CHAPTER VII

GENERAL THEORY OF DYNAMICAL SYSTEMS

1. Introductory remarks.* The final aim of the theory of the motions of a dynamical system must be directed toward the qualitative determination of all possible types of motions and of the interrelation of these motions.

The present chapter represents an attempt to formulate a theory of this kind.

As has been seen in the preceding chapters, for a very general class of dynamical systems the totality of states of motion may be set into one-to-one correspondence with the points, P, of a closed n-dimensional manifold, M, in such wise that for suitable coördinates $x_1, \cdots, x_n$, the differential equations of motion may be written

$$dx_i/dt = X_i\,(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

in the vicinity of any point of M, where the X_i are n real analytic functions and where t denotes the time. The motions are then presented as curves lying in M. One and only one such curve of motion passes through each point P_0 of M, and the position of a point P on this curve varies analytically with the variation of P_0 and the interval of time to pass from P_0 to P. As t changes, each point of M moves along its curve of motion and there arises a steady fluid motion of M into itself.

By thus eliminating singularities and the infinite region, it is evident that we are directing attention to a restricted class of dynamical problems, namely those of 'non-singular' type.

* Sections 1–4 are taken directly from my paper *Über gewisse zentrale Bewegungen dynamischer Systeme*, Göttinger Nachrichten (1926). The remainder is closely related to my papers, *Quelques théorèmes sur les mouvements des systèmes dynamiques*, Bull. Soc. Math. France, vol. 40 (1912), *Surface Transformations and Their Dynamical Applications*, Acta Mathematica, vol. 43 (1912), in particular sections 54–57.

However, most of the theorems for this class of problems admit of easy generalization to the singular case. The problem of three bodies, treated in chapter IX, is of singular type.

The differential equations of classical dynamics are more special, and in particular possess an invariant n-dimensional integral over M. In consequence any small molecule, σ, of M about a point P_0 at time t_0, must subsequently overlap its first position σ_0. This fact may be deduced as follows: Suppose that at a time t_1 following τ units after t_0, the molecule is in an entirely distinct position, and consider its positions at times τ, 2τ, $\cdots$ after the initial instant t_0. These positions cannot be entirely distinct from one another; for if v denotes the value of the invariant integral over the molecule in its initial position, this value will be the same in such subsequent positions, and since the value of the invariant integral over M is finite, say V, the number of distinct positions cannot exceed V/v. Hence some i-th and j-th molecules overlap ($i<j$). But if these overlap, then in the corresponding positions $(j-i)\tau$ units of time earlier, they will still do so. It follows that the $(j-i)$-th position of σ overlaps σ_0. By this argument and its natural extension, Poincaré* proved that in general the motions of such more special dynamical systems will recur infinitely often to the neighborhood of an initial state, and so will possess a kind of stability 'in the sense of Poisson'.

It will be our first aim in this chapter to show that with an arbitrary dynamical system not so restricted, there is associated always a closed set of 'central motions' which do possess this property of regional recurrence, towards which all other motions of the system in general tend asymptotically.

2. Wandering and non-wandering motions. Consider an arbitrary point P_0 of the manifold M of states of motion. Let σ be an open continuum of small diameter† ϵ, containing

* *Méthodes nouvelles de la Mécanique céleste*, vol. 3, chap. 26.

† It is evident that distance may be defined in an appropriate fashion in M. The diameter of a set of points is merely the upper limit of distances between pairs of points of the set.

P_0. As the time t increases, this 'molecule' σ moves. It may happen that P_0 represents a state of equilibrium; in that case the molecule will always continue to overlap P_0; we exclude this case for the moment. In any other case σ will move outside of itself if σ_0 is small enough, since the velocity components dx_i/dt are approximately the same as at P_0 throughout the molecule. If it is possible to take ϵ so small that σ never again overlaps its first position, we shall call P_0 a 'wandering point', and the corresponding motion a 'wandering motion'. In the contrary case P_0 will be termed a non-wandering point, and the corresponding motion a non-wandering motion. With this second class we naturally include equilibrium points and the degenerate corresponding motions.

There is an apparent asymmetry between the increasing and decreasing directions of the time t, as far as these definitions go. But it is easily seen that there is no asymmetry in actuality. In fact if the image of σ intersects itself after τ units of time, it does τ units of time earlier; for the overlapping molecules σ and σ_τ (i. e. taken τ units afterward) occupy the positions $\sigma_{-\tau}$ and σ respectively, τ units of time earlier, and these continue to overlap.

Thus the wandering point P_0 is characterized by the fact that the corresponding molecule σ describes an n-dimensional tube which never overlaps itself as t changes from $-\infty$ to $+\infty$. For this reason the characterization as 'wandering' seems legitimate, since P_0 never recurs to the infinitesimal neighborhood of any points once passed.

The set W of wandering points of M is made up of curves of motion filling open n-dimensional continua. The set M_1 of non-wandering points of M is made up of the complementary closed set of curves of motion.

For the reasons just presented all of this statement is obvious, except perhaps for the assertion that W is open and consequently M_1 is closed. But if P_0 is a wandering point, so evidently are all the points of the molecule σ including P_0. This shows at once that W is made up of open continua, and hence that M_1 is closed.

If the set M_1 of non-wandering motions contains points which are not limit points of the set W, these form a sub-set M_1' of curves of motion filling up open n-dimensional continua and possessing the property of regional recurrence.

It is clear that M_1' is made up of a set of curves of motion, for since Q of M_1' is not in the immediate neighborhood of any curve of motion of W the same will be true of all other points on the curve of motion containing Q. Also a sufficiently small molecule containing Q will be entirely in M_1' so that M_1' is made up of open n-dimensional continua of non-wandering points. Hence the property of regional recurrence is obvious.

Evidently the set $M_1 - M_1' = M_1''$ is merely the set of boundary points of the n-dimensional open continua W, M_1'. It is made up of a closed set of complete motions, of less than n dimensions.

As time increases or decreases, every wandering point approaches the set M_1 of non-wandering curves of motion.

The proof of this fundamental property of wandering motions is not difficult.

Consider any small open neighborhood of M_1, within which M_1 lies, and the complementary closed set C made up exclusively of the points W. About each point of C can be constructed a small molecule σ, such that the molecule never overlaps itself as time changes. Hence a finite set of these molecules can be found which cover C completely. A moving point can enter any one of these molecules (held fixed) only once, and can stay within it only a short interval of time. It is thus obvious that after a finite time the moving point lies always inside of the arbitrary neighborhood of M_1. Hence any moving point must approach M_1, as stated.

A more detailed study reveals certain further characteristics of the mode of approach of the wandering motions to the non-wandering motions. Since in the above discussion the moving point enters one of the fixed molecules covering C only once and stays in it for only a brief period of time, the following facts are obvious.

*Any wandering motion remains outside of a prescribed neighborhood of M_1 only a finite time T, and goes out of this neighborhood only a finite number of times N, where N and T are uniformly limited, once the neighborhood is chosen.**

3. **The sequence $M, M_1, M_2, \cdots$.** Having arrived at the closed set of non-wandering points M_1, between points of which distance may be defined as in M, we are in a position to define wandering and non-wandering points relative to M_1, (instead of M) as follows. Choose an arbitrary point P_0 of M_1, and an open continuum σ, of small diameter, containing P_0 and certain other points of M_1. Setting aside the case when P_0 is an equilibrium point, and choosing the diameter of σ small enough, we see that this molecular part of M_1 will move outside of itself away from its initial position. If the diameter can be chosen so small that the part of M_1 in σ never again overlaps itself, we may say that P_0 is a wandering point of M_1 (though of course non-wandering with respect to M by definition of M_1). Other points, including equilibrium points, in M_1 may be termed non-wandering points of M_1.

It is clear that the parallelism is complete. The non-wandering points M_2 relative to M_1 form a closed set of motions toward which any point P of the set of wandering points W_1 is asymptotic as time increases or decreases, and similar uniformity properties relative to M_1 hold.

The same process may now be repeated with respect to M_2 as a basis, and thus are defined M_3 and W_2. Continuing in this manner we obtain $M, M_1, M_2, \cdots$. We regard the process as terminating if any M_{i+1} is the same as M_i, in which case there are no points W_i of course. In case the process does not terminate in this way the sequence of distinct closed sets $M, M_1, M_2, \cdots$, each within its predecessors, defines a unique limiting set M_ω, which is evidently closed

* To render the count of exits precise, it is desirable to take each covering molecule so as to have at most one segment cut off by any neighboring stream line, and then to consider the neighborhood outside of the covering molecules as the given neighborhood of M_1.

and composed of curves of motion. Further application of the process yields $M_{\omega+1}, M_{\omega+2}, \cdots$. Thus are defined successively

$$\begin{array}{llll} M, & M_1, & M_2, & \cdots, \\ M_\omega, & M_{\omega+1}, & M_{\omega+2}, & \cdots, \\ M_{2\omega}, & M_{2\omega+1}, & M_{2\omega+2}, & \cdots, \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ M_{\omega^2}, & M_{\omega^2+1}, & M_{\omega^2+2}, & \cdots, \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{array}$$

in accordance with the well known theory of transfinite ordinals given by Cantor.

But these form an ordered set of distinct closed point sets each with an immediate successor and contained in all of its predecessors. Such a set is certainly numerable. Hence the process does eventually terminate in some M_r.

Thus there exists a well-ordered, terminating set of distinct closed sets

$$M,\ M_1,\ M_2,\ \cdots,\ M_\omega,\ \cdots,\ M_r,$$

in which an element M_{p+1} immediately following M_p consists of the non-wandering motions relative to M_p, while an element M_p without an immediate predecessor is the limit of its predecessors. The wandering points W_p of M_p tend asymptotically toward the non-wandering motions M_{p+1} in such wise that the total time outside of a given neighborhood of M_{p+1}, as well as the number of exits from this neighborhood, are uniformly limited.

The final set obtained, M_r, is the set of 'central motions'. It is obvious that these have the property of regional recurrence since there are no wandering points W_r. From this property it may be inferred by the method of Poincaré (loc. cit.) that in any arbitrary neighborhood of a point of M_r there is a motion which enters this region infinitely often in past and future.

In fact, if there is an isolated curve of motion of M_r in the region, the curve must be closed and this motion must be itself periodic, since the motion is non-wandering. In this case the periodic motion itself has the desired property. If there is no isolated motion, we can select a neighborhood of the given point as small as we please which overlaps it in M_r again at least once at some later time. Thus we get two points P, Q, on the same curve of motion of M_r, both in the molecule about the given point, but not near together in time. Next we choose a still smaller molecule about P, so small that every point P' in this molecule continues to meet the original molecule at a point Q' near to Q. But by choosing P' suitably, we can find a point R' before P' in time, lying in the same molecule as P', Q'. Thus we obtain an arc $R'\,P'\,Q'$ of a curve of motion of M_r such that the three points R', P', Q' lie in the given molecule taken at the outset at three different times. Next by choosing a still smaller molecule about P' we are led to an arc $R''\,P''\,Q''\,S''$, then to $T^{(3)}\,R^{(3)}\,P^{(3)}\,Q^{(3)}\,S^{(3)}$, and so on. The limit of the points $P, P', P'', \cdots$, will be a point of M_r in the given molecule, which traverses that molecule infinitely often in the past and in the future.

It is obvious that the periodic motions in the dynamical problem must lie in the set of central motions. The motions defined later (section 7) as 'recurrent' will also.

4. Some properties of the central motions. It is easy to see that every point of M approaches within an arbitrarily given neighborhood of the central motions at least once within every sufficiently large fixed interval of time. For such a point certainly approaches M_2 in this manner inasmuch as every motion of M_1 approaches M_2 uniformly often, and therefore every motion near M_1 does also. Continuing in this manner we see that this uniformity property holds for $M_1, M_2, \cdots$. If the series continues to M_ω, the same property must be true for M_ω. Indeed an M_n can then be found in any neighborhood of M_ω since M_ω is the inner limiting set of the closed sets M_n. Since a point approaches an arbitrarily given neighborhood of M_n uniformly often, it is therefore evident that it does the same for M_ω. By continuing indefinitely in this manner with $M_{\omega+1}, \cdots, M_{\omega^2}, \cdots, M_r$ we arrive at the stated conclusion.

Now the motions of M were seen to approach those of M_1 in a definite manner, while in turn those of M_1 approach M_2 similarly. By combining these results it would be possible to make certain statements as to the mode of approach of any point of M to M_2. This method of description might be carried on to $M_3, \cdots, M_\omega, \cdots$. But for the general purpose of this paper we shall merely establish a simpler result for all the sets M_p.

Let us define the 'probability' that an arc PQ lies in a given region Σ as the ratio of the time interval for the part that does lie in Σ to the total interval.

The probability that an arc of a curve of motion lies inside of an arbitrary neighborhood of a set M_p, and in particular the set of central motions M_r, approaches unity uniformly as the interval of time for such an arc increases indefinitely.

From what was proved at the outset concerning M_1, the probability that any arc lies in a given neighborhood of M_1 approaches unity uniformly as the interval of time increases indefinitely.

To prove a like result for M_2, we recall that any arbitrary motion of M_1 is represented by a curve which lies inside of a prescribed neighborhood of M_2 except for a limited number of arcs corresponding to a limited total interval. Now any long arc sufficiently near to M_1 will share the property of the motion of M_1, of being inside the prescribed neighborhood except for a limited number of arcs of limited total length, provided that first the length of the long arc is taken arbitrarily large, and then the neighborhood of M_1 within which it is to lie is chosen suitably. Furthermore if a point is near enough to M_1 it will evidently lie on such a long arc of a curve of motion.

Hence, since the probability that an arc of M lies within the prescribed neighborhood of M_1 approaches unity as the time interval is taken longer and longer, and since every point in such a neighborhood of M_1 is part of a long arc near to M_2 except for a finite number of pieces of limited total length, it is obvious that the probability that an arc in M lies within a given neighborhood of M_2 approaches unity uniformly as the time interval for the arc becomes larger and larger.

This same argument is evidently applicable for $M_3, M_4, \cdots$. For M_ω we need only note that since M_ω is the inner limiting set defined by $M_1, M_2, \cdots$, it will be approached as a limit by this set in such a way that for a sufficiently large n every point of M_n will be within distance ε of some point of M_ω. Since the probability that an arc of an arbitrary motion for a sufficiently long interval is within distance ε of M_n is at least $1-\delta$ where δ is small, the probability that it is within distance 2ε of M_ω is at least $1-\delta$.

Evidently this reasoning admits of indefinite continuation and leads to the desired conclusion.

5. Concerning the role of the central motions. It is obvious then that a first problem concerning the properties of dynamical systems is the determination of the central motions.

For the equations of classical dynamics the central motions are obviously the totality of motions, at least for the case without singularities to which we are now confining our attention. In fact the property of stability in the sense of Poisson involves that of regional recurrence, characteristic of the central motions.

The superior usefulness of the equations of classical type may very well be a reflection of the fact that the central motions are the most probable motions, rather than any consequence of the laws of nature.

6. Groups of motions. Consider now any curve of motion in M with a point P_t moving on it. The points P_t constitute the 'point group' of the given motion.

Every limit point of the set P_t for $\lim t = +\infty$ will be termed an ω limit point of the motion, and every limit point of the set P_t for $\lim t = -\infty$ will be termed an α limit point.

In all cases the limit points of either class form a closed point set.

The set of ω (α) *limit points of any motion* P_t *form a closed connected set of complete motions. The distance of* P_t *from this limit set approaches* 0 *for* $\lim t = +\infty$ ($-\infty$).

In fact let P^* be an ω limit point which P_t approaches for $\lim t = +\infty$, and let P^{**} be a point of the curve of motion through P^*, after an interval c. Evidently P_{t+c} will approach P^{**} in the same sense. That is to say, P^{**} is an ω limit point if P^* is. From this argument we infer that all points of the point group of P^* are ω limit points.

To establish that the distance from P_t to this set approaches 0, we employ an indirect argument. If P_t did not approach the set of ω limit points uniformly for $\lim t = +\infty$, it would be possible to select an infinite set of indefinitely increasing values of t, such that P_t would be distant from any ω limit point by at least a definite positive quantity d. There would then be at least one limit point P_1 of the set P_t, and this point would be at least d distant from any ω limit point. By definition, however, P_1 is an ω limit point, so that a contradiction results.

It is obvious that the ω limit set is connected inasmuch as it is approached uniformly by the point P_t as t becomes infinite, while P_t moves along its curve of motion continuously.

If we consider the groups of motions $M_1, M_2, \cdots$ which lead to the central motions M_r, it is obvious that the α and ω limit motions of a motion in M_p will form part of M_{p+1}.

7. **Recurrent motions.** Consider now an arbitrary, closed, connected set Σ of complete motions. It was observed above that the α or ω limit motions of any motion form such a set of motions. More generally, if we take any connected set of complete motions and adjoin to it the limit points, we obtain an enlarged set Σ.

If a set Σ contains no proper sub-set Σ' of the same type we shall say that Σ is a 'minimal set of motions'. In this case if P is any point of Σ, its α and ω limit points form closed sets in Σ, which must therefore coincide with Σ.

By definition any complete point group in a minimal set forms a recurrent point group and any motion in that group is called 'recurrent'.

All recurrent motions belong to the central motions. In fact the α and ω limit points of any such motion in M_p form a set Σ in M_{p+1}, which must coincide with the minimal set, so that no point of the set can be in M_p but not in M_{p+1}. Hence the minimal set corresponding to the recurrent motion lies in M_r.

In all cases but the simplest one, in which Σ consists of a single closed curve, a minimal set Σ contains a non-denumerable perfect set of curves of motion. For suppose a minimal set to have an isolated curve of motion. A point P_t on this curve has the points of the curve as its α or ω limit points. Hence this curve must be closed and constitutes the minimal set Σ.

In order that a point group generated by a motion P_t be recurrent, it is necessary and sufficient that for any positive quantity ε, however small, there exists a positive quantity T so large that any arc $P_t P_{t+T}$ of the curve of motion has points within distance ε of every point of the curve of motion.

This condition is necessary.

If not there is a recurrent point group Σ generated by P_t and a positive ε such that a sequence of arcs $P_t P_{t+2T}$ (T, arbitrarily large) can be found for each of which no point of the arc which comes with distance ε of a corresponding point Q of Σ. As T increases, the point Q has at least one limit point Q^*, and thus it is clear that for a properly taken subset of the sequence $P_t P_{t+2T}$, no point lies within distance $\varepsilon/2$ of Q^*. Consider the sequence of middle points P_{t+T} of such arcs. For a limiting position P^*, we infer that every point of the complete point group of P^* is at distance at least $\varepsilon/2$ from Q^*. Hence P^* defines a closed set of point groups lying within the closed minimal set Σ defining the given recurrent motion, but forming only part of it, and in particular not containing Q^*. This is absurd by the very definition of a minimal set.

To prove the condition sufficient, we note first that the sets of α and ω limit points of a point group satisfying this condition must coincide. We need only take $t = 0$ in the arbitrary arc $P_t P_{t+T}$ to see the truth of this fact. Call the set of these common α and ω limit points, Σ.

If the set Σ is not minimal it would contain a proper subset Σ' of the same sort to which some point Q of Σ would not belong. Now, when the point P_t approaches sufficiently near to a point of Σ', it will remain very near to this closed connected set of complete motions for an arbitrarily long interval of time and so will not approach all of Σ in this interval, as demanded. Thus the assumed condition would not be satisfied by the point group generated by P_t.

Hence Σ is minimal, and the motion is recurrent.

Clearly all recurrent motions are central motions, but of course central motions need not be recurrent. Indeed in the case of the differential equations of classical dynamics, all the motions are central, but need not be recurrent.

8. Arbitrary motions and the recurrent motions. The importance of the motions of recurrent type for the consideration of any arbitrary motion is evidenced by the following result:

There exists at least one recurrent group of motions in the $\omega(\alpha)$ limit motions of any given motion.

Let Σ denote the closed set of ω limit points of the given motion. We need to prove that the set Σ contains a minimal sub-set.

Divide M into a large number of small regions of maximum span not greater than ϵ, an assigned positive constant. Among the motions of Σ there will be one which enters a least number of these small regions of M under indefinite increase and decrease of t. Let Σ_1 be the corresponding closed set of complete limit motions. This set is part of Σ and lies wholly in the same small regions. Divide these small regions into regions of maximum span $\epsilon/2$. Among the motions of Σ_1 there will be one which enters a least set of these smaller regions of M under indefinite increase and decrease of t.

Define Σ_2 as the corresponding closed set of limit motions, which will be part of Σ_1.

Proceeding in this way we determine an infinite sequence $\Sigma_1, \Sigma_2, \cdots$ of closed connected sets of complete motions, each set being contained in its predecessors. Now let P_n be any point whatever of Σ_n, and let P^* be a limit point of the set P_n. The point P^* belongs to Σ of course since it is a limit point of points of Σ. Furthermore since P_n is contained in Σ_m $(m \leqq n)$, the limit point P^* lies in all of the regions $\Sigma, \Sigma_1, \cdots$. Likewise since the complete curve of motion through P_n is contained in Σ_m $(m \leqq n)$, the complete curve of motion through P^* lies in $\Sigma, \Sigma_1, \cdots$. Hence, in accordance with the property by which these regions were defined, the curve through P^* must enter all of the sub-regions at every stage, and hence its limit points must consist of all the points Σ_r common to $\Sigma, \Sigma_1, \Sigma_2, \cdots$.

The same argument shows that any motion lying in Σ_r has this complete set as its set of α or ω limit points. In other words the set Σ_r forms the desired minimal set.

The following further result shows that either a point P_t generates a recurrent motion, or else that it successively approaches and recedes from such recurrent motions uniformly often:

For any $\varepsilon > 0$ there exists an interval T so large that any arc $P_t P_{t+T}$ in M contains at least one point within distance ε of some group of recurrent motions.

The proof is immediate.

If the theorem is not true it is possible to obtain arcs $P_t P_{t+2T}$, not coming within distance ε of a recurrent point group for T arbitrarily large. Let then P_{t+T} denote the middle point of such an arc. If P^* is a limit point of the points P_{t+T} for $\lim t = +\infty$, evidently the complete curve through P^* has none of its points within distance ε of any recurrent point group. But the set of α and ω limit points of P^* each contain a minimal set. Thus a contradiction appears, since every motion in a minimal set is by definition recurrent.

9. Density of the special central motions. It is evident that the structure of the set of central motions M_r is of vital theoretic importance. Now this closed set of motions has been seen to be characterized by the property of regional recurrence, and thus the existence of an n-dimensional invariant volume integral for the equations of classical dynamics insures that the totality M_r is M itself in this case.

We propose to establish some simple properties of the set of central motions.

The set M_r is made up of one or more connected parts, each of which contains at least one minimal set of recurrent motions.

Any central motion whose α or ω limit points do not fill up all of the connected part of M_r on which the motion lies will be termed a 'special' central motion. A recurrent motion is special according to this definition unless the corresponding minimal set constitutes all of the connected part of M_r to which the recurrent motion belongs.

In particular then for classical dynamics, the special motions are those which do not pass arbitrarily near all possible states of motion, either as time increases or else as time decreases.

The special central motions are everywhere dense on any connected part of the set M_r of central motions, unless that part is made up of a single minimal set of recurrent motions.

For the case of classical dynamics ($M_r = M$) the special motions are thus dense throughout M, unless M is made up of a single minimal set of recurrent motions.

In establishing this result we shall take M_r as M itself, but it will be evident that the proof applies equally well to any connected part of M_r provided that by a region of M_r is understood any connected part of the set M_r, no point of which is a limit point of points not belonging to the region but in M_r.

Suppose if possible that there is a closed region E no point of which belongs to a special motion. Now there exists at least one set of recurrent motions Σ in M, which are special motions in the case under consideration and so fall entirely within the complementary region $F = M - E$.

Consider all of the points within a distance ϵ of Σ where ϵ is chosen so that the ϵ neighborhood of Σ lies entirely within F.

As t increases indefinitely, this ϵ neighborhood moves, and two possibilities arise: either (1) no points of the ϵ neighborhood go outside of F for ϵ sufficiently small or (2) at least one point of the neighborhood finally emerges from F, no matter how small ϵ is chosen.

The second alternative is easily disposed of. Let ϵ approach 0 and consider a sequence of arcs PQ of curves of motion in F such that P lies in the ϵ neighborhood of Σ, while Q lies on the boundary of F and corresponds to a later time t. Evidently the half curve of motion in the sense of *decreasing* time through any limiting position $\bar{Q}$ of Q for $\lim \epsilon = 0$, lies wholly in F, and constitutes a special motion of the type whose existence was denied. Thus we may confine our attention to the first alternative.

But in the first case choose ϵ as large as possible so that the set of motions passing through the ϵ neighborhood of Σ continue to lie wholly in F as t increases. It is apparent that the upper limit of values ϵ for which this is true is also a permissible value of ϵ. The points on these motions and limit motions constitute an augmented region R within which Σ lies. No motion of R when continued in the sense of decreasing time can emerge at a point P on the boundary of F, for then the motion through P for *increasing* time would be a special motion leaving E at P, and thereafter lying within F. For the same reason R must lie wholly within F. But now if we consider the points within the ϵ' neighborhood of R, some of these must emerge at a later point Q from F; otherwise the region R would not correspond to a maximum value of ϵ. Thus there arises a limiting point $\bar{Q}$, the motion through which is special of course, and remains in F with decreasing t. Thus a contradiction follows in all cases.

A slight extension of this reasoning enables us to establish the following more precise result:

If a connected region of M_r contains a motion within it, there exists at least one special motion passing through a point of its boundary and lying within the region as t increases (decreases).

Suppose that we consider an ϵ neighborhood of the complete motion within the region F. If t decreases indefinitely, this region moves and our earlier argument shows that the fact stated must be true unless no points of this ϵ neighborhood ever go outside of F with decrease of time, for ϵ small.

If we consider the ϵ neighborhood together with all of the part of F into which it moves as t decreases we obtain an augmented region. This augmented region must lie in F when t increases as well as decreases, and it must be invariant. For if a point of the augmented region moves outside of itself to a point Q as t increases, then a small enough molecule about Q would never overlap Q again as t decreases, contrary to the property of regional recurrence.

If we take ϵ as the upper limiting value, we obtain an invariant region R in F made up of complete motions. By considering points in the ϵ' neighborhood of R and letting t decrease, we find as before that some of the motions in this neighborhood must finally leave F at a point Q. By allowing ϵ' to approach 0, a limit point $\bar{Q}$ is found through which passes a motion lying in F for increasing t, as desired.

10. Recurrent motions and semi-asymptotic central motions. We shall say that a motion is positively (negatively) 'semi-asymptotic' to a minimal set of recurrent motions in case that set is the only minimal set among its ω (α) limit motions. With this definition in mind, we can state the following conclusion:

Either there are other recurrent motions in the immediate vicinity of a recurrent motion, or there exist central motions positively (negatively) semi-asymptotic to the recurrent motion.

The proof is immediate. Choose a small neighborhood of the given minimal set of recurrent motions. By the preceding section there will be a motion entering this neighborhood at a point P, and remaining within it subsequently. If this motion has other minimal sets besides the given minimal set in its set of ω limit points for ϵ arbitrarily small, the statement made is true. In the contrary case the motion through P will be positively semi-asymptotic to the given recurrent motion, also in accordance with the result stated.

11. Transitivity and intransitivity. Let us consider a 'molecule' about an arbitrary point in some connected part of the manifold of central motions, M_r. As t increases, this molecule moves in accordance with the differential equations and will sweep out a tube in M which must ultimately overlap itself because of the property of regional recurrence. Let R denote the tubular region so described together with its limit points. As t increases, the end of the tube R moves into the tube, and the region R is carried into all or part of itself. But, because of the property of regional recurrence, this region cannot move into part of itself, and so is carried precisely into itself, as t increases, or decreases. Thus the complete tube formed from the molecule by allowing t to vary in either direction yields the same region R, made up of complete motions.

Now two possibilities arise: either for every point P of M, and any molecule about the point, a region R is obtained which coincides with M, or for some point P and choice of an enclosing molecule, R is only part of M.

In the first case we shall call the connected part of the set of central motions M_r of 'transitive' type, whereas in the second case it is of 'intransitive' type.

For the problem of classical dynamics, transitivity means that any small molecule ultimately sweeps out the entire manifold M of states of motion (except for nowhere dense motions), whereas this is not true in the intransitive case.

A necessary and sufficient condition for the intransitivity of a connected set of central motions is that there exists an invariant closed region in the set, forming only part of it.

The characteristic of the intransitive case of classical dynamics is thus that there exist invariant n-dimensional continua of complete motions filling only part of M.

Evidently the stated condition is necessary since such an invariant region is found in the region R described above. On the other hand, if there is an invariant region R in M_r, a molecule lying within R must always continue to lie in R for increasing or decreasing t, so that the motions are intransitive.

In the intransitive case all motions are special. For an arbitrary motion either lies in such an invariant sub-continuum, or in the complementary set of invariant sub-continua, or on the boundary of the given invariant sub-continuum.

In case a connected part of the set of central motions M_r is transitive, there will exist motions which, as t either increases or decreases, ultimately pass arbitrarily near all points of the manifold of states of motion.

For definiteness we shall take $M_r = M$ in our demonstration. Moreover we make the preliminary observation that every molecule must fill M with increasing t; else it would define an invariant partial region R of M of the type excluded in the transitive case. Hence there exist arcs of curves of motion which go with increasing t from the neighborhood of a given point P to that of a second given point Q.

Let us begin by choosing a positive quantity d less than 1, and any numerable set of points P_k, $(k = 1, 2, \cdots)$, which is everywhere dense in M. It is clear that if a second set $\bar{P}_k$, $(k = 1, 2, \cdots)$ is assigned such that P_k is distant at most d^k from P_k for $k = 1, 2, \cdots$, then the second set will also be everywhere dense in M.

To obtain a motion which is not special we may proceed in the following manner. Within the d neighborhood of P_1, the point P_1' and an arc $P_1' P_2'$ of a curve of motion can be found such that P_2' lies in the d^2 neighborhood of P_2. Mark now about P_1' a smaller neighborhood, lying within the d neighborhood of P_1 and such that if P_1'' varies anywhere within it, the point P_2'' of the curve of motion through P_1''

still may be taken to vary continuously within the d^2 neighborhood of P_2. This is obviously possible.

Now within this smaller neighborhood of P_1' a point P_1'' may be selected so that the point Q_2'' of the arc of a curve of motion $Q_2''\,P_1''$ lies in the d^2 neighborhood of P_2. Thus an arc of a curve of motion $Q_2''\,P_1''\,P_2''$ is obtained, such that Q_2'' is within the d^2 neighborhood of P_2 while P_1'', P_2'' are within the d and d^2 neighborhoods of P_1 and P_2 respectively. Furthermore, we can take a neighborhood of P_1 still in the d neighborhood of P_1, so small that as P_1''' varies continuously within this neighborhood, both P_2''' and Q_2''' of an arc $Q_2'''\,P_1'''\,P_2'''$ vary continuously within the d^2 neighborhood of P_2.

By another similar step we may fix upon a P_3''' of an arc $Q_2'''\,P_1'''\,P_2'''\,P_3'''$ so that P_3''' lies within the d^3 neighborhood of P_3. By still another step we obtain an arc $Q_3^{(4)} \cdots P_3^{(4)}$, and so we may proceed indefinitely. In this way we construct at the k-th stage an arc of a curve of motion

$$Q_{k-1}^{(k)} \cdots P_1^{(k)}\,P_2^{(k)} \cdots P_{k-1}^{(k)}$$

such that $P_k^{(j)}$ and $Q_k^{(j)}$ lie in the d^k neighborhood of P_k.

It is clear that by a limiting process we arrive at a curve of motion

$$\cdots Q_2^*\,Q_1^*\,P_1^*\,P_2^*\,P_3^* \cdots$$

in which P_k^* and Q_k^* lie within the d^k neighborhood of P_k. Consequently the sets P_1^*, P_2^*, $\cdots$ and Q_1^*, Q_2^*, $\cdots$ are everywhere dense in M. Hence the α limit and the ω limit motions make up all of M, and the motion itself is not special.

In the following chapter (section 11) an example of a nonsingular geodesic problem of transitive type is given. It seems probable that, in general, after the obvious reductions by means of known integrals are effected, the problems of classical dynamics are of transitive type.

Between the general transitive case and the highly specialized cases of completely integrable type, there is a prodigious variety of intermediate possibilities, dependent on the particular properties of the differential equations.

In the following chapter we consider the case of a system with two degrees of freedom. Unfortunately it does not seem to be the fact that the methods there employed admit of simple extension to the case of more degrees of freedom. The problem of three bodies, treated in chapter IX, is extremely instructive as an instance of this more complicated case, although it is of singular type.