcal{H}$. In particular, ***complex projective*** n***-space*** $\mathbb{CP}^n$ will result when this construction is applied to $\mathbb{C}^n$. Recall from Example (f) of §2.3 that $\mathcal{H}$ is a symplectic vector space relative to the quantum-mechanical symplectic form

$$\Omega(\psi_1, \psi_2) = -2\hbar \operatorname{Im} \langle \psi_1, \psi_2 \rangle ,$$

where $\langle \, , \rangle$ is the Hermitian inner product on $\mathcal{H}$, $\hbar$ is Planck's constant, and $\psi_1, \psi_2 \in \mathcal{H}$. Recall also that $\mathbb{P}\mathcal{H}$ is the space of complex lines through the origin in $\mathcal{H}$. Denote by $\pi : \mathcal{H}\backslash\{0\} \to \mathbb{P}\mathcal{H}$ the canonical projection that sends a vector $\psi \in \mathcal{H}\backslash\{0\}$ to the complex line it spans, denoted by $[\psi]$ when thought of as a point in $\mathbb{P}\mathcal{H}$ and by $\mathbb{C}\psi$ when interpreted as a subspace of $\mathcal{H}$. The space $\mathbb{P}\mathcal{H}$ is a smooth complex manifold, π is a smooth map, and the tangent space $T_{[\psi]}\mathbb{P}\mathcal{H}$ is isomorphic to $\mathcal{H}/\mathbb{C}\psi$. Thus, the map π is a surjective submersion. (Submersions were discussed in Chapter 4, see also Abraham, Marsden, and Ratiu [1988, Chapter 3].) Since the kernel of

$$T_\psi \pi : \mathcal{H} \to T_{[\psi]}\mathbb{P}\mathcal{H}$$

is $\mathbb{C}\psi$, the map $T_\psi\pi|(\mathbb{C}\psi)^\perp$ is a complex linear isomorphism from $(\mathbb{C}\psi)^\perp$ to $T_\psi\mathbb{P}\mathcal{H}$ that depends on the chosen representative ψ in $[\psi]$.

If $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator, that is, U is invertible and

$$\langle U\psi_1, U\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle$$

for all $\psi_1, \psi_2 \in \mathcal{H}$, then the rule $[U][\psi] := [U\psi]$ defines a biholomorphic diffeomorphism on $\mathbb{P}\mathcal{H}$.

Proposition 5.3.1.

(i) *If $[\psi] \in \mathbb{P}\mathcal{H}$, $\|\psi\| = 1$, and $\varphi_1, \varphi_2 \in (\mathbb{C}\psi)^\perp$, the formula*

$$\langle T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2) \rangle = 2\hbar \, \langle \varphi_1, \varphi_2 \rangle \tag{5.3.5}$$

gives a well-defined strong Hermitian inner product on $T_{[\psi]}\mathbb{P}\mathcal{H}$, that is, the left-hand side does not depend on the choice of ψ in $[\psi]$. The dependence on $[\psi]$ is smooth, and so (5.3.5) *defines a Hermitian metric on $\mathbb{P}\mathcal{H}$ called the* ***Fubini–Study metric****. This metric is invariant under the action of the maps $[U]$, for all unitary operators U on $\mathcal{H}$.*

(ii) *For* $[\psi] \in \mathbb{P}\mathcal{H}$, $\|\psi\| = 1$, *and* $\varphi_1, \varphi_2 \in (\mathbb{C}\psi)^\perp$,

$$g_{[\psi]}(T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)) = 2\hbar\,\mathrm{Re}\,\langle\varphi_1, \varphi_2\rangle \tag{5.3.6}$$

defines a strong Riemannian metric on $\mathbb{P}\mathcal{H}$ *invariant under all transformations* $[U]$.

(iii) *For* $[\psi] \in \mathbb{P}\mathcal{H}$, $\|\psi\| = 1$, *and* $\varphi_1, \varphi_2 \in (\mathbb{C}\psi)^\perp$,

$$\Omega_{[\psi]}(T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)) = -2\hbar\,\mathrm{Im}\,\langle\varphi_1, \varphi_2\rangle \tag{5.3.7}$$

defines a strong symplectic form on $\mathbb{P}\mathcal{H}$ *invariant under all transformations* $[U]$.

Proof. We first prove (i).[1] If $\lambda \in \mathbb{C}\backslash\{0\}$, then $\pi(\lambda(\psi + t\varphi)) = \pi(\psi + t\varphi)$, and since

$$(T_{\lambda\psi}\pi)(\lambda\varphi) = \frac{d}{dt}\pi(\lambda\psi + t\lambda\varphi)\Big|_{t=0} = \frac{d}{dt}\pi(\psi + t\varphi)\Big|_{t=0} = (T_\psi\pi)(\varphi),$$

we get $(T_{\lambda\psi}\pi)(\lambda\varphi) = (T_\psi\pi)(\varphi)$. Thus, if $\|\lambda\psi\| = \|\psi\| = 1$, it follows that $|\lambda| = 1$. We have, by (5.3.5),

$$\begin{aligned}\langle(T_{\lambda\psi}\pi)(\lambda\varphi_1), (T_{\lambda\psi}\pi)(\lambda\varphi_2)\rangle &= 2\hbar\,\langle\lambda\varphi_1, \lambda\varphi_2\rangle = 2\hbar|\lambda|^2\,\langle\varphi_1, \varphi_2\rangle \\ &= 2\hbar\,\langle\varphi_1, \varphi_2\rangle = \langle(T_\psi\pi)(\varphi_1), (T_\psi\pi)(\varphi_2)\rangle\,.\end{aligned}$$

This shows that the definition (5.3.5) of the Hermitian inner product is independent of the normalized representative $\psi \in [\psi]$ chosen in order to define it. This Hermitian inner product is strong, since it coincides with the inner product on the complex Hilbert space $(\mathbb{C}\psi)^\perp$.

A straightforward computation (see Exercise 5.3-3) shows that for $\psi \in \mathcal{H}\backslash\{0\}$ and $\varphi_1, \varphi_2 \in \mathcal{H}$ arbitrary, the Hermitian metric is given by

$$\langle T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)\rangle = 2\hbar\|\psi\|^{-2}(\langle\varphi_1, \varphi_2\rangle - \|\psi\|^{-2}\,\langle\varphi_1, \psi\rangle\,\langle\psi, \varphi_2\rangle). \tag{5.3.8}$$

Since the right-hand side is smooth in $\psi \in \mathcal{H}\backslash\{0\}$ and this formula drops to $\mathbb{P}\mathcal{H}$, it follows that (5.3.5) is smooth in $[\psi]$.

If U is a unitary map on $\mathcal{H}$ and $[U]$ is the induced map on $\mathbb{P}\mathcal{H}$, we have

$$\begin{aligned}T_{[\psi]}[U] \cdot T_\psi\pi(\varphi) = T_{[\psi]}[U] \cdot \frac{d}{dt}[\psi + t\varphi]\Big|_{t=0} &= \frac{d}{dt}[U][\psi + t\varphi]\Big|_{t=0} \\ = \frac{d}{dt}[U(\psi + t\varphi)]\Big|_{t=0} &= T_{U\psi}\pi(U\varphi).\end{aligned}$$

[1] One can give a conceptually cleaner, but more advanced, approach to this process using general reduction theory. The proof given here is by a direct argument.

Therefore, since $\|U\psi\| = \|\psi\| = 1$ and $\langle U\varphi_j, U\psi\rangle = 0$, we get by (5.3.5),

$$\begin{aligned}\left\langle T_{[\psi]}[U] \cdot T_\psi \pi(\varphi_1), T_{[\psi]}[U] \cdot T_\psi \pi(\varphi_2)\right\rangle &= \langle T_{U\psi}\pi(U\varphi_1), T_{U\psi}\pi(U\varphi_2)\rangle \\ &= \langle U\varphi_1, U\varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle \\ &= \langle T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)\rangle ,\end{aligned}$$

which proves the invariance of the Hermitian metric under the action of the transformation $[U]$.

Part (ii) is obvious as the real part of the Hermitian metric (5.3.5).

Finally, we prove (iii). From the invariance of the metric it follows that the form Ω is also invariant under the action of unitary maps, that is, $[U]^*\Omega = \Omega$. So, also $[U]^*\mathbf{d}\Omega = \mathbf{d}\Omega$. Now consider the unitary map U_0 on $\mathcal{H}$ defined by $U_0\psi = \psi$ and $U_0 = -$Identity on $(\mathbb{C}\psi)^\perp$. Then from $[U_0]^*\Omega = \Omega$ we have for $\varphi_1, \varphi_2, \varphi_3 \in (\mathbb{C}\psi)^\perp$,

$$\begin{aligned}&\mathbf{d}\Omega([\psi])(T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2), T_\psi\pi(\varphi_3)) \\ &\quad = \mathbf{d}\Omega([\psi])(T_{[\psi]}[U_0] \cdot T_\psi\pi(\varphi_1), T_{[\psi]}[U_0] \cdot T_\psi\pi(\varphi_2), T_{[\psi]}[U_0] \cdot T_\psi\pi(\varphi_3)).\end{aligned}$$

But

$$T_{[\psi]}[U_0] \cdot T_\psi\pi(\varphi) = T_\psi\pi(-\varphi) = -T_\psi\pi(\varphi),$$

which implies by trilinearity of $\mathbf{d}\Omega$ that $\mathbf{d}\Omega = 0$.

The symplectic form Ω is strongly nondegenerate, since on $T_{[\psi]}\mathbb{P}\mathcal{H}$ it restricts to the corresponding quantum-mechanical symplectic form on the Hilbert space $(\mathbb{C}\psi)^\perp$. ■

The results above prove that $\mathbb{P}\mathcal{H}$ is an infinite-dimensional Kähler manifold on which the unitary group $U(\mathcal{H})$ acts by isometries. This can be generalized to Grassmannian manifolds of finite- (or infinite-) dimensional subspaces of $\mathcal{H}$, and even more, to flag manifolds (see Besse [1987] and Pressley and Segal [1986]).

Exercises

◇ **5.3-1.** On $\mathbb{C}^n$, show that $\Omega = -\mathbf{d}\Theta$, where $\Theta(z) \cdot w = \frac{1}{2}\operatorname{Im}\langle z, w\rangle$.

◇ **5.3-2.** Let P be a manifold that is both symplectic, with symplectic form Ω, and Riemannian, with strong metric g.

(a) Show that P has an almost complex structure $\mathbb{J}$ such that $\Omega(u, v) = g(\mathbb{J}u, v)$ if and only if

$$\Omega(\nabla F, v) = -g(X_F, v)$$

for all $F \in \mathcal{F}(P)$.

(b) Under the hypothesis of (a), show that a Hamiltonian vector field X_H is locally a gradient if and only if $\pounds_{\nabla H}\Omega = 0$.

◇ **5.3-3.** Show that for any vectors $\varphi_1, \varphi_2 \in \mathcal{H}$ and $\psi \neq 0$ the Fubini–Study metric can be written

$$\langle T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)\rangle = 2\hbar\|\psi\|^{-2}(\langle\varphi_1,\varphi_2\rangle - \|\psi\|^{-2}\langle\varphi_1,\psi\rangle\langle\psi,\varphi_2\rangle).$$

Conclude that the Riemannian metric and symplectic form are given by

$$g_{[\psi]}(T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)) = \frac{2\hbar}{\|\psi\|^4}\operatorname{Re}(\langle\varphi_1,\varphi_2\rangle\|\psi\|^2 - \langle\varphi_1,\psi\rangle\langle\psi,\varphi_2\rangle)$$

and

$$\Omega_{[\psi]}(T_\psi\pi(\varphi_1), T_\psi\pi(\varphi_2)) = -\frac{2\hbar}{\|\psi\|^4}\operatorname{Im}(\langle\varphi_1,\varphi_2\rangle\|\psi\|^2 - \langle\varphi_1,\psi\rangle\langle\psi,\varphi_2\rangle).$$

◇ **5.3-4.** Prove that $\mathbf{d}\Omega = 0$ on $\mathbb{P}\mathcal{H}$ directly without using the invariance under the maps $[U]$, for U a unitary operator on $\mathcal{H}$.

◇ **5.3-5.** For $\mathbb{C}^{n+1}$, show that in a projective chart of $\mathbb{CP}^n$ the symplectic form Ω is given by

$$-i\hbar(1+|z|^2)^{-1}(\mathbf{d}\sigma + (1+|z|^2)^{-1}\sigma\wedge\overline{\sigma}),$$

where $\mathbf{d}|z|^2 = \sigma + \overline{\sigma}$ (explicitly, $\sigma = \sum_{i=1}^n z_i\mathbf{d}\overline{z}_i$). Use this to show that $\mathbf{d}\Omega = 0$. (Use the general formula in Exercise 5.3-3.)

5.4 Hamiltonian Systems

With the geometry of symplectic manifolds now available, we are ready to study Hamiltonian dynamics in this setting.

Definition 5.4.1. *Let (P,Ω) be a symplectic manifold. A vector field X on P is called* ***Hamiltonian*** *if there is a function $H : P \to \mathbb{R}$ such that*

$$\mathbf{i}_X\Omega = \mathbf{d}H; \tag{5.4.1}$$

that is, for all $v \in T_zP$, we have the identity

$$\Omega_z(X(z), v) = \mathbf{d}H(z)\cdot v.$$

In this case we write X_H for X. The set of all Hamiltonian vector fields on P is denoted by $\mathfrak{X}_{\mathrm{Ham}}(P)$. ***Hamilton's equations*** *are the evolution equations*

$$\dot{z} = X_H(z).$$

In finite dimensions, Hamilton's equations in canonical coordinates are

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}.$$

Vector Fields and Flows. A vector field X is called ***locally Hamiltonian*** if $\mathbf{i}_X \Omega$ is closed. This is equivalent to $\pounds_X \Omega = 0$, where $\pounds_X \Omega$ denotes Lie differentiation of Ω along X, because

$$\pounds_X \Omega = \mathbf{i}_X \mathbf{d}\Omega + \mathbf{d}\mathbf{i}_X \Omega = \mathbf{d}\mathbf{i}_X \Omega.$$

If X is locally Hamiltonian, it follows from the Poincaré lemma that there locally exists a function H such that $\mathbf{i}_X \Omega = \mathbf{d}H$, so locally $X = X_H$, and thus the terminology is consistent. In a symplectic vector space, we have seen in Chapter 2 that the condition that $\mathbf{i}_X \Omega$ be closed is equivalent to $\mathbf{D}X(z)$ being Ω-skew. Thus, the definition of locally Hamiltonian is an intrinsic generalization of what we did in the vector space case.

The flow φ_t of a locally Hamiltonian vector field X satisfies $\varphi_t^* \Omega = \Omega$, since

$$\frac{d}{dt} \varphi_t^* \Omega = \varphi_t^* \pounds_X \Omega = 0,$$

and thus we have proved the following:

Proposition 5.4.2. *The flow φ_t of a vector field X consists of symplectic transformations (that is, for each t, we have $\varphi_t^* \Omega = \Omega$ where defined) if and only if X is locally Hamiltonian.*

A constant vector field on the torus $\mathbb{T}^2$ gives an example of a locally Hamiltonian vector field that is not Hamiltonian. (See Exercise 5.4-1.)

Using the straightening out theorem (see, for example, Abraham, Marsden, and Ratiu [1988, Section 4.1]) it is easy to see that on an even-dimensional manifold *any* vector field is locally Hamiltonian near points where it is nonzero, relative to *some* symplectic form. However, it is not so simple to get a general criterion of this sort that is global, covering singular points as well.

Energy Conservation. If X_H is Hamiltonian with flow φ_t, then by the chain rule,

$$\begin{aligned} \frac{d}{dt}(H\varphi_t(z)) &= \mathbf{d}H(\varphi_t(z)) \cdot X_H(\varphi_t(z)) \\ &= \Omega\left(X_H(\varphi_t(z)), X_H(\varphi_t(z))\right) = 0, \end{aligned} \tag{5.4.2}$$

since Ω is skew. Thus $H \circ \varphi_t$ is constant in t. We have proved the following:

Proposition 5.4.3 (Conservation of Energy)**.** *If φ_t is the flow of X_H on the symplectic manifold P, then $H \circ \varphi_t = H$ (where defined).*

Transformation of Hamiltonian Systems. As in the vector space case, we have the following results.

Proposition 5.4.4. *A diffeomorphism $\varphi : P_1 \to P_2$ of symplectic manifolds is symplectic if and only if it satisfies*

$$\varphi^* X_H = X_{H \circ \varphi} \tag{5.4.3}$$

for all functions $H : U \to \mathbb{R}$ *(such that* X_H *is defined) where* U *is any open subset of* P_2*.*

Proof. The statement (5.4.3) means that for each $z \in P$,

$$T_{\varphi(z)}\varphi^{-1} \cdot X_H(\varphi(z)) = X_{H\circ\varphi}(z),$$

that is,

$$X_H(\varphi(z)) = T_z\varphi \cdot X_{H\circ\varphi}(z).$$

In other words,

$$\Omega(\varphi(z))(X_H(\varphi(z)), T_z\varphi \cdot v) = \Omega(\varphi(z))(T_z\varphi \cdot X_{H\circ\varphi}(z), T_z\varphi \cdot v)$$

for all $v \in T_zP$. If φ is symplectic, this becomes

$$\mathbf{d}H(\varphi(z)) \cdot [T_z\varphi \cdot v] = \mathbf{d}(H \circ \varphi)(z) \cdot v,$$

which is true by the chain rule. Thus, if φ is symplectic, then (5.4.3) holds. The converse is proved in the same way. ■

The same qualifications on technicalities pertinent to the infinite-dimensional case that were discussed for vector spaces apply to the present context as well. For instance, given H, there is no *a priori* guarantee that X_H exists: We usually assume it abstractly and verify it in examples. Also, we may wish to deal with X_H's that have dense domains rather than everywhere defined smooth vector fields. These technicalities are important, but they do not affect many of the main goals of this book. We shall, for simplicity, deal only with everywhere defined vector fields and refer the reader to Chernoff and Marsden [1974] and Marsden and Hughes [1983] for the general case. We shall also tacitly restrict our attention to functions that *have* Hamiltonian vector fields. Of course, in the finite-dimensional case these technical problems disappear.

Exercises

◇ **5.4-1.** Let X be a constant nonzero vector field on the two-torus. Show that X is locally Hamiltonian but is not globally Hamiltonian.

◇ **5.4-2.** Show that the bracket of two locally Hamiltonian vector fields on a symplectic manifold (P, Ω) is globally Hamiltonian.

◇ **5.4-3.** Consider the equations on $\mathbb{C}^2$ given by

$$\begin{aligned}\dot{z}_1 &= -iw_1z_1 + ip\overline{z}_2 + iz_1(a|z_1|^2 + b|z_2|^2),\\ \dot{z}_2 &= -iw_2z_2 + iq\overline{z}_1 + iz_2(c|z_1|^2 + d|z_2|^2),\end{aligned}$$

where $w_1, w_2, p, q, a, b, c, d$ are real. Show that this system is Hamiltonian if and only if $p = q$ and $b = c$ with

$$H = \frac{1}{2}\left(w_2|z_2|^2 + w_1|z_1|^2\right) - p\ \mathrm{Re}(z_1z_2) - \frac{a}{4}|z_1|^4 - \frac{b}{2}|z_1z_2|^2 - \frac{d}{4}|z_2|^4.$$

◇ **5.4-4.** Let (P, Ω) be a symplectic manifold and $\varphi : S \longrightarrow P$ an immersion. The immersion φ is called a ***coisotropic immersion*** if $T_s\varphi(T_sS)$ is a coisotropic subspace of $T_{\varphi(s)}P$ for every $s \in S$. This means that

$$[T_s\varphi(T_sS)]^{\Omega(\varphi(s))} \subset T_s\varphi(T_sS)$$

for every $s \in S$ (see Exercise 2.3-5). If (P, Ω) is a strong symplectic manifold, show that $\varphi : S \longrightarrow P$ is a coisotropic immersion if and only if $X_H(\varphi(s)) \in T_s\varphi(T_sS)$ for all $s \in S$, all open neighborhoods U of $\varphi(s)$ in P, and all smooth functions $H : U \longrightarrow \mathbb{R}$ satisfying $H|\varphi(S) \cap U = \text{constant}$.

5.5 Poisson Brackets on Symplectic Manifolds

Analogous to the vector space treatment, we define the ***Poisson bracket*** of two functions $F, G : P \to \mathbb{R}$ by

$$\{F, G\}(z) = \Omega(X_F(z), X_G(z)). \tag{5.5.1}$$

From Proposition 5.4.4 we get (see the proof of Proposition 2.7.5) the following result.

Proposition 5.5.1. *A diffeomorphism $\varphi : P_1 \to P_2$ is symplectic if and only if*

$$\{F, G\} \circ \varphi = \{F \circ \varphi, G \circ \varphi\} \tag{5.5.2}$$

for all functions $F, G \in \mathcal{F}(U)$, where U is an arbitrary open subset of P_2.

Using this, Proposition 5.4.2 shows that the following statement holds.

Proposition 5.5.2. *If φ_t is the flow of a Hamiltonian vector field X_H (or a locally Hamiltonian vector field), then*

$$\varphi_t^*\{F, G\} = \{\varphi_t^*F, \varphi_t^*G\}$$

for all $F, G \in \mathcal{F}(P)$ (or restricted to an open set if the flow is not everywhere defined).

Corollary 5.5.3. *The following derivation identity holds:*

$$X_H[\{F, G\}] = \{X_H[F], G\} + \{F, X_H[G]\}, \tag{5.5.3}$$

where we use the notation $X_H[F] = \pounds_{X_H}F$ for the derivative of F in the direction X_H.

Proof. Differentiate the identity

$$\varphi_t^*\{F, G\} = \{\varphi_t^* F, \varphi_t^* G\}$$

in t at $t = 0$, where φ_t is the flow of X_H. The left-hand side clearly gives the left side of (5.5.3). To evaluate the right-hand side, first notice that

$$\begin{aligned} \Omega_z^\flat \left[\left. \frac{d}{dt} \right|_{t=0} X_{\varphi_t^* F}(z) \right] &= \left. \frac{d}{dt} \right|_{t=0} \Omega_z^\flat X_{\varphi_t^* F}(z) \\ &= \left. \frac{d}{dt} \right|_{t=0} \mathbf{d}(\varphi_t^* F)(z) \\ &= (\mathbf{d} X_H[F])(z) = \Omega_z^\flat (X_{X_H[F]}(z)). \end{aligned}$$

Thus,

$$\left. \frac{d}{dt} \right|_{t=0} X_{\varphi_t^* F} = X_{X_H[F]}.$$

Therefore,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \{\varphi_t^* F, \varphi_t^* G\} &= \left. \frac{d}{dt} \right|_{t=0} \Omega_z (X_{\varphi_t^* F}(z), X_{\varphi_t^* G}(z)) \\ &= \Omega_z (X_{X_H[F]}, X_G(z)) + \Omega_z (X_F(z), X_{X_H[G]}(z)) \\ &= \{X_H[F], G\}(z) + \{F, X_H[G]\}(z). \end{aligned}$$

■

Lie Algebras and Jacobi's Identity. The above development leads to important insight into Poisson brackets.

Proposition 5.5.4. *The functions $\mathcal{F}(P)$ form a Lie algebra under the Poisson bracket.*

Proof. Since $\{F, G\}$ is obviously real bilinear and skew-symmetric, the only thing to check is Jacobi's identity. From

$$\{F, G\} = \mathbf{i}_{X_F} \Omega(X_G) = \mathbf{d}F(X_G) = X_G[F],$$

we have

$$\{\{F, G\}, H\} = X_H[\{F, G\}],$$

and so by Corollary 5.5.3 we get

$$\begin{aligned} \{\{F, G\}, H\} &= \{X_H[F], G\} + \{F, X_H[G]\} \\ &= \{\{F, H\}, G\} + \{F, \{G, H\}\}, \end{aligned} \tag{5.5.4}$$

which is Jacobi's identity. ■

This derivation gives us additional insight: *Jacobi's identity is just the infinitesimal statement of* φ_t *being canonical*.

In the same spirit, one can check that *if* Ω *is a nondegenerate two-form with the Poisson bracket defined by* (5.5.1)*, then the Poisson bracket satisfies the Jacobi identity if and only if* Ω *is closed* (see Exercise 5.5-1).

The ***Poisson bracket–Lie derivative identity***

$$\{F, G\} = X_G[F] = -X_F[G] \tag{5.5.5}$$

we derived in this proof will be useful.

Proposition 5.5.5. *The set of Hamiltonian vector fields* $\mathfrak{X}_{\mathrm{Ham}}(P)$ *is a Lie subalgebra of* $\mathfrak{X}(P)$*, and in fact,*

$$[X_F, X_G] = -X_{\{F,G\}}. \tag{5.5.6}$$

Proof. As derivations,

$$\begin{aligned}
[X_F, X_G][H] &= X_F X_G[H] - X_G X_F[H] \\
&= X_F[\{H, G\}] - X_G[\{H, F\}] \\
&= \{\{H, G\}, F\} - \{\{H, F\}, G\} \\
&= -\{H, \{F, G\}\} = -X_{\{F,G\}}[H],
\end{aligned}$$

by Jacobi's identity. ■

Proposition 5.5.6. *We have*

$$\frac{d}{dt}(F \circ \varphi_t) = \{F \circ \varphi_t, H\} = \{F, H\} \circ \varphi_t, \tag{5.5.7}$$

where φ_t *is the flow of* X_H *and* $F \in \mathcal{F}(P)$*.*

Proof. By (5.5.5) and the chain rule,

$$\frac{d}{dt}(F \circ \varphi_t)(z) = \mathbf{d}F(\varphi_t(z)) \cdot X_H(\varphi_t(z)) = \{F, H\}(\varphi_t(z)).$$

Since φ_t is symplectic, this becomes

$$\{F \circ \varphi_t, H \circ \varphi_t\}(z),$$

which also equals $\{F \circ \varphi_t, H\}(z)$ by conservation of energy. This proves (5.5.7). ■

Equations in Poisson Bracket Form. Equation (5.5.7), often written more compactly as

$$\dot{F} = \{F, H\}, \tag{5.5.8}$$

is called the ***equation of motion in Poisson bracket form***. We indicated in Chapter 1 why the formulation (5.5.8) is important.

Corollary 5.5.7. *$F \in \mathcal{F}(P)$ is a constant of the motion for X_H if and only if $\{F, H\} = 0$.*

Proposition 5.5.8. *Assume that the functions f, g, and $\{f, g\}$ are integrable relative to the Liouville volume $\Lambda \in \Omega^{2n}(P)$ on a $2n$-dimensional symplectic manifold (P, Ω). Then*

$$\int_P \{f, g\}\Lambda = \int_{\partial P} f\mathbf{i}_{X_g}\Lambda = -\int_{\partial P} g\mathbf{i}_{X_f}\Lambda.$$

Proof. Since $\pounds_{X_g}\Omega = 0$, it follows that $\pounds_{X_g}\Lambda = 0$, so that $\operatorname{div}(fX_g) = X_g[f] = \{f, g\}$. Therefore, by Stokes' theorem,

$$\int_P \{f, g\}\Lambda = \int_P \operatorname{div}(fX_g)\Lambda = \int_P \pounds_{fX_g}\Lambda = \int_P \mathbf{d}\mathbf{i}_{fX_g}\Lambda = \int_{\partial P} f\mathbf{i}_{X_g}\Lambda,$$

the second equality following by skew-symmetry of the Poisson bracket. ■

Corollary 5.5.9. *Assume that $f, g, h \in \mathcal{F}(P)$ have compact support or decay fast enough such that they and their Poisson brackets are L^2 integrable relative to the Liouville volume on a $2n$-dimensional symplectic manifold (P, Ω). Assume also that at least one of f and g vanish on ∂P if $\partial P \neq \varnothing$. Then the L^2-inner product is bi-invariant on the Lie algebra $(\mathcal{F}(P), \{\,,\})$, that is,*

$$\int_P f\{g, h\}\Lambda = \int_P \{f, g\}h\Lambda.$$

Proof. From $\{hf, g\} = h\{f, g\} + f\{h, g\}$ we get

$$0 = \int_P \{hf, g\}\Lambda = \int_P h\{f, g\}\Lambda + \int_P f\{h, g\}\Lambda.$$

However, from Proposition 5.5.8, the integral of $\{hf, g\}$ over P vanishes, since one of f or g vanishes on ∂P. The corollary then follows. ■

Exercises

◇ **5.5-1.** Let Ω be a nondegenerate two-form on a manifold P. Form Hamiltonian vector fields and the Poisson bracket using the same definitions as in the symplectic case. Show that Jacobi's identity holds if and only if the two-form Ω is closed.

◇ **5.5-2.** Let P be a compact boundaryless symplectic manifold. Show that the space of functions $\mathcal{F}_0(P) = \{\, f \in \mathcal{F}(P) \mid \int_P f\Lambda = 0 \,\}$ is a Lie subalgebra of $(\mathcal{F}(P), \{\,,\})$ isomorphic to the Lie algebra of Hamiltonian vector fields on P.

◇ **5.5-3.** Using the complex notation $z^j = q^j + ip_j$, show that the symplectic form on $\mathbb{C}^n$ may be written as

$$\Omega = \frac{i}{2} \sum_{k=1}^{n} dz^k \wedge d\bar{z}^k,$$

and the Poisson bracket may be written

$$\{F, G\} = \frac{2}{i} \sum_{k=1}^{n} \left(\frac{\partial F}{\partial z^k} \frac{\partial G}{\partial \bar{z}^k} - \frac{\partial G}{\partial z^k} \frac{\partial F}{\partial \bar{z}^k} \right).$$

◇ **5.5-4.** Let $J : \mathbb{C}^2 \to \mathbb{R}$ be defined by

$$J = \frac{1}{2}(|z_1|^2 - |z_2|^2).$$

Show that

$$\{H, J\} = 0,$$

where H is given in Exercise 5.4-3.

◇ **5.5-5.** Let (P, Ω) be a $2n$-dimensional symplectic manifold. Show that the Poisson bracket may be defined by

$$\{F, G\}\Omega^n = \gamma \mathbf{d}F \wedge \mathbf{d}G \wedge \Omega^{n-1}$$

for a suitable constant γ.

◇ **5.5-6.** Let $\varphi : S \longrightarrow P$ be a coisotropic immersion (see Exercise 5.4-4). Let $F, H : P \longrightarrow \mathbb{R}$ be smooth functions such that $\mathbf{d}(\varphi^* F)(s)$, $\mathbf{d}(\varphi^* H)(s)$ vanish on $(T_s\varphi)^{-1}([T_s\varphi(T_s S)]^{\Omega(\varphi(s))})$ for all $s \in S$. Show that $\varphi^*\{F, H\}$ depends only on $\varphi^* F$ and $\varphi^* H$.