CHAPTER VIII

THE CASE OF TWO DEGREES OF FREEDOM

1. Formal classification of periodic motions. In chapter VI we studied dynamical systems of Hamiltonian type ($m = 2$) in a preliminary way, with particular reference to the periodic motions. We propose now not only to obtain a more complete idea as to the existence and distribution of these periodic motions, but also of the various other types of motion.

For systems of this type the manifold of states of motion is four-dimensional at the outset, with coördinates p_1, q_1, p_2, q_2. However, by specification of the constant of energy, $H = h$, a three-dimensional analytic sub-manifold is defined, and it is a particular such manifold M which is the manifold of states of motion under consideration. In other words, by use of the energy integral the system of differential equations is reduced from the fourth to the third order.

In order to limit attention to a definite type of case we assume that M is non-singular, i. e. closed and analytic, and furthermore we exclude the possibility of equilibrium in M, since equilibrium cannot arise for a general value of h.

Consider now a periodic motion, which will be represented by a closed curve in M. Imagine that curve to be cut by an analytic surface S. If a point P of S is followed along its corresponding curve of motion in the sense of increasing time, it will intersect S again at a point P_1. We write $P_1 = T(P)$, thereby defining a one-to-one, analytic transformation of S into itself, at least in the neighborhood of the given periodic motion. The transformation T leaves invariant the point corresponding to the periodic motion. For periodic motions near to the given periodic motion, but represented by curves making k circuits before closing, there will be a corresponding set of k points, $P, T(P), \cdots, T^{k-1}(P)$ with $T^k(P) = P$, where the meaning of the notation is manifest.

There is a particular and important case in which we can specify the characteristic properties of the transformation T, on the basis of our earlier work. This is the case in which the Hamiltonian problem is derived from a Lagrangian problem with principal function quadratic in the velocities (chapter VI, sections 1–3).

Let us recall the precise method of selecting the coördinates in this case. In the first place the Lagrangian coördinates q_1, q_2 are so selected that q_1 is an angular coördinate reducing to $2\pi t/\tau$ (τ, the period) along the periodic motion, while q_2 is 0 along it. Then, from the differential equations, $\partial H/\partial p_1$ is not 0 along the periodic motion, and we can solve the equation $H = h$ for p_1 in the form

$$p_1 + K(q_1, p_2, q_2, h) = 0,$$

where K is analytic in its four arguments and is periodic of period 2π in q_1. Hence q_1, p_2, q_2 constitute a suitable set of coördinates for M in the torus-shaped vicinity of the given periodic motion. In these variables the equations of Hamiltonian type

$$\frac{dp_2}{dq_1} = -\frac{\partial K}{\partial q_2}, \quad \frac{dq_2}{dq_1} = \frac{\partial K}{\partial p_2} \tag{1}$$

subsist. The equations (1) enable us to express the coördinates p_2, q_2 of any curve of motion in terms of the angular coördinate q_1; and t may then be found by a simple integration.

A final modification is to use $p_2 - p_2^0$ as coördinate instead of p_2. Here $p_2^0(t)$ is the expression for p_2 along the given curve where t is thought of as replaced by its value $q_1\tau/2\pi$ along the periodic motion. If at the same time we modify K by adding a term $q_2\, dp_2^0/dq_1$, the form (1) is preserved, and the periodic motion corresponds to $p_2 = q_2 = 0$. Furthermore K remains periodic of period 2π in q_1.

In this way the nature of the coördinates employed in the reduction to generalized equilibrium becomes apparent. These coördinates have the advantage that when the 'plane' $q_1 = 0$ is taken as the surface of section S, the transformation T becomes area-preserving. Furthermore, if the periodic motion is of general stable type, so that the multipliers $\pm\lambda$ in (1) are pure imaginary and incommensurable with $\sqrt{-1}$, it was seen (loc. cit.) that T may be given the normal form

$$(2) \qquad \begin{aligned} u_1 &= u_0 \cos(\sigma + s r_0^2) - v_0 \sin(\sigma + s r_0^2) + \Phi \\ v_1 &= u_0 \sin(\sigma + s r_0^2) + v_0 \cos(\sigma + s r_0^2) + \Psi \end{aligned} \qquad (r_0^2 = u_0^2 + v_0^2),$$

in suitably chosen variables u, v, where Φ, Ψ are given by convergent power series in u_0, v_0 starting off with terms of arbitrarily high degree, at least if a certain quantity $l = \sqrt{-1}\, s/2\pi$ does not vanish.

It should be remarked that the choice of the surface S does not affect the transformation T obtained except by a change of variables.

The detailed study of the transformation T on the basis of the area-preserving property and the normal form (2) enabled us to infer that infinitely many periodic motions exist in the immediate vicinity of the given periodic motion of stable type.

In the case when the periodic motion is of general unstable type the multiplier λ is either real, or $2\lambda - \sqrt{-1}$ 7
may be taken real. The same method as was employed in the stable case (chapter III, sections 6—9) leads to an analogous formal solution, and to a real normal form for T

$$(3) \qquad u_1 = \mu u e^{l r_0^2} + \Phi, \qquad v_1 = \frac{1}{\mu} v e^{-l r_0^2} + \Psi \qquad (\mu \neq \pm 1), \qquad 8$$

in the case $l \neq 0$, where Φ, Ψ are of the same type as in (2).

This general unstable case is very simple to treat analytically.* There will be two invariant analytic curves through

* Cf. my paper, *Surface Transformations and Their Dynamical Applications,* Acta Mathematica, vol. 43 (1912), section 27, or the paper by Hadamard, *Sur l'itération et les solutions asymptotiques des équations différentielles,* Bull. Soc. Math. France, vol. 29 (1901).

the origin, which may be taken to be the u and v axes. Points along one of these invariant curves approach the origin upon successive iteration of T; points on the other invariant curve leave the vicinity of the origin; while points not on these curves first approach and then recede from the origin.

If then we interpret this situation in the manifold M near a periodic motion of unstable type, we find that there are two invariant analytic surfaces through the curve of periodic motion, one of these corresponding to an analytic family of positively asymptotic motions, the other to an analytic family of negatively asymptotic motions. Other nearby motions first approach and then recede from the given periodic motion of unstable type.

It is clear that there can be no periodic motions whatsoever lying wholly near to the periodic motion of unstable type, in contrast with the fact that there must be periodic motions near to any periodic motion of general stable type ($l \neq 0$).

In this way it is seen how fundamentally the classes of periodic motions of stable and unstable type differ from one another.

We are now prepared to state to what extent the apparent limitations introduced are necessary.

In the first place not all the first partial derivatives of H can vanish at any point of the periodic motion, so that the manifold $H = h$ is a regular analytic three-dimensional manifold M along the periodic motion. If variables p, q, r, h are selected as coördinates instead of p_1, q_1, p_2, q_2, the invariant integral of ordinary four-dimensional volume takes the form

$$\int\int\int\int \varphi \, dp \, dq \, dr \, dh$$

where $\varphi > 0$ is analytic in p, q, r, h. Hence $\int\int\int \varphi \, dp \, dq \, dr$ is invariant in M.

It follows further that T will leave a double integral $\int\int \psi \, du \, dv$ invariant, where u, v are coördinates in S, and

$\psi > 0$ is analytic in u and v. The argument is essentially that of chapter VI, section 1. This fact alone suffices to lead to the normal forms (2), (3), and to the conclusions cited above.*

Hence the Hamiltonian problem need not be restricted in this manner.

In fact it is found that for the most general transformation T with such an invariant double integral $\int\int \psi\, du\, dv$, there is always a formally invariant function $\Omega(u, v)$ given by a formal power series in u, v. We may define the case in which the equation $\Omega = 0$ yields real formal invariant curves as of unstable type. In this case there are always asymptotic invariant analytic families of motions (or analytic families of periodic motions containing the given periodic motion). 9
All other nearby motions approach and then recede from the given motion. Thus there are no nearby periodic motions, except those that belong to the same analytic family as the given periodic motion, if there are such.

If $\Omega = 0$ yields no real formal invariant curve of this kind, the periodic motion may be called of stable type. In the general stable case treated above, Ω is r^2, to terms of higher order. When σ is incommensurable with 2π, while s, together with some but not all of the set of analogous constants perhaps, vanishes, no essential modification is required except that the term sr_0^2 in (2) is replaced by a term $s^{(k)} r_0^{2k}$. If, however, all of these constants vanish, the normal form (2) holds with $s = 0$, and for these irregular periodic motions it is no longer possible to apply the reasoning by which the existence of infinitely many nearby periodic motions was established.

On the other hand, no essential difficulty arises in the case of stable type when σ is 0 or $\pm\pi$ or, more generally, is commensurable with 2π; this is the case when the given periodic motion is multiple, at least when taken as

* See my paper (loc. cit.) for justification of the fact stated as well as of what follows.

described a certain number, k, of times. It is only necessary to consider T^k in place of T, for which the number σ is also 0. Here the invariant function Ω starts off with higher degree terms than the second, and casual inspection indicates that T is analogous to a rotation through an angle which vanishes at the origin but increases (or decreases) with distance from the origin. It would therefore seem highly probable that in this case too there must be infinitely many neighboring periodic motions, although the analytic details need to be carried through.

Consequently it appears that in very general stable cases, and probably in all cases except the highly exceptional case when T is equivalent formally to a pure rotation through an angle incommensurable with 2π, this property will continue to hold. This exceptional case is that in which the function M in the formal solution reduces to its first term λ.

Hence in the most general case of unstable type ($m = 2$) *the phenomenon of asymptotic analytic families of motions (or at least of analytic families of periodic motions containing the given motion) is characteristic. Other nearby motions approach and then recede from the given periodic motion.*

In the most general stable case, except the highly degenerate case where σ is incommensurable with 2π and the formal series involve no variable periods, there will be neighboring periodic motions.

It is to be emphasized that the second of these conclusions has been formulated without completion of a detailed proof, such as I have not yet had the opportunity to effect.

The degenerate case of stable type includes a real exception, as the example of chapter VI, section 4, shows, and should be further studied. Moreover the formal series break down in the stable case when λ is commensurable with $\sqrt{-1}$. These must be replaced by much more complicated types of series, a suggestion for the structure of which may perhaps be found in my paper referred to above; the earlier definition of complete formal stability will need to be extended so as to permit of indefinitely large periods.

Between the non-specialized dynamical problem and the highly exceptional integrable case, there exists an enormous variety of intermediate cases. In order to possess the analytical weapons with which to treat all cases whatsoever, it will undoubtedly be necessary to treat the question of the stability and instability of analytic families of periodic motions in much the same way as that which is outlined for the periodic motions above. While the individual periodic motions in such a family are to be regarded as unstable, this fact yields no information as to how nearby motions behave with respect to the family of motions as a whole.

In order to avoid complication then, rather than because of any essential mathematical difficulty, we propose to deal mainly with the class of dynamical problems for which every periodic motion and its multiples are simple with $l \neq 0$. Such 11
systems will be termed 'non-integrable systems of general type'. The integrable case will be treated separately (section 13), while indications as to the nature of the result in the intermediate cases will be given.

2. Distribution of periodic motions of stable type. Our first aim will be to establish the following result:

For non-integrable Hamiltonian systems of general type ($m = 2$), *the set of periodic motions of general stable type is dense on itself in M.*

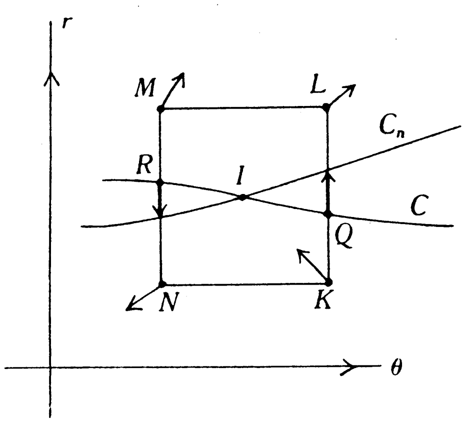
It will be observed that this result constitutes a slight improvement over the result of chapter VI, sections 1–3, according to which other periodic motions, stable or unstable, lie near such a periodic motion of stable type.

To begin with, we recall the facts developed in the lemma of chapter VI, section 1. It was found there that an arbitrarily small vicinity of the origin, $r \leqq \varrho$ (r, θ, polar coördinates) can be selected at pleasure, and then an integer n such that (1) all the points of $r \leqq \varrho$ remain in the region $r \leqq 2\varrho$ under $T, T^2, \cdots, T^n$ and (2) $\partial \theta_n / \partial r_0$ is positive for $r \leqq \varrho$, θ_n being at least 2π greater for $r = \varrho$ than for $r = 0$. It is easy to extend the argument to show that $\partial r_n / \partial r_0$, $\partial \theta_n / \partial \theta_0$ are positive under the same circumstances.

In this way the curve

$$\theta_n - \theta_0 - 2k\pi = 0,$$

where k is so chosen that the left-hand member is negative but not less than -2π for $r = 0$, will have one and only one point (r, θ) on each radius vector, with $r \leqq \varrho$. Thus the equation written defines an analytic curve C encircling the origin and meeting each radius vector only once. But, by the defining property of C, each point P of C goes into a point P_n on the same radius vector; thus the curve C_n is also met only once by any radius vector. Also, because of the area-preserving property of T^n, C_n and C will intersect in at least two points, and these are obviously invariant points of T^n. In the case under consideration C_n and C cannot coincide, for C would then correspond to an analytic family of multiple periodic motions.



We propose to consider more closely the indices of these invariant points. Let us regard r, θ as rectangular coördinates and consider the adjoining figure in which I is an invariant point at which the curve C_n passes from within C to outside of C, as a moving point describes C in the sense of increasing θ.

If a point P makes a positive circuit of I, for instance around a rectangle $KLMN$, the vector PP_n will have a component to the right above C, a component to the left below C, as follows from the facts noted above. At the points Q and R the vector PP_n is directed upwards and downwards respectively. It is therefore apparent that during the circuit, the vector PP_n rotates through an angle $+2\pi$, so that the index of I is $+1$; while the invariant point J, at which C_n crosses C in the opposite sense has an index -1.

Now by hypothesis the periodic motions corresponding to I and J are not multiple. If of stable type, the number σ is not commensurable with 2π. The corresponding normal 12
forms are either of the type (3), in which μ is positive or negative but not ± 1, or of the type (2).

By the aid of these forms it is easy to determine the respective indices. In the first and second cases, the slope of the vector PP_n is

$$\frac{v_n - v}{u_n - u} = -\frac{1}{\mu}\,\frac{v + \cdots}{u + \cdots},$$

where only the first order terms are indicated explicitly in the numerator and denominator. It thus appears that if μ is negative the rotation is the same as that of the vector drawn from the invariant point I at the origin to the point (u, v), i. e., 2π. Hence the index is $+1$ if μ is negative; likewise the index is obviously -1 if μ is positive. Moreover, in the third case of stable type when the number σ is by hypothesis incommensurable with 2π in (2), T^n is approximately a rotation through an angle incommensurable with 2π near the origin, so that the vector PP_n rotates through 2π during such a circuit. Hence the index is $+1$ if the motion is of stable type.

We infer then that J corresponds to a periodic motion of unstable type, while it is not clear as yet whether I is of stable or unstable type.

As a matter of fact, however, under the conditions stated I must be stable. The numbers μ are the roots of the characteristic equation, which takes the form

$$\begin{vmatrix} \dfrac{\partial r_n}{\partial r_0} - \mu, & \dfrac{\partial r_n}{\partial \theta_0} \\ \dfrac{\partial \theta_n}{\partial r_0}, & \dfrac{\partial \theta_n}{\partial \theta_0} - \mu \end{vmatrix} = 0$$

when the variables r, θ are used. Since the roots are the reciprocals of one another, this equation reduces to

$$\mu^2 - \left(\frac{\partial r_n}{\partial r_0} + \frac{\partial \theta_n}{\partial \theta_0}\right)\mu + 1 = 0,$$

13 in which we are assured that the coefficient of μ is negative. Consequently μ is positive and I corresponds to a periodic motion of stable type.

This disposes of the general case when there are no multiple periodic motions and $l \neq 0$.

If the original motion is of stable type but not of that highly exceptional type when there are no variable periods in the formal series, it seems to me that analogous results are to be expected, i. e. that there will exist nearby periodic motions of stable type.

This exceptional case merits particular attention; it is conceivable that it can only arise for integrable dynamical problems.

3. Distribution of quasi-periodic motions. Let us suppose that there exists at least one periodic motion of stable type for the Hamiltonian system under consideration, taken of non-integrable general type. This motion is represented by a closed curve C in the manifold M of states of motion.

Now select any such closed curve C_1 of motion of stable type. Very near to it can be found a closed curve C_2 of motion of stable type which makes k_1 circuits of C before closing. Next choose a closed curve of motion C_3 of stable type very near to C_2 and making k_2 circuits of C_2, and so $k_1 k_2$ circuits of C_1, before closing. Thus we obtain a sequence of closed curves C_n, $(n = 1, 2, \cdots)$, which can evidently be chosen so as to tend toward a definite geometric limiting set C as n becomes infinite, merely by restricting sufficiently the successive neighborhoods of $C_1, C_2, \cdots$. Furthermore, we can prevent C from being itself one of the numerable set of closed curves of motion by the same process. For instance at the nth stage we might confine attention to a neighborhood of C_n which is so small as to contain no closed curve of motion (other than C_n) of length less than n; there are only a finite number of such motions of course.

It is interesting to inquire into the analytic form of the set C. Let q_1 be the angular coördinate in M which increases by 2π when a single circuit of C is made. Then p_1, p_2, q_2 may be thought of as appropriate coördinates of this motion (section 1), and we may write

$$p_1 = f_1(q_1), \quad p_2 = g_1(q_1), \quad q_2 = h_1(q_1), \quad t = \int k_1(q_1)\, dq_1$$

as the equations of the periodic motion C_1, in which $f_1, g_1, h_1, k_1 > 0$ are analytic periodic functions of q_1 of periodic 2π. For C_2 we have likewise

$$p_2 = f_2(q_1), \quad p_2 = g_2(q_1), \quad q_2 = h_2(q_1), \quad t = \int k_2(q_1)\, dq_1$$

where f_2, g_2, h_2, k_2, are analytic periodic functions of q_1 of period $2k_1\pi$. Thus we form in succession a sequence of functions f_n, g_n, h_n, k_n, periodic of period $2k_1 \cdots k_{n-1}\pi$ in q_1, corresponding to the periodic motions $C_n, (n = 1, 2, \cdots)$. If we take points $q_1 = 0$ so as to approach a limit, it is obvious that f_n, g_n, h_n, θ_n approach limits f, g, h, θ, where the limits are approached uniformly for all values of q_1.

If there exists a single periodic motion of stable type for a non-integrable Hamiltonian problem of general type, there will exist infinitely many nearby motions, quasi-periodic but not periodic, with coördinates of the form

$$p_1 = \lim_{n \doteq \infty} f_n(q_1), \quad p_2 = \lim_{n \doteq \infty} g_n(q_1),$$
$$q_2 = \lim_{n \doteq \infty} h_n(q_1), \quad t = \int \lim_{n \doteq \infty} k_n(q_1)\, dq_1,$$

where f_n, g_n, h_n, k_n are analytic periodic functions of q_1 with periods $2\pi k_1 \cdots k_{n-1}$, $k_1, \cdots, k_n$ being positive integers which may be taken greater than 1. The convergence is uniform for all values of q_1.

Evidently there is a non-denumerable number of such quasi-periodic motions, and the coördinates are functions of the type treated by Bohr. It is clear that they constitute a class of recurrent motions of a new type.

4. Stability and instability. For the consideration of the periodic motions of stable type in dynamical problems, a fundamental division of cases must be made. It may happen that all motions sufficiently near the given periodic motion remain in a small neighborhood for all time. This is the simpler of the two cases, in which case the periodic motion in question may be termed 'stable'. The other possibility is that for some small fixed neighborhood of the given periodic motion, there may be found motions which are arbitrarily near the given periodic motion at the outset but ultimately pass out of the fixed neighborhood. In this case the periodic motion in question may be termed 'unstable'.

Evidently the classification here effected may be made not only for periodic motions but also for recurrent motions of any type. Stability in this fundamental qualitative sense is not to be confused with the 'complete formal stability' introduced earlier, and a periodic motion 'of stable type' may or may not be stable.

The transformation T of the surface S yields an immediate simple condition for stability.

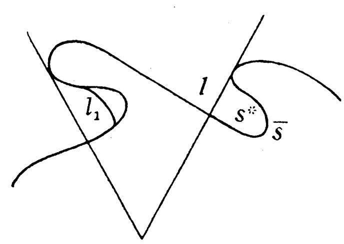
Consider a small region s of S about the invariant point, and its images $s_1, s_2, \cdots$ under successive applications of the transformation T. All of these contain the invariant point as interior point. The infinite set of regions $s, s_1, \cdots$ will lie in the vicinity of the invariant point, according to the hypothesis of stability. These regions taken together occlude a certain neighborhood $\bar{s}$ of the origin, which is taken into all or part of itself by the transformation T since the set $s, s_1, s_2, \cdots$ is taken into $s_1, s_2, \cdots$. But $\bar{s}$ cannot be taken into a part of itself because of the existence of an invariant area integral. Hence $\bar{s}$ yields an invariant area in S, corresponding to which there is an invariant torus-shaped region of M enclosing the curve of the given periodic motion in its interior.

A necessary and sufficient condition for stability is the existence of infinitely many invariant torus-shaped regions closing down upon the curve of the given periodic motion in the manifold M of states of motion.

5. The stable case. Zones of instability. What then is the nature of the boundary of such an invariant torus-shaped region in M surrounding the given closed curve of stable periodic motion? In answering this question we naturally turn to consider the nature of the corresponding invariant closed curve in S forming the outer boundary of invariant region s.

Let us assume that the periodic motion, although of general stable type is not such that the formal series involve no variable periods (section 1). In this case a normal form (2) with $s \neq 0$ (or a similar form) can be used. We may assume that s is positive, for if s is negative for T the corresponding quantity, $-s$, is positive for T^{-1}. This normal form (2) shows that the counter-clockwise rotation about the invariant point increases with radial distance if r is sufficiently small.

No region $\bar{s}$ lying very near to the invariant point can be met more than once by some radial line. In fact let s^* denote the part of the plane formed by the radial lines extended to the most distant points on the boundary of $\bar{s}$. The regions of s^* not forming part of $\bar{s}$ are of one of two possible types: either they are bounded by the boundary of $\bar{s}$ and a piece of a radial line on the left, or by the boundary of $\bar{s}$ and a piece of a radial line on the right. But the transformation T evidently takes a region bounded on the left by such a radial line l into a region bounded by $\bar{s}$ and the image l_1 of this radial line l. Since the angular coördinate increases with the radius, it is geometrically evident (see figure) that



the image of such a region will necessarily fall within a region of the same type.

But this situation is not possible since the transformation admits an invariant area integral, and consequently no set of regions can be taken into part of itself by T. Thus there exist no regions bounded on the left by such radial lines.

Similarly the use of the inverse transformation T^{-1} shows that there are no such regions bounded on the right by such radial lines.

Hence the boundary of the invariant region s is met only once by any radius sector.

The impossibility of a radial segment forming part of the boundary of $\bar{s}$ is obvious, so that there is actually only one point of intersection with each radial line.

A more elaborate consideration based on the normal form shows that the boundary curve $r = f(\theta)$ is one for which the difference quotient

$$[f(\theta_1) - f(\theta_2)]/(\theta_1 - \theta_2)$$

is bounded and indeed small for invariant regions near enough to the invariant point.* The truth of the fact seems almost obvious if one observes that T rotates positively directions differing by any considerable amount from the directions perpendicular to the radial direction into other directions differing still more from that direction.

Our conclusion may thus be summarized in the form:

For a stable periodic motion of general stable type and with variable periods in the formal series, the invariant torus-shaped regions are such that their intersections with the analytic surface of section S may be represented in the form $r = f(\theta)$, where r, θ are polar coördinates with the invariant point at the origin, and where f is a continuous periodic function of
14 *θ of period 2π for which the difference quotient is bounded.*

The curves of motion on the boundary of such a torus-shaped region form a closed invariant family. In any such

* For details see my paper (loc. cit.), sections 42–48.

closed invariant family of motions near the given stable periodic motion, there is of course at least one recurrent motion.

If the rotation number $\varkappa$ along the corresponding invariant curve on S is incommensurable with 2π the surface of the torus may represent a single minimal set of recurrent motions. In this case the coördinates and the time can be expressed in terms of continuous doubly periodic functions. In order to make this clear, let us first select angular coördinates θ, φ on the torus as follows. The coördinate φ will be taken to vanish in S, and to increase proportionately with the time along each curve of motion, the factor of proportionality being so taken as to increase φ by 2π between successive intersections with S. The coördinate θ will be defined along the invariant curve on S so that the transformation T takes the form $\theta_1 = \theta + \varkappa$ where $\varkappa$ is the rotation number specified. Elsewhere on the torus the variable θ may be defined as the θ of the corresponding point on S diminished by $\varkappa\varphi/2\pi$ in order to make θ single-valued on the torus. In these coördinates the equation of a curve of motion is $\theta - \theta_0 = \varkappa\varphi/2\pi$. Moreover the coördinates p_1, q_1, p_2, q_2 are doubly-periodic continuous functions of θ, φ, and $dt/d\varphi$ is also. Hence we may write

$$p_1 = f(\varkappa\varphi/2\pi, \varphi), \quad q_1 = g(\varkappa\varphi/2\pi, \varphi), \quad p_2 = h(\varkappa\varphi/2\pi, \varphi),$$
$$q_2 = k(\varkappa\varphi/2\pi, \varphi), \quad t = \int l(\varkappa\varphi/2\pi, \varphi)\, d\varphi,$$

where f, g, h, k, l are continuous doubly periodic functions of period 2π in their two arguments.

There is a second possibility to be considered also. The minimal set of curves of motion may correspond to a perfect nowhere dense set of points on the invariant curve; all other curves of motion on the surface of the torus-shaped region will then approach this minimal set of recurrent motions asymptotically as the time t either increases or decreases.*

* For proof of these facts and reference to the prior work of Poincaré, see my paper *Quelques théorèmes générales sur le mouvement des systèmes dynamiques*, Bull. Soc. Math. France, vol. 40 (1912).

When, however, the rotation number is not incommensurable with 2π, but is $2p\pi/q$ (p, q, relatively prime integers), there will of necessity exist points of the invariant curve which are invariant under T^q. It can be proved that the entire curve is then made up of analytic arcs terminated by points invariant under T^q, while the interior points of such arcs tend asymptotically towards these invariant points upon iteration of T or its inverse.* We are thus led to the following conclusion.

Any such closed invariant family of motions near the given stable periodic motion of general stable type and with variable periods in the formal series is characterized by a rotation number. If this number is incommensurable with 2π, either the family consists of a single minimal set of recurrent motions of continuous type, or it contains a perfect nowhere dense minimal set of recurrent motions of discontinuous type which all other motions of the family approach asymptotically as t increases or decreases. If this number is commensurable with 2π, there exists one or more closed periodic motions in the family, while the other motions form analytic branches asymptotic to these periodic motions.

It may be observed that this is a result concerning invariant sub-manifolds of the manifold M and, in particular, concerning the central motions in this sub-manifold. Incidentally it appears that, although in dynamical systems of classical type, all the motions are central with reference to the whole manifold, the same is not necessarily true of invariant sub-manifolds, so that the concept of central motions continues to play a part even in the problems of classical dynamics.

Any two of these closed families must be entirely distinct from one another, except when both have the same rotation number, commensurable with 2π. Clearly the rotation number, which measures the mean angular rotation, must be the same for two intersecting families. To establish that this number must be commensurable with 2π, we note that since the two families have at least one motion in common although they do not coincide, the two corresponding curves, $r = f_1(\theta)$,

* See my paper in the Acta Mathematica, loc. cit., sections 42–48.

$r = f_2(\theta)$ in S will enclose one or more areas between them, each bounded by a single arc of either curve. Under iteration of T this area must ultimately overlap itself and so coincide with itself, the two arcs going into themselves of course. The two common end points of the two arcs will then be invariant, and hence the rotation number is commensurable with 2π.

Conversely, any two families with distinct rotation numbers must be entirely distinct, the one further away from the periodic motion having the greater rotation number.

By convention let us consider all of the invariant families with the same rotation number commensurable with 2π, as forming a single family. This is natural since any two of the constituent families must then intersect. The outermost boundary of the corresponding network of invariant curves on S, and the innermost boundary curve cannot be wholly distinct of course, since then they would correspond to distinct rotation numbers. This augmented family is clearly composed of a finite number of periodic motions and of certain analytic families of asymptotic motions, according to the statement made above.

Consider now an infinite expanding or contracting sequence of such invariant families. The sequence evidently defines a limiting invariant family, provided it does not close down upon the invariant point, nor expand beyond the neighborhood of a stable periodic motion to which attention is confined.

These invariant families of motions are entirely distinct from one another, with rotation numbers that increase (or decrease) with the distance from the stable periodic motion, and form a closed sequence.

In case the sequence of invariant families of motions contains a pair of successive members, the region of M within the outer of the two corresponding torus-shaped regions and outside the inner one may be called a 'zone of instability'.* On the surface S the zone corresponds to a

* In section 8 the question of the existence of such zones of instability is briefly considered.

ring-shaped region lying between two successive invariant curves. Such regions will certainly exist unless the invariant families fill up the neighborhood of the stable periodic motion completely, aside from the regions occluded by the invariant families with a rotation number commensurable with 2π.

Many of the methods here employed might be used to develop further details concerning the sequence of invariant families, the zones of instability between them and their relation to nearby periodic motions (see sections 8, 9). We shall merely establish the following property:

In any zone of instability about the given stable periodic motion there exist motions passing from an assigned arbitrarily small vicinity of a motion of either of the bounding invariant families into a like arbitrary assigned vicinity of the other bounding invariant family.

In fact consider the inner boundary of the corresponding ring in S, and a small area which abuts on some arbitrary point of that boundary in the ring. Upon indefinite iteration of T an invariant part of the surface S is defined, made up of the part of it formed by the interior of this inner boundary, together with this boundary, the small region and all of its images. The boundary of the invariant part of S so defined must coincide with the outer boundary of the ring, since the inner and outer boundaries are successive invariant curves. But this means that the images of the small area extend arbitrarily near to the outer invariant curve, which is what we wished to prove.

6. A criterion for stability. It is an easy matter to give an analytic criterion for stability.

Let

$$u_1 = f(u, v), \qquad v_1 = g(u, v)$$

be the equations defining the transformation T of the surface S in the vicinity of the invariant point and let $r = F(\theta)$ be the equation of one of the invariant curves in polar coördinates. According to the conclusions reached above, F is then a single-valued continuous fuuction of θ_1 of period

2π and with bounded difference quotient. Since this curve is invariant under T it may equally well be written in the form $r_1 = F(\theta_1)$, where the expressions for r_1, θ_1 in terms of r, θ are to be obtained for the coördinate relations above. If then in the modified equations so obtained we replace r by $F(\theta)$, we are lead to an identity and to the following simple criterion:

In order that a periodic motion of general stable type and with variable periods in the formal series be actually stable, it is necessary and sufficient that a certain related functional equation

$$f^2(F\cos\theta, F\sin\theta) + g^2(F\cos\theta, F\sin\theta) = F^2\left(\tan^{-1}\frac{g(F\cos\theta, F\sin\theta)}{f(F\cos\theta, F\sin\theta)}\right)$$

admits of continuous solutions $F(\theta)$, *periodic of period* 2π *with* $|F|$ *arbitrarily small, but not* 0.

7. The problem of stability. An outstanding question in dynamics is whether or not the complete formal stability of a periodic motion of stable type assures stability in the fundamental qualitative sense defined above.

The analytic criteria which distinguish the stable from the unstable case are exceedingly delicate. There are two types of questions which present themselves here. Does formal stability assure such actual stability? If not, does formal 15
stability assure actual stability in important special cases such as the restricted problem of three bodies?

It appears to be certain that in the general case there is instability, although no proof of this conjecture has been obtained. The second of the above questions is much the more difficult one, and is at bottom arithmetic in character; it may be compared to the question of determining whether or not an assigned number is trancendental.

8. The unstable case. Asymptotic families. Let us turn to a like consideration of unstable periodic motions of general stable type and with variable periods in the formal 16 series. In this case there exist no invariant families of curves of the type present in the stable case, at least if attention be restricted to a sufficiently small neighborhood of the given periodic motion. Corresponding to this fact there will be no invariant curves enclosing the invariant point on the surface S.

In such a region of instability about an unstable periodic motion of stable type, there exist two connected families of motions reaching to the boundary of the region, which remain indefinitely within it as t increases and decreases respectively.

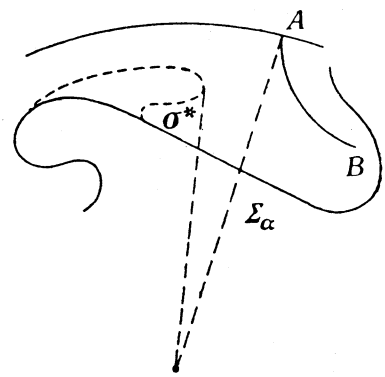
To prove this fact we consider as usual the transformation T of the surface S in the corresponding neighborhood of the invariant point. Let σ be a very small region met by each radius vector once and only once. The images $\sigma_n (n = 1, 2, \cdots)$ of σ under T^n must always contain the invariant point within them, and for some value of n must finally extend to the boundary of S; otherwise they would occlude an invariant region $\bar{\sigma}$, whose boundary would be an invariant curve of the excluded type, according to the argument of section 4. The points of σ_n remain in S for n iterations of T^{-1} at least.

Now take the diameter of σ smaller and smaller. The limiting closed set thereby obtained will be connected with the invariant point and the boundary, and must remain in S under all iterations of T^{-1}. If we had started with T^{-1} instead of T in our reasoning, we would have obtained a second similar set remaining within S under all iterations of T. Obviously these two connected sets of points correspond to two connected families of motions possessing the properties specified.

Let us add the hypothesis that the periodic motion is of general stable type with variable periods in the formal series. In that case it has been observed that the transformation T rotates the tangent directions to a curve in a counter clockwise direction, relatively to the radial direction, except for tangent directions nearly perpendicular to the radial direction at the point.

With this property in mind, let us consider the total set Σ_α of points remaining in S under all iterations of T^{-1}, and connected with the invariant point by points of the same kind. According to what has just been established, the set Σ_α extends to the boundary of S.

Imagine any regular curve AB drawn from a point A on the boundary of S, starting in the inward radial direction at A, and never turning to the right of the radial direction. All the points outside of Σ_α are accessible from the boundary of S along such curves AB which have no point in common with Σ_α (see figure). To prove this 'left-handed accessibility' of Σ_α from the boundary of S, we suppose if possible that there are one or more inaccessible regions (see the region σ^* of the figure), which will evidently be partially bounded by radial segments which they lie to the right of in the inward radial direction. The transformation T^{-1} will evidently take these inaccessible regions into parts of themselves, since it rotates radial directions in the clockwise sense relative to the radial direction. But this yields an impossibility, because of the existence of the invariant area integral.



Let the unstable periodic motion be of general stable type with variable periods in the formal series, and suppose for definiteness that the rotation increases away from the motion. Then the closed connected families of motion, Σ_α and Σ_ω, remaining in the region of instability as t decreases or increases respectively and reaching to its boundary, are respectively left-handedly accessible and right-handedly accessible from that boundary.

Let us return to the representation of Σ_α on S. The set Σ_α on S must wind indefinitely often to the right about the invariant point from its intersection with the boundary of S. In order to establish this fact it is convenient to take the polar coördinates θ and r as rectangular coördinates with the θ and r axes directed to the left and upwards respectively. The surface S then appears as an infinite strip, and the region in the strip to the right of and above the connected set Σ_α cannot extend to the right of the point of Σ_α on the boundary of S; otherwise there would clearly be inaccessible regions of the excluded type.

If then Σ_α does not extend indefinitely far to the left, it will be entirely included between two vertical lines in this representation. But the work of section 2 shows that two points, one on $r = 0$ and another nearby, move at sufficiently different rates in the direction of the θ axis so as to separate by an arbitrarily large amount. Hence on sufficient iteration of T^{-1} the curve Σ_α (whose images under T^{-1} all lie in S) must spread over an arbitrarily large strip in the θ direction and so intersect Σ_α. Thus Σ_α and this image will contain an area which must remain in S under all iterations of T^{-1}. But this would lead us to an invariant area σ in S as before. Hence Σ_α extends indefinitely far in the direction of the negative θ axis. It follows that Σ_α winds indefinitely often to the right about the invariant point, while Σ_ω winds indefinitely often in the opposite sense. Evidently then the two sets Σ_α and Σ_ω intersect infinitely often.

In conclusion we observe that Σ_α must tend uniformly towards the periodic motion under iteration of T^{-1}. Otherwise the limit points of Σ_α under iteration of T^{-1} would yield a closed set Σ'_α, connected with the invariant point and remaining in S under the iteration of T as well as of T^{-1}. The set of such points would then constitute an invariant region $\bar{\sigma}$, whose boundary would be met once and only once by every radius vector according to the argument of section 5. By hypothesis there is no such invariant region.

We may summarize these conclusions in the following way:

The families Σ_α and Σ_ω of motions wind indefinitely often to right or left about the periodic motion, according as they are left-handedly or right-handedly accessible, and so intersect in infinitely many common motions. The motions Σ_α and Σ_ω

are respectively negatively and positively asymptotic to the given periodic motion, while the infinitely many common motions are doubly asymptotic to the given periodic motion.

Entirely analogous considerations to those used above may be applied to any zone of instability. There will exist positively and negatively asymptotic connected sets attached to either boundary and reaching to the vicinity of the opposite boundary. The two sets intersect infinitely often. Furthermore, by the kind of argument employed in the following section, it could be proved that the set positively asymptotic to one boundary intersects the set negatively asymptotic to the other. In consequence there must exist infinitely many motions positively and negatively asymptotic to the two boundaries in any one of the four possible ways.

9. Distribution of motions asymptotic to periodic motions. Hitherto in this chapter we have limited attention to the vicinity of a periodic motion. We turn now to the consideration of the totality of motions in M, which we take to be a closed analytic manifold.

In doing so we shall assume that there exists a surface of section S of genus one, and a corresponding transformation T. The boundaries of S are to correspond to periodic motions of general stable type, and S is cut in the same sense by every curve of motion in M, at least once in any interval of time of sufficient length. Any point P of S if followed along the curve of motion in the sense of increasing time meets it again at P_1, and we write $P_1 = T(P)$, thereby defining a one-to-one analytic transformation of S into itself, which we take to be continuous along the boundaries.

We shall not attempt here to give conditions for the explicit construction of a surface of section S. The details seem to be special to each case (see chapter VI) and not particularly illuminating. Such surfaces of section S and associated transformations T exist in very wide classes of problems.

Moreover we propose to introduce the working hypothesis that the dynamical system is transitive. This hypothesis is certainly satisfied in certain cases as the example given later (section 11) shows, and in all probability is satisfied unless exceptional conditions obtain. However, since the presence of a single stable periodic motion obviously would bring about intransitivity, the conjecture just made can not be established unless the problem of stability is also solved.

If we interpret the hypothesis of transitivity on the surface of section S, it means that, given arbitrary points P_0 and Q_0 of S, then points P and Q arbitrarily near to P_0 and Q_0 respectively and an integer n can be found such that $Q = T^n(P)$.

Suppose now that there exists a single periodic motion of general stable type and with variable periods in the formal series. Since this motion is unstable there exists a network formed by the corresponding connected sets Σ_α and Σ_ω.

Let us consider a boundary of this network which cuts off a small neighborhood of the invariant point. Such a boundary is afforded by the part of S inaccessible from without a given region enclosing the invariant point but without intersection with the corresponding branches Σ_α and Σ_ω. This boundary is not made up wholly of Σ_α or Σ_ω. Otherwise under indefinite iteration of T^{-1} or of T respectively the images of the boundary would continue to lie in that part of S, and an invariant part of S would be defined. Instead it is obvious that under iteration of T^{-1}, for instance, the part Σ_α of the boundary approaches the invariant point, while the part Σ_ω must finally extend into every part of S whatsoever. Otherwise the part of S reached and occluded would also yield an invariant part of S under T, such as is excluded by the hypothesis of transitivity.

Consequently the sets Σ_α and Σ_ω, connected with the invariant point under consideration and asymptotic to it under iteration of T^{-1} and T respectively, are both everywhere dense on the surface of section S.

We have seen (sections 1, 2) that near such a periodic motion of stable type there are infinitely many other periodic motions of stable type which correspond to invariant points of S under some iterate T^n of T. Consider any such periodic motion which is of general stable type with variable periods in the formal series. It has a set Σ'_α and Σ'_ω, both of which must also be dense throughout S.

Now the sets Σ_α and Σ'_α have no points in common, inasmuch as a motion cannot be negatively asymptotic to two different periodic motions. Similarly the sets Σ_ω and Σ'_ω have no points in common.

Hence it is apparent that the sets Σ_α and Σ'_ω have infinitely many points in common, as have the sets Σ_ω and Σ'_α.

In the transitive case of two degrees of freedom when there exists a periodic motion of general stable type having variable periods in the formal series, there will exist infinitely many other periodic motions of stable type. The motions positively or negatively asymptotic to any periodic motion of this infinite set which is of general stable type with variable periods in the formal series, form a set everywhere dense in S. There exist infinitely many motions positively asymptotic to any one of these periodic motions, and at the same time negatively asymptotic to any other periodic motion of the same set, or even to the same periodic motion.

We shall consider next the periodic motions of unstable type and the motions asymptotic to them.

As has been remarked, there exist analytic families of motions positively and negatively asymptotic to such a periodic motion. In the simplest case, to which we may limit attention, there are two corresponding analytic invariant curves through the invariant point, one yielding the positively asymptotic motions and the other the negatively asymptotic motions (section 1). The two arcs of the same invariant curve ending at the invariant point cannot intersect of course, no matter how far extended.

On the contrary, two arcs belonging one to each invariant curve may intersect. In this case the same argument is applicable as was made for the analogous Σ_α, Σ_ω curves above, to show that these two invariant curves are each everywhere dense in S and intersect infinitely often.

Furthermore we can argue as before and obtain the conclusion that infinitely many motions exist, negatively asymptotic to an assigned periodic motion of general stable type with variable periods in the formal series or to an assigned periodic motion of general unstable type with doubly asymptotic motions, and at the same time positively asymptotic to an assigned periodic motion of one of these two types.

It is evident that if two arcs of positively and negatively asymptotic types for such a periodic motion of unstable type intersect the network of one motion of stable type, they will intersect all such networks, and also each other.

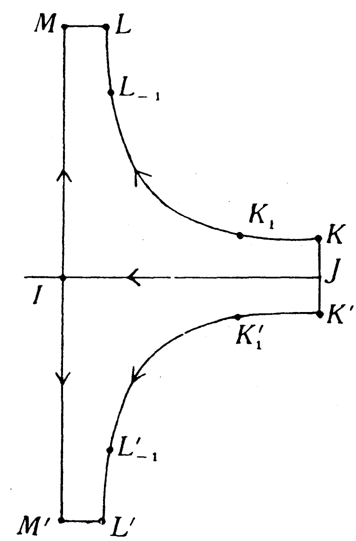
If then we can establish that all the four arcs attached to the invariant point intersect these networks it is obvious that we can extend our previous conclusions about motions asymptotic to the periodic motions of stable type to those of unstable type. We shall prove that this is the case provided that (1) the asymptotic analytic arc from one such invariant point of unstable type is not identical to that from a second such invariant point and (2) there is no periodic motion of general stable type with invariable periods in the formal series. The cases when one of these two assumptions fails to hold are to be regarded as highly exceptional. The argument is made only in the case when there are no multiple periodic motions, although the result holds under much wider conditions.

In order to establish this fact, let us suppose that some one of the four arcs belonging to the periodic motion of unstable type does not intersect these networks, and show that a contradiction results.

By extending this arc indefinitely, we obtain a connected limiting set Σ on S. This set Σ is clearly invariant under T. Furthermore, in consequence of this fact and the hypothesis of transitivity Σ cannot enclose an area. Hence, by a well known theorem due to Brouwer, there exists an invariant point of Σ under T. This must correspond to a periodic motion of unstable type, since by hypothesis the extended asymptotic arc cannot approach a periodic motion of stable type with variable periods in the formal series.

Now it is obvious that Σ must contain at least two of the asymptotic arcs of opposite type to this invariant point unless the original extended arc coincides with one of these arcs. But this possibility was also excluded.

It is clear that these new arcs in Σ do not intersect the networks belonging to the periodic motions of stable type, and so we may take any one of them, thus deriving a set Σ_1 within Σ. In this way a process is set up which must finally yield a Σ^* in Σ containing a minimum number of invariant points corresponding to a periodic motion of unstable type, and of asymptotic arcs. Any arc of Σ^* in such a set extended from an invariant point must then have itself as limit point, and there will be at least two such arcs of opposite type to the same invariant point.



Let us consider separately the cases when there are two, three and four such arcs to the invariant point I (see figure).

To take the first case, let IJ and IM be the positively and negatively asymptotic arcs in Σ^*. Now IM extended must return to the vicinity of I and can only do it along IJ. Let $IJKLM$ be constructed as follows: JK is a short curvilinear arc which meets IJ at J; KL is an arc made up of an arc KK_1 joining K to its image K_1 near to IJ and of its images K_1K_2, $K_2K_3, \cdots$ till the point L is reached near to IM, with LM a short segment. The curvilinear polygon $IJK'L'M'$ is similarly constructed on the opposite side of IJ. The extended arc IM cannot approach IJ along $JK'K_1'$ for then evidently IM' would also lie in Σ^*.

Hence IM produced intersects JKK_1. Likewise IJ produced intersects MLL_{-1}. The *analysis situs* of the figure shows that IM and IJ extended will intersect, contrary to hypothesis.

Next suppose that three arcs, say IJ of one type and IM, IM' of the other, lie in Σ^*. It follows as before that IJ extended meets MLL_{-1} and $M'L'L'_{-1}$. But if there are no intersections of IJ extended with IM or IM', it is apparent from the *analysis situs* of the figure that IM intersects $JK'K'_1$, and also that IM' intersects JKK_1. Thus IM and IM' would necessarily intersect which is impossible since these asymptotic arcs are of the same type.

There remains to consider the case when all four arcs lie in Σ^*. Here IM must intersect $K_1KJK'K'_1$ if extended. But IM cannot intersect $JK'K'_1$, for then IJ if extended could not approach IM', according to the *analysis situs* of the figure. Hence IM must intersect JKK_1, and similarly IM' must intersect $JK'K'_1$. But the same argument applied to IM as to IJ shows that IJ must intersect MLL_{-1}. Hence IJ and IM intersect, which is impossible.

We are now prepared to state our conclusion:

Suppose that there is a surface of section S of genus one for a dynamical problem with two degrees of freedom, of transitive type. Suppose furthermore that all of the periodic motions of general stable type involve variable periods in the formal series, and that no two analytic asymptotic families attached to different periodic motions of unstable type coincide.

Under these circumstance, all of these asymptotic families are dense in M, and there will exist infinitely many motions asymptotic in either sense to any assigned periodic motions, whether of stable or of unstable type.

The special hypothesis made that the periodic motions are not multiple is not essential to the method of attack, but was made in order to restrict attention to the general case, for the sake of simplicity.

The above result makes plain a certain analogy between the motions of stable and unstable type.

It has been proved earlier that in the neighborhood of a periodic motion of general stable type and with variable periods in the formal series, there are nearby periodic motions both of stable and unstable type. Are there also periodic motions which approach arbitrarily near to any periodic motion of unstable type? Of course such a motion cannot remain near throughout a period. The answer is in the affirmative. It may be proved that when the two asymptotic analytic branches of a periodic motion of unstable type intersect, there will be infinitely many periodic motions coming into an arbitrarily small vicinity of the corresponding doubly asymptotic motion and of the given periodic motion of unstable type.* 17

Thus it appears that in a certain sense the totality of periodic motions, whether of stable or unstable type, will be dense on itself in very general cases. The conjecture of Poincaré that these periodic motions are everywhere dense has been seen not to be always true (chapter VI. section 4), but doubtless holds in very general cases also.

10. On other types of motion. Thus far we have only considered periodic motions, the quasi-periodic motions, and certain other simple types of recurrent motions, among the various types of recurrent motions. Such recurrent motions almost certainly form an endless hierarchy of more and more complicated types, even for non-integrable dynamical systems with two degrees of freedom such as we have been considering. Among motions asymptotic to recurrent motions in either sense we have only considered those asymptotic to periodic motions. We have not considered other special motions, nor non-special motions. The general methods used in this and the preceding chapter yield a variety of results in this connection.†

We shall not attempt to go further in this direction. The

* See my forthcoming paper in the Acta Mathematica, *On the Periodic Motions of Dynamical Systems.*

† See my paper *Surface Transformations and Their Dynamical Applications*, Acta Mathematica, vol. 43 (1922), sections 54–73.

example taken up section 11 will give some idea of the complexity of the situation to be expected.

18 **11. A transitive dynamical problem.** The problems of dynamics usually called 'integrable' are those of intransitive type, and the motions are represented by curves which lie upon invariant analytic manifolds in M, of one or two dimensions. For example, in the case of the problem of two bodies, every motion is periodic, and these invariant manifolds in M are closed curves. For the integrable cases (sections 12, 13), the special analytic relations are sufficient to yield complete information about the motions and their interrelations.

Any non-integrable problem of transitive type might, however, be considered to be 'solved', in case a special algorithm of sufficient power could be devised for dealing with it.

I propose here to develop such an algorithm for the transitive geodesic problem on a special analytic surface with negative curvature. The results obtained seem likely to be typical of the genéral transitive case in many respects, and can be readily extended to any analytic closed surface with negative curvature. It will only be possible to give an intuitive justification for the results. For the technique, I may refer to the notable work of Hadamard* and Morse,† the methods and ideas of which perform the principal role in the case here treated.

It seems improbable that any analogous algorithm exists in the geodesic problem on closed analytic surfaces with positive curvature.

The particular surface which we consider is defined by the equation

$$z^2 = 1 - e^2 \sin^2 \frac{1}{2} x \sin^2 \frac{1}{2} y \qquad (e > 1)$$

* *Les surfaces à courbures opposées et leur lignes géodesiques*, Journ. de Math., ser. 5, vol. 4 (1898).

† *Recurrent Geodesics on a Surface of Negative Curvature*, Trans. Amer. Math. Soc., vol. 22 (1921); *A Fundamental Class of Geodesics on Any Closed Surface of Genus Greater Than One*, Trans. Amer. Math. Soc., vol. 26 (1924).

in which x, y, z are rectangular coördinates, and where we shall make the convention that all points

$$(x \pm 2k\pi,\ y \pm 2l\pi) \qquad (k, l = 0, 1, 2, \cdots)$$

correspond to the same point of the surface. This convention is legitimate inasmuch as the linear group of translations

$$\bar{x} = x \pm 2k\pi, \qquad \bar{y} = y \pm 2l\pi, \qquad \bar{z} = z,$$

takes the surface into itself. A fundamental domain for x, y is then the square given by

$$0 \leqq x < 2\pi, \qquad 0 \leqq y < 2\pi.$$

For any z this equation may be written

$$\sin^2 \frac{1}{2} x \sin^2 \frac{1}{2} y = (1 - z^2)/e^2.$$

Hence for $|z_0| < 1$ the trace of the given surface in the plane $z = z_0$ consists of a convex, symmetrical, analytic oval within the fundamental square, with center at the mid-point of the square. As z_0 increases or decreases from $z_0 = 0$ towards $z_0 = \pm 1$, this oval expands analytically, and becomes the fundamental square for $z_0 = \pm 1$. It is easily verified directly, but is also obvious from the above qualitative considerations, that this surface is everywhere analytic, with curvature which is negative except at the points on the surface corresponding to the corner of the limiting squares, $z = \pm 1$.

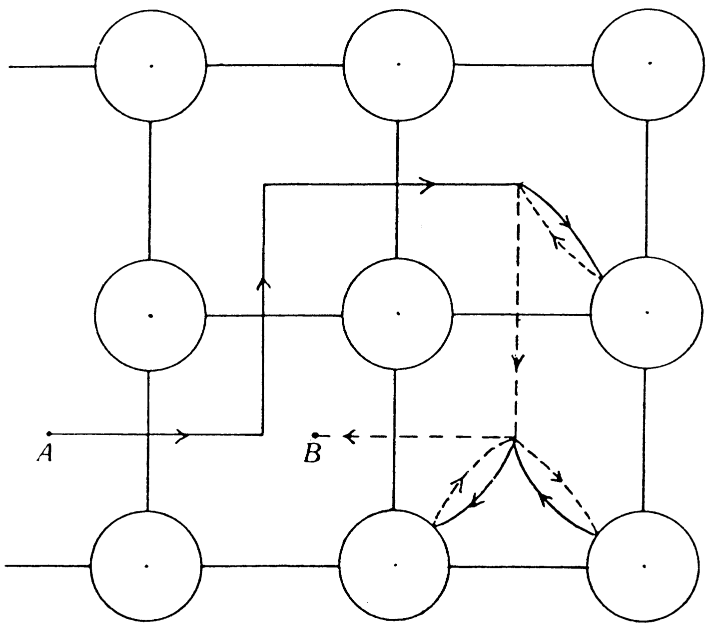
The connectivity of this closed surface is readily determined. If the equatorial oval $z = 0$ were capped, the upper half of the surface would be homeomorphic with a square in which pairs of opposite points on parallel sides are identical, i. e. with an anchor ring. When this cap is removed we find that the upper half of the surface is homeomorphic with the surface of an anchor ring having a hole in it. The lower half is of the same type. Thus the total surface is homeomorphic with a double anchor ring of genus 2.

A fundamental property for such a surface of negative curvature is that there is one and one only geodesic arc AB of given type, from the standpoint of *analysis situs*, which joins two assigned points of the surface. For example, take any closed curve circling around one copy of the surface. There is only one closed geodesic of this type, which of course is the oval $z=0$. If we use the complete representation in x, y, z space by means of the surface and the infinitely many congruent copies, the above result means that any continuous line in this surface joining a fixed point A to a fixed point B can be continuously deformed into a unique geodesic arc AB.

Let us seek to introduce a symbolism adequate to express the type of any arc AB. Consider the projection of the given surface upon the equatorial plane. This will cover doubly all of that plane except the part inside any one of the geodesic circles in the equatorial plane. In the adjoining figure these circles are then the images of these geodesic circles, and likewise the horizontal and vertical segments of the square net work having the centers of these circles as vertices, will obviously correspond to closed geodesics.

Suppose that the projection of A lies within a square of the network as well as that of B. As a point P moves from A to B along AB, its projection traces a continuous curve in the x, y plane. Moreover if this projection is given, together with the points on the circles where P moves from a region of the surface $z>0$ to a region $z<0$, then the curve AB is determined.

Let the symbol x denote a crossing of a vertical segment in the positive x direction, and x^{-1} a crossing in the opposite direction; likewise let the symbols y and y^{-1} denote crossings of a horizontal segment in the directions of the positive y axis and the negative y axis respectively. From a point within a square there are four accessible quadrants of circles, in the lower left-hand corner, lower right-hand corner, upper right-hand corner and upper left-hand corner. Denote the corresponding crossings of the geodesic circles toward the positive z axis by w_1, w_2, w_3, w_4, respectively, and also the oppositely directed crossings by $w_1^{-1}, w_2^{-1}, w_3^{-1}, w_4^{-1}$ respectively. It is then geometrically evident that any arc AB corresponds to a symbol formed by a finite set of these 12 symbols written in the same order as the corresponding crossings. Conversely, if such a symbol is given, restricted solely in that after any w_i follows a w_i^{-1}, and vice versa, there will be a unique



corresponding path. The reason for this restriction is that the point P moves from a region $z<0$ to a region $z>0$, and then from a region $z>0$ to a region $z<0$.

To every allowable deformation of the path AB (A and B being fixed of course) will correspond a modification of the symbol. The class of symbols obtained from one another in this way may be called 'equivalent'. It thus becomes important to determine the legitimate type of modification of a symbol, and a normal form of each class of equivalent symbols.

The legitimate operations are of two types. The first allows us to insert or remove any pair of elements $a a^{-1}$ or $a^{-1} a$ in the symbol, since this corresponds to a deformation over a boundary. The second operation allows us to replace such a symbol as $w_3 y$ by $y w_2$; or $w_3^{-1} y$ by $y w_2^{-1}$, or $w_2 y^{-1}$ by $y^{-1} w_3$, or $w_2^{-1} y^{-1}$ by $y^{-1} w_3^{-1}$; these are the changes possible in the symbol when a point P of AB is deformed through the point common to the quadrants w_2, w_3 of a geodesic circle. There will be similar operations at the point common to w_3, w_4, at the point common to w_4, w_1, and at the point common to w_1, w_2.

In order to obtain a normal form we reduce the number of elements in the symbol as far as possible by the following three processes. First we strike out any pair $a a^{-1}$ or $a^{-1} a$. Secondly we replace any triple such as $y w_2 y^{-1}$ by its equivalent w_3; here we have

$$y w_2 y^{-1} = w_3 (y y^{-1}) = w_3$$

of course. For each of the sixteen operations of the second type specified above there will be two corresponding triples of the form $p w_i p^{-1}$ or $p w_i^{-1} p^{-1}$ where p is x, x^{-1}, y or y^{-1}, which can be replaced by a single letter w_j or w_j^{-1}. Thirdly we replace any triple such as $w_3 y w_2^{-1}$ by y. For each of the sixteen operations of the second type there will also be two corresponding triples of the same sort which can be replaced by a single letter x, x^{-1}, y or y^{-1}.

Lastly whereever possible we invert all ordered pairs such as $y^{-1} w_2$ made up of an element x, x^{-1}, y, y^{-1} followed by element w_i or w_i^{-1}, so that an element of the second type appears first. Thus $y^{-1} w_2$ is replaced by $w_3 y^{-1}$.

The normal form will be defined as that which results when all of these processes have been carried out. We propose to show that the normal form is unique. For this purpose we divide the symbol into components made up of the elements x, x^{-1}, y, y^{-1}, and into those made up of the letters w_i, w_i^{-1}. For the arc AB of the figure the symbol is evidently

$$xy\,x w_2\, y^{-1}\, w_2^{-1} w_1 x^{-1}.$$

It is in normal form and breaks up into the four components

$$xyx,\ w_2,\ y^{-1},\ w_2^{-1} w_1,\ x_3^{-1}.$$

We will argue that this particular normal form for the symbol of AB is unique, but the method used is obviously general. The first component evidently gives the least number of squares passed by the projection of a point P moving from A to B along a path of this type, before the point P crosses the equatorial plane, and it identifies them uniquely in proper order. Hence any other normal symbol for AB must have the same first component xyx as this one. Similarly the second component describes uniquely the greatest number of passages which P can make across $z = 0$ in the same copy of the surface always cutting distinct geodesic circles in succession. In this case there is only one element w_2 in the second component. Thus the second component is w_2 in any other normal symbol.

In general, only the inevitable crossings are made of the sides of the squares and of the geodesic circles, while the latter are made as soon as possible. This is the geometric interpretation of the operations reducing to normal form, and may be made the basis of the proof of uniqueness.

The condition that a symbol is in this normal form merely requires that certain sequences of elements are not to occur. The forbidden sequences are obviously

$$\begin{array}{llll} xx^{-1}, x^{-1}x, & yy^{-1}, y^{-1}y, & w_i w_i^{-1}, w_i^{-1} w_i & (i = 1, 2, 3, 4), \\ xw_1, xw_4, & xw_1^{-1}, xw_4^{-1}, & x^{-1}w_2, x^{-1}w_3, & x^{-1}w_2^{-1}, x^{-1}w_3^{-1}, \\ yw_1, yw_2, & yw_1^{-1}, yw_2^{-1}, & y^{-1}w_3, y^{-1}w_4, & y^{-1}w_3^{-1}, y^{-1}w_4^{-1}. \end{array}$$

The infinite symbol corresponding to a complete geodesic will obviously possess the same minimum property as the normal symbol, since the geodesic arc will cross the auxiliary geodesics a minimum number of times. We will call this symbol the 'reduced symbol'. The letters x, x^{-1}, y, y^{-1} follow each other in the reduced symbol ordered precisely as in the normal symbol. But the exact position of the elements w_i can not be determined without a more explicit knowledge of the given surface, and would be different for other surfaces of the same general type.

For the reduced symbol the points of crossing may be associated with the corresponding elements of the symbol, and intermediate points may be indicated by placing an ordinary real number u, $(0 \leqq u \leqq 1)$, between two successive elements α, β, thereby indicating that the point P_u lies the fractional part u of the way from the crossing α to the crossing β along the geodesic arc.

In this way we obtain a symbol for a 'state of motion' in the problem, by adjoining the number u to the complete reduced symbol. Continuous variation of the state of motion will vary u continuously if u be regarded as periodic of period 1. The corresponding variation of the symbol itself is 'continuous' in that a slight variation of the state of motion can produce only changes in a distant element of the reduced symbol, or else will introduce allowable changes of order in successive elements. The corresponding variation in the normal symbol is in general continuous also, but here a component of the form $w_i ccc \cdots$ where c is a given set of elements is to be regarded as the same as $ccc \cdots$. The two ends evidently vary continuously, but no other changes of order may occur anywhere in the symbol.

We are now prepared to take up the question of the types of motion and their interrelation.

In the first place the periodic motions, corresponding to the closed geodesics, are evidently in one-to-one correspondence with the normal types of finite symbols, in which cyclic order is not considered. The two periodic motions corresponding to the senses in which one and the same closed geodesic may be traversed will correspond to such a symbol and the same symbol taken in the reversed order. The normal symbol for the complete geodesic is evidently the partial symbol taken an infinite number of times. Since the partial symbol may be arbitrarily chosen, the corresponding periodic motion can be made to approach an arbitrary geodesic motion.

The periodic motions are everywhere dense in the totality of motions. 19

Next let us consider the motions which are positively asymptotic to a given periodic motion. For this to happen, the normal symbol of the motion under consideration must of course be the symbol of the periodic motion from a certain point on, and this condition is sufficient. In order to be negatively asymptotic to the periodic motion generated by the finite symbol p, and positively asymptotic to that generated by q, the symbol must repeat the partial symbol p sufficiently far to the left and q sufficiently far to the right. The intermediate part of the symbol can be taken arbitrarily. If the symbols p and q are the same, a motion doubly asymptotic to one and the same periodic motion may be defined. Since the intermediate part of the symbol is arbitrary the motions of either type are everywhere dense, but must be numerable.

This reasoning shows the existence of a continuous family of positively or negatively asymptotic motions to a given periodic motion. There exist also infinitely many periodic motions positively asymptotic to one assigned periodic motion and negatively asymptotic to a second assigned periodic motion, and these motions are everywhere dense although numerable.

More generally, if a motion is merely to be asymptotic in one sense to any given motion, only one end of the symbol is assigned.

A similar conclusion holds then concerning the motions asymptotic to any two given motions in assigned senses.

There exist also motions semi-asymptotic to the given motions, so that while not actually asymptotic to them the deviations become increasingly infrequent as the time increases or decreases indefinitely.

To make up the corresponding symbol we need merely write down a normal symbol such that in one direction it is in increasing proportion made up of increasing components of the symbol of one of the given motions taken further and further out in its symbol, while in the opposite direction it is similarly made up principally of components of the symbol of the other given motion.

Let us next pass to the general recurrent motion not of periodic type. Evidently the corresponding normal symbol is characterized by the fact that given an arbitrary positive integer n, there exists a second integer N so great that every sequence of n figures in the symbol can be found at least once in any N successive figures.

Morse (loc. cit.) has given a specific method of construction of such a symbol.

There is probably a hierarchy of such recurrent motions dependent in degree of complication upon the way in which N varies with n. Here I wish merely to indicate a possible method which will lead to the discovery of recurrent motions not of periodic type for the system under consideration. Let $f(x_1, \cdots, x_p)$ be any function which is analytic and periodic of period 1 in $x_1, \cdots, x_p$, $(p > 1)$. If $c_1, \cdots, c_p$ are p quantities without linear relation of commensurability between them, then $f(c_1 \lambda, \cdots, c_p \lambda)$ is a quasi-periodic function of λ. Suppose now that we let $\{a\}$ denote the integral part of the least residue of a taken modulo q, so that $\{a\}$ is one of the numbers $0, 1, \cdots, q-1$. The function $\{f(c_1 \lambda, \cdots, c_p \lambda)\}$ will then yield a doubly infinite sequence of the integers $0, 1, \cdots, q-1$ for λ an integer, which will possess the characteristic recurrence property desired, and will not be of periodic type unless f happens to be very nearly periodic.

Suppose for example that we take $q = 8$ and identify $0, 1, \cdots, 7$ with x, x^{-1}, y, y^{-1}, w_1 or w_1^{-1}, w_2 or w_2^{-1}, w_3 or w_3^{-1}, w_4 or w_4^{-1}. The apparent ambiguity is unimportant since the symbols w_i, w_i^{-1} alternate. Then we should obtain a symbol for a recurrent motion on our surface, corresponding to any function f.

It is in the nature of this proposed method of construction of recurrent non-periodic motions that they possess a certain

p underlying periods, and may very well be of continuous type. The example proposed by Morse was established by him to be of discontinuous type.

The theory of chapter VII *shows that every motion has a certain set of recurrent motions among its closed connected set of α and ω limit motions.*

The method there used for constructing such a recurrent motion has its symbolic counterpart, whereby at least one normal symbol of recurrent type can be obtained from either end of any normal symbol.

The question arises in how far the closed connected α and ω sets may be assigned at will. Now it is apparent that since the given motion approaches the ω limit motions asymptotically as time increases, it is possible for a sequence of indefinitely long arcs

$$AB,\ B'C,\ C'D,\ \cdots$$

of the limit motions to be found such that B' is very near to B, C' to C, etc., and such that these arcs taken together trace out arbitrarily closely all of the set of limit motions. To this end we may use the given motion in which each long segment MN, NP, PQ, $\cdots$ is near the ω limit set, so that segments AB, $B'C$, $\cdots$ of this set near MN, NP, $\cdots$ respectively exist, and so near all of the ω limit set. Let us say that any such closed connected set of motions is 'cyclic'.

Such α and ω limit sets are necessarily cyclic. Conversely given any two cyclic sets of motions, there exist in the case at hand everywhere dense motions which have precisely these sets as α and ω limit sets.

In fact, construct a symbol such that in one direction it contains a succession of symbols attached to arcs $AB, B'C, C'D, \cdots$ of the ω cyclic set as above, in which these symbols increase in length while at the same time the distances BB', CC', $\cdots$ become smaller and smaller; and do the same in the symbol with respect to the α cyclic set, but operating in the reverse direction. In between the two parts an arbitrary finite symbol may be inserted. Such a symbol evidently corresponds to a motion with the desired property, and these motions are clearly everywhere dense because of the presence of the arbitrary symbol.

Finally there exist non-special motions so that the dynamical problem in hand is transitive.

In order to obtain non-special motions we need only write down normal symbols which contain all allowable sequences of symbols.

The set of non-special motions is of course measurable in the sense of Lebesgue, and it is natural to conjecture that its measure is that of the totality of motions in M, i. e. that the special motions are of measure 0. I have not been able to establish this conjecture.

Thus there is an enormous complexity of types of motion in this geodesic problem on a closed analytic surface of negative curvature; but nevertheless a specific algorithm exists which suffices to describe adequately this complication by means of certain symbols.

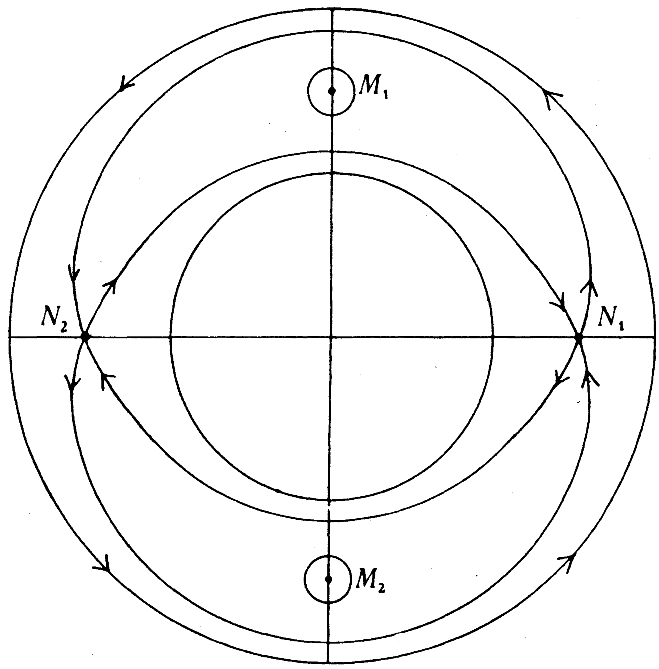
Of course the above problem differs from the most interesting class of dynamical problem, typified by the geodesic problem on a convex surface, in that all of the periodic motions are of unstable type. Nevertheless it seems to be typical of the general case in many respects.

12. An integrable case. A well-known integrable problem, discovered by Jacobi, is afforded by the geodesics on a convex ellipsoid.* If this ellipsoid flattens to a limiting elliptical form, a special integrable case of the billiard ball problem (chapter VI, section 6) results. This example is still more concrete inasmuch as the geodesics become broken straight lines with vertices lying on the ellipse and making equal angles with the normal at any vertex.

It is a fact of elementary geometry that if a single segment of such a broken line passes through one focus of the ellipse, alternate segments will continue to pass through alternate

* A very suggestive treatment of certain integrable cases will be found in a paper by Whittaker, *On the Adelphic Integral of the Differential Equations of Dynamics*, Proc. Royal Soc. Edinburgh, vol. 47 (1917).

foci, no matter how far produced. A further well-known fact is that if the system of confocal conics containing the limiting ellipse be drawn, successive segments (or these segments extended) will be tangent to a particular conic of the set, which may be an ellipse or a hyperbola; if these points of tangency lie on an ellipse, the billiard ball will



continue to go around the table in one and the sense indefinitely; if these points lie on a hyperbola, the successive points of tangency lie on the two branches in alternation, while the successive segments are between its branches; the major and minor axes constitute two limiting cases of periodic motion.

Suppose now that we make use of the coördinates φ and θ already employed chapter VI, section 7, for setting up the ring transformation T. Here φ denotes an angular coördinate of period 2π which measures arc length along the ellipse while θ measures the angle of projection which the rebounding billiard ball makes with the positive direction of the tangent at the given point of projection on the ellipse.

For any (θ, φ) where $\theta + \pi$, φ may be taken as essentially polar coördinates on the ring-shaped surface of section (figure), there is one and only one state of projection of the ball, and, as time increases there is a next following state of projection (θ_1, φ_1). The transformation T is then defined as that which carries (θ, φ) into (θ_1, φ_1). We shall not undertake to write down the explicit analytic formulas involved although these can be obtained either directly or as a limiting case of formulas arising in the geodesic problem on an ellipsoid. Such explicit formulas are not necessary for our purpose.

We propose to determine qualitatively the character of the transformation T in this integrable case.

In the first place the motion of a billiard ball around the table in either of the two possible senses evidently corresponds to a succession of points on a single closed analytic curve lying near $\theta = 0$ or $\theta = \pi$ respectively, according as φ increases or decreases; the two limiting cases $\theta = 0$ and $\theta = \pi$ correspond to rolling motion of the billiard ball along the elliptical boundary in the two possible senses. Thus we get two analytic families of closed curves which abut on $\theta = 0$ and $\theta = \pi$, and which are invariant under the transformation T. According to the results obtained in chapter VI, the transformation T leaves invariant the points on $\theta = 0$ but rotates the points on $\theta = \pi$ through an angle 2π.

Secondly, if we consider a state of projection which is associated with tangency to a hyperbola, there is one and only one such point of tangency lying on the straight line formed by the segment described by the billiard ball, or that segment extended, and the point of tangency may be continuously varied over the complete hyperbola. There are two kinds of projection, however, corresponding to the values of φ on either of the two elliptical arcs of the given ellipse which lie between the two branches of the hyperbola under consideration. Here each hyperbola gives two closed analytic curves within the ring, and, as the hyperbolas tend to approach the minor axis of the ellipse, we obtain two analytic families of curves closing down upon two points $M_1 = (\pi/2, \pi/2)$, $M_2 = (\pi/2, 3\pi/2)$, corresponding to motion along the minor axis.

Evidently the limiting case of any of these four types of motion is that referred to at the outset when the straight line segments pass through the foci. But the states of projection through either focus correspond to a single closed analytic curve, and these two curves have two points in common, namely the points $N_1 = (\pi/2, 0)$, $N_2 = (\pi/2, \pi)$, corresponding to projection along the major axis.

In this way the points of the ring which represent each state of projection are seen to be divided up as represented in the figure above.

The transformation T leaves the points of the inner boundary invariant and rotates the analytic curves which abut upon it through an angle which increases with distance from the boundary, inasmuch as if θ is increased while φ is held fast, it appears that φ_1 is thereby increased. For the limiting curve of this family made up of two analytic arcs which end at N_1 and N_2, it appears that the transformation T rotates N_1 into N_2 and N_2 into N_1 in a positive sense, while interchanging the two arcs through N_1, N_2.

Similarly the transformation T advances the points of the outer boundary by an angle 2π and rotates the analytic curves which abut upon it by an angle less than 2π, which decreases as the distance from this boundary increases. The limiting curve of this family is made up of two analytic arcs which end at N_1 and N_2, and the transformation T rotates N_1 into N_2, and N_2 into N_1, and interchanges the two arcs.

Examination of the motions which pass through the foci shows that these tend asymptotically toward the major axis in either sense. This agrees with the fact that all points within the inner arcs $N_1 N_2$ are advanced by an angle less than π while those on the outer arcs $N_1 N_2$ are advanced by an angle more than π.

It is clear that there are no invariant points under T.

Let us consider now the various types of motion, and to begin with, one of the type corresponding to a curve of the analytic family abutting upon $\varphi = 0$.

Let us adopt analytic variables (φ, ψ) where ψ varies with the curve of the family. The transformation T takes the form

$$\varphi_1 = F(\varphi, \psi), \qquad \psi_1 = \psi,$$

with Jacobian determinant

$$J = \partial F / \partial \varphi > 0.$$

If the invariant area integral is

$$\int\int I \, d\varphi \, d\psi$$

in these variables (see section 1), we will have

$$I(\varphi_1, \psi_1) \, \partial F / \partial \varphi = I(\varphi, \psi),$$

so that the integral of $\int I \, d\varphi$ over any arc $\psi = \text{const.}$, and over the transformed arc has the same value.

Now write

$$\varkappa / 2\pi = \int_0^{\varphi} I(\varphi, \psi) \, d\varphi \Big/ \int_0^{2\pi} I(\varphi, \psi) \, d\varphi,$$

thereby introducing a new analytic angular variable $\varkappa$ of period 2π which can replace φ. The transformation T will take the form

$$\varkappa_1 = \varkappa + \alpha(\psi), \qquad \psi_1 = \psi$$

in these special variables. Here $\alpha(\psi)$ is an analytic function of ψ.

Hence the transformation of each invariant curve of the analytic family which abuts on $\theta = 0$ is essentially a rotation of that curve into itself through an angle α which varies analytically with the curve and increases from 0 along $\theta = 0$ toward a limiting value π. But this variable ψ is not to be regarded as defined along the limiting non-analytic curve of course.

Consequently, if $\alpha(\psi)$ is commensurable with 2π, say $\alpha = 2\pi p/q$, every point of the invariant curve corresponds to a polygonal periodic motion of the billiard ball going p times around the ellipse and having q vertices. Here p is taken relatively prime to q, and p, q take all values for which $p < q/2$.

If $\alpha(\psi)$ is incommensurable with 2π, the entire curve corresponds to a single minimal set of recurrent motions of continuous type.

Furthermore, along the analytic family of invariant curves which abut on $\theta = \pi$, the transformation T is essentially a rotation with rotation number β say, which varies analytically from one invariant curve to another, and which diminishes from 2π at the boundary towards a limiting value π. In this case we have a similar distribution of periodic motions and motions of recurrent type.

If now we turn to the two analytic families of curves which abut on the points M_1 and M_2 respectively, and which are interchanged by T, it is apparent that it is desirable to consider T^2 rather than T, inasmuch as T^2 will leave every curve of either family invariant, and no point on any of these curves can be invariant save under an even iteration of T.

In the same way as before, it follows that the transformation of each of these invariant curves into itself under T^2 is essentially a rotation, of which the rotation number γ varies analytically from curve to curve; at either invariant point M_1, M_2, the value of γ is merely the rotation number for the corresponding stable periodic motion along the minor axis while γ tends toward a limiting value along the limiting non-analytic curve of the family.

If γ is commensurable with 2π, say $\gamma = 2\pi p/q$, every point of the invariant curve corresponds to a polygonal periodic motion going $2q$ times across the ellipse and oscillating in direction p times about the minor axis. If γ

is incommensurable with 2π, the curve corresponds to a single minimal set of recurrent type.

Evidently the points N_1, N_2 correspond to periodic motion along the major axis, and are of unstable type with analytic asymptotic branches represented by the invariant curves through these points. Similarly M_1, M_2 correspond to periodic motion of stable type along the minor axis.

Thus it is seen that the analytic weapons at our disposal have put us in a position to determine the possible types of motion and their interrelations.

Not only so, but the other natural questions which arise can be answered without difficulty.

For example, in the case of motion around the table in either sense is there a unique number which may be properly termed the mean angular rotation? The answer is affirmative in the periodic case, and can also be shown to be so in the non-periodic case. For note that if n denotes the number of vertices passed in any interval of time and if f_n denotes the corresponding increase in the coördinate x defined earlier then we have clearly

$$\lim_{n=\infty} f_n / n = \alpha$$

where α is the rotation number. Furthermore, if n is large the vertices will be distributed with approximately equal density for x between 0 and 2π. But the time from any vertex to the next is proportional to the length of the segment and so of the form $l(x)$, where l is analytic and periodic in x of period 2π. Hence the total time between the n vertices is

$$l(x) + l(x_1) + \cdots + l(x_{n-1}),$$

which is approximately given by

$$\frac{n}{2\pi}\int_0^{2\pi} l(x)\, dx.$$

Thus we conclude that there is a mean angular rotation which has the value

$$\frac{2\pi\,\alpha(\psi)}{\int_0^{2\pi} l(x)\,dx}$$

where α and l are definite analytic functions.

13. **The concept of integrability.** Questions concerning the integrability of a given dynamical problem possess great interest. It is a well known fact that for certain problems, auxiliary analytic relations can be deduced by means of which the solutions of the system of differential equations can be satisfactorily treated, in which case the system may be said to be 'integrable'. When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest. Let us consider briefly the concept of integrability, not forgetting the dictum of Poincaré, that a system of differential equations is only more or less integrable.

Let us note that in the particular problem just treated there are four periodic motions which play a special role, namely the motions along the two axes of the ellipse and the two motions of rolling around the ellipse.

All other periodic motions fall in analytic families of such motions and so are of highly degenerate type from a formal point of view. But these special motions are isolated and of general type. About these points there will be the usual formal series developments for the coördinates*, and these expressions may of course be taken to converge and to be analytically extended throughout a certain domain of the motion; in fact these properties are merely the counterpart of similar properties of the integrable transformation T, according to which it rotates certain invariant curves surrounding the invariant points in a specific manner.

Let us note also that in this problem four suitable neighborhoods of the four fundamental periodic motions yield the entire manifold M; in fact the two families of motions around

* Of course a discrete integer n figures in the formal series instead of the time t.

the ellipses, the family of motions across the ellipse and the single periodic motion along the major axis yield the totality of motions.

These facts suggest the following (not wholly precise) definition of integrability, based upon a local property and and a non-local property:

A given system of analytic differential equations on a closed analytic manifold M will be said to be integrable if there exists a finite set of periodic motions such that the corresponding complete formal series developments may be taken to converge, and to provide a corresponding analytic representation for every possible motion.

Using this definition as a kind of norm, some reflections suggest themselves.

In the first place it is natural to define 'local integrability' in the vicinity of a periodic motion of general stable type, as that in which the formal series may be taken to converge. Hence the motion is stable in the integrable case, and the explicit formulas yield complete information as to the character of nearby motions.

Now it is conceivable that, although these series do not converge, they may represent asymptotically functions continuous together with some or all of their derivatives near the periodic motions, with the aid of which the differential system can be transformed into a normal form like that of chapter III, section 13, in which $M_1, \cdots, M_n$ are functions of $\xi_1 \eta_1, \cdots, \xi_m \eta_m$, continuous together with certain of their derivatives. Here the qualitative behavior is the same as in the case of convergence. Shall the differential system be called 'locally integrable' in this more general case?

Furthermore, the qualitative behavior of the motions near a periodic motion of general unstable type in the case of two degrees of freedom is essentially independent of convergence or divergence. Shall we call every system locally integrable near such a periodic motion of unstable type?

If the differential system involves a parameter μ, we may inquire into that kind of local integrability in which the formal series are required not only to converge but also to be analytic in μ. It was in this sense that Poincaré established the non-existence of further uniform integrals in the problem of three bodies.* But evidently this definition is logically distinct from that presented in the above definition. Even if a system be non-integrable in this sense, it might perhaps be integrable according to our definition for each particular value of μ.

As far as I am aware, the local non-integrability of no single dynamical problem in the sense formulated above has been hitherto established. We shall, however, establish it in the following way for $m = 1$. 20

Let us suppose, if possible, that every Hamiltonian problem is locally integrable in the vicinity of a generalized equilibrium point of general stable type (see chapter III). The use of the normal variables shows then that the associated transformation T is essentially a rotation through a variable angle.

Along the invariant analytic curves with rational rotation numbers, all the motions will be periodic in the integrable case, with the same period $2k\pi$. It is this fact upon which we base our argument.

Furthermore, it is in the nature of the transformation T that there are no other periodic motions near equilibrium with this same period $2k\pi$, since the rotation constantly increases with the distance, and the invariant family is represented by a curve which meets every radius vector only once.

Now let us write

$$H = H_0 + \mu H_1$$

where H_0 is the given value of the principal function, μ is a small parameter, and H_1 is an analytic function of p_1, q_1 and t, periodic in t of period 2π, which is free of terms up to the fourth degree in p_1, q_1 at $p_1 = q_1 = 0$, but is otherwise arbitrary. It is readily verified that the multiplier λ and the constant l are independent of μ in the modified problem.

* *Méthodes nouvelles de la Mécanique céleste*, vol. 1, chap. 5.

Furthermore we may assume that H_0 is in normal form

$$H_0 = \lambda p_1 q_1 + \frac{1}{2} l p_1^2 q_1^2,$$

since the normal variables are available. Along any motion of the analytic family, the equations of variation have one periodic solution of period $2k\pi$, and only one, as direct computation shows.

Let μ vary from 0. There are two possibilities. Either the curve representing the periodic analytic family for $\mu = 0$ can be continued analytically, in which case there is a nearby curve for $\mu \neq 0$. Or there will only be a finite number of periodic motions of this period for μ small in absolute value.

In the first case there must obviously be a periodic solution of the equations of variation as to μ, which are obtained by adding non-homogeneous terms $-\partial H_1/\partial q_1$, $\partial H_1/\partial p_1$ to the respective right-hand members of the equations of variation referred to above. But, since $\partial H_1/\partial q_1$, $\partial H_1/\partial p_1$ can be taken almost at pleasure, along any particular periodic motion, the ordinary explicit formulas for the variations δp_1, δq_1 show that no periodic solution will exist in general. In fact the functions δp_1, δq_1 can be expressed as integrals linear in these arbitrary functions, augmented by the general solution of the homogeneous system, one part of which is periodic. Thus there are two conditions to fulfill and only one essential constant, and the condition for compatibility demands that a certain integral over a range $2k\pi$ vanishes, in which the integrand involves $\partial H_1/\partial q_1$, $\partial H_1/\partial p_1$ linearly. Evidently this cannot in general be the case.

Consequently for H_1 suitably chosen and then μ taken arbitrarily small, there will only be a finite number of periodic motions with this period.

But by hypothesis the modified system is integrable. By an other much slighter modification of H we can destroy an analytic periodic family much nearer to the position of generalized equilibrium while not introducing further periodic motions of period $2k\pi$.

Continuing in this manner we set up a limiting admissible principal function H, for which λ and l are unmodified, but for which there are no analytic periodic families belonging to rotation numbers as near to that of the generalized equilibrium as desired. This limiting problem cannot therefore be locally integrable in the specified sense.

Since there are only a denumerable set of periods $2k\pi$ $(k = 1, 2, \cdots)$ which enter into consideration, it is readily seen that no periodic analytic families near equilibrium will exist for suitable H.

*In a locally integrable Hamiltonian problem near generalized
equilibrium of general stable type, $l \neq 0$, there will exist in-
finitely many nearby analytic families of motions periodic in* 21
a multiple of the fundamental period.

In general a Hamiltonian problem near such a periodic motion will be locally non-integrable, and will possess no analytic families of nearby periodic motions.

It would be possible perhaps, and of considerable interest, to use the same method to show that nearby invariant families asymptotic in opposite senses to one and the same periodic motion do not exist in general. This would eliminate the possibility of invariant families belonging to a rational rotation number, and would establish that in general there is either complete instability or zonal instability:

The same method allows us to establish that multiple periodic motions do not exist in general for dynamical problems of this type.