CHAPTER III

FORMAL ASPECTS OF DYNAMICS

1. Introductory remarks. In the preceding chapter it was pointed out that a system of ordinary differential equations

$$(1) \qquad dx_i/dt = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

was devoid of invariantive characteristics under the general group of one-to-one, analytic transformations

$$(2) \qquad x_i = \varphi_i(\bar{x}_1, \cdots, \bar{x}_n) \qquad (i = 1, \cdots, n),$$

provided that attention be confined to the vicinity of any point x_i^0. This was done under the assumption that the functions X_i were analytic and did not all vanish at the point x_i^0.

The simplest case in which invariantive characteristics can be expected to arise is that of a point of equilibrium, the condition for which is $X_i^0 = 0$, $(i = 1, \cdots, n)$. Another important case, which may be regarded as including this one, is that associated with the neighborhood of a periodic solution of (1) of period τ

$$x_i = f_i(t) \qquad (i = 1, \cdots, n).$$

If we write then

$$x_i = f_i(t) + \bar{x}_i \qquad (i = 1, \cdots, n),$$

the equations take the more general form

$$(3) \qquad dx_i/dt = X_i(x_1, \cdots, x_n, t) \quad (i = 1, \cdots, n),$$

where the functions X_i are analytic in $x_1, \cdots, x_n$ and t, periodic in t with the period τ of the motion, and vanish at the origin in the new $x_1, \cdots, x_n$ space for all values of t. We shall treat the question of the invariantive characteristics for the 'generalized equilibrium problem' defined by a system of this more general type.

In the present chapter we shall in the main restrict attention to those purely formal properties which have no regard to the convergence or divergence of the series employed, and which take for granted an equilibrium point or a known periodic motion. By doing so we shall be able to develop to a considerable extent the formal significance of the Lagrangian, Hamiltonian and Pfaffian types of equations employed in dynamics. In order to do this we propose first of all to develop the characteristics of what may be called the general case of the equilibrium problem, and then to pass on to the more special types referred to, so that a comparison may be effected.

2. The formal group. As a matter of notational convenience, the equilibrium point of (3) will be kept at the origin in all cases. The type of transformation considered will be

$$x_i = \sum_{j=1}^{n} a_{ij}(t)\,\bar{x}_j + \frac{1}{2}\sum_{j,k=1}^{n} a_{ijk}(t)\,\bar{x}_j\,\bar{x}_k + \cdots \tag{4}$$

where the real coefficients $a_{ij}, a_{ijk}, \cdots$ are periodic analytic functions of t with period τ, such that the determinant $|a_{ij}|$ is not zero for any t. Evidently two of these transformations performed successively may be united into a single composite transformation, while the inverse of such a transformation is another of the same type. Furthermore, the form (3) of the differential equations is clearly maintained under this group of transformations.

Imagine now that divergent series appear in (4). The right-hand members $\bar{X}_i$ of the transformed differential equations will then be given as definite formal power series in $\bar{x}_1, \cdots, \bar{x}_n$ with coefficients analytic in t and of period τ, and these series will lack constant terms. Thus along with the formal group we obtain corresponding formal differential equations. Now it is to be particularly stressed that the ordinary laws for composition of transformations, and for deriving the associated differential equations, hold in the case of divergence as well as in that of convergence. This is an obvious consequence of the fact that when the formal series involved are broken off at terms of high degree, actual transformations and actual differential equations are obtained for which these formal laws are valid. By including terms of higher and higher degree the lower degree terms in the differential equations are not affected. Proceeding thus to the formal limiting case, we infer that the usual purely formal relations will continue to hold in the case of divergence.

In many cases it is convenient to introduce a slight extension of the above formal group so as to take care of certain pairs of variables x_i and x_j in a special way. In fact it is convenient to introduce conjugate variables

$$\xi = x_i + \sqrt{-1}\, x_j, \qquad \eta = x_i - \sqrt{-1}\, x_j$$

so that if x_i and x_j are real, ξ and η are conjugate imaginaries, and conversely. At the same time the transformation from $x_1, \cdots, x_n$ to $\bar{x}_1, \cdots, \bar{x}_n$ may be expressed in terms of the conjugate pairs such as ξ, η and the corresponding transformed variables $\bar{\xi}$, $\bar{\eta}$. The series involved are then characterized by the property that if the conjugate variables ξ and η are given conjugate imaginary values, while those not so paired are real, then the same will hold for the new variables. If we go back to the underlying transtormation belonging to the formal group, the new variables $\bar{x}_1, \cdots, \bar{x}_n$ will be real if $x_1, \cdots, x_n$ are.

It is not difficult to determine the characteristics of the formal series which appear in the transformation of such conjugate pairs of variables. If the transformation within the original group be written

$$\begin{aligned} x_i &= f_i(\bar{x}_1, \bar{y}_1, \cdots, \bar{x}_s, \bar{y}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n), \\ y_i &= g_i(\bar{x}_1, \bar{y}_1, \cdots, \bar{x}_s, \bar{y}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n), \\ &\qquad\qquad\qquad\qquad (i = 1, \cdots, s), \\ x_i &= h_i(\bar{x}_1, \bar{y}_1, \cdots, \bar{x}_s, \bar{y}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n), \\ &\qquad\qquad\qquad\qquad (i = 2s+1, \cdots, n), \end{aligned}$$

where $x_1, y_1, \cdots, x_s, y_s$ are the s pairs of associated variables, then the conjugate variables are

$$\xi_i = x_i + \sqrt{-1}\, y_i, \qquad \eta_i = x_i - \sqrt{-1}\, y_i \qquad (i = 1, \cdots, s)$$

with like definitions of $\bar{\xi}_i, \bar{\eta}_i$ in terms of $\bar{x}_i, \bar{y}_i$. Hence we have explicitly

$$\begin{aligned} \xi_i &= f_i((\bar{\xi}_1 + \bar{\eta}_1)/2, (\bar{\xi}_1 - \bar{\eta}_1)/2\sqrt{-1}, \cdots, \bar{x}_{2s+1}, \cdots, \bar{x}_n) \\ &+ g_i((\bar{\xi}_1 + \bar{\eta}_1)/2, (\bar{\xi}_1 - \bar{\eta}_1)/2\sqrt{-1}, \cdots, \bar{x}_{2s+1}, \cdots, \bar{x}_n) \sqrt{-1} \end{aligned}$$

for $i = 1, \cdots, s$ with like formulas for $\eta_i, (i = 1, \cdots, s)$ and for $x_i, (i = 2s+1, \cdots, n)$, when we employ the modified variables. Clearly the series on the right have the properties of the transformations of the formal group except that the periodic coefficients are in general complex.

If we examine the form of these series we perceive at once that they posses the following additional characteristic property. If the pairs $\bar{\xi}_i, \bar{\eta}_i$ be interchanged in the series on the right, and if at the same time the periodic coefficients be replaced by their conjugates, then the series for $\xi_i, \eta_i, (i = 1, \cdots, s)$ are interchanged, while those for $x_i, (i = 2s+1, \cdots, n)$ are unaltered.

It is readily proved that this necessary formal property is also sufficient. In fact suppose that x_i, y_i are real quantities and define $\bar{\xi}_i, \bar{\eta}_i$ as before so that $\bar{\xi}_i, \bar{\eta}_i$ are conjugate complex quantities. Write

$$\begin{aligned} \xi_i &= \varphi_i(\bar{\xi}_1, \cdots, \bar{\eta}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n, t) \qquad (i = 1, \cdots, s), \\ \eta_i &= \psi_i(\bar{\xi}_1, \cdots, \bar{\eta}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n, t) \qquad (i = 1, \cdots, s), \\ x_i &= \chi_i(\bar{\xi}_1, \cdots, \bar{\eta}_s, \bar{x}_{2s+1}, \cdots, \bar{x}_n, t) \qquad (i = 2s+1, \cdots, n), \end{aligned}$$

where the series $\varphi_i, \psi_i, \chi_i$ are assumed to have the stated formal property and, at the outset, to be convergent. If then we use the * as superscript to denote the 'conjugate of', we find, for instance,

$$\begin{aligned}\xi_i^* &= \varphi_i^*(\bar{\xi}_1^*, \cdots, \bar{\eta}_s^*, x_{2s+1}, \cdots, x_n, t)\\ &= \varphi_i^*(\bar{\eta}_1, \cdots, \bar{\xi}_s, x_{2s+1}, \cdots, x_n, t)\\ &= \psi_i(\bar{\xi}_1, \cdots, \bar{\eta}_s, x_{2s+1}, \cdots, x_n, t) = \eta_i;\end{aligned}$$

here φ_i^* designates the formal series obtained from φ_i by replacing the periodic coefficients by their conjugates. Hence ξ_i and η_i are conjugates also. The same kind of argument shows that x_i must be real for $i = 2s+1, \cdots, n$.

In case the series of the underlying formal gronp are divergent, the series belonging to conjugate variables may also be divergent. But of course the above formal property is still maintained as may be proved by breaking of the formal series at terms of high degree and then applying the above argument.

As a very simple instance of the change from real to conjugate variables let us suppose that the transformation in real form involves one pair of variables $\bar{x}$, $\bar{y}$ as follows:

$$\begin{aligned}x &= \bar{x}\cos(\theta+t-c\bar{r}^2)-\bar{y}\sin(\theta+t-c\bar{r}^2) \qquad (\bar{r}^2 = \bar{x}^2+\bar{y}^2),\\ y &= \bar{x}\sin(\theta+t-c\bar{r}^2)+\bar{y}\cos(\theta+t-c\bar{r}^2).\end{aligned}$$

Evidently this is a transformation of the admitted type with $\tau = 2\pi$. Introducing the corresponding pair of conjugate variables ξ, η we find readily

$$\xi = \bar{\xi}e^{\sqrt{-1}(\theta+t-c\bar{\xi}\bar{\eta})}, \qquad \eta = \bar{\eta}e^{-\sqrt{-1}(\theta+t-c\bar{\xi}\bar{\eta})},$$

and the characteristic property of the series on the right is at once verified.

3. Formal solutions. Suppose that we substitute in the system (3) of differential equations under consideration

$$x_i = F_i(t, c_1, \cdots, c_n) \qquad (i = 1, \cdots, n), \tag{5}$$

where the F_i are formal power series in the n arbitrary constants $c_1, \cdots, c_n$ without constant terms but such that the determinant $|\partial F_i/\partial c_j|$ is not zero for any t at the origin, and where the coefficients in these series are real and analytic in t; the series may converge or diverge. It may happen that the n equations (3) so obtained will be satisfied identically by these series x_i in the formal sense. In this case we shall say that the series (5) give the 'general formal solution' of the given differential system.

In a special case the coefficients in F_i may happen to be periodic in t of period τ. There would then be defined a corresponding transformation

$$x_i = F_i(t, y_1, \cdots, y_n) \qquad (i = 1, \cdots, n)$$

in the formal group. If the associated formal differential system is

$$dy_i/dt = Y_i(y_1, \cdots, y_n, t) \qquad (i = 1, \cdots, n),$$

we have then the formal identities

$$\frac{\partial F_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j} Y_j \equiv X_i(F_1, \cdots, F_n, t) \qquad (i = 1, \cdots, n),$$

in which the arguments of F_i are $y_1, \cdots, y_n, t$ of course. But the precise meaning of the hypothesis that the F_i yield a formal solution is that

$$\partial F_i/\partial t \equiv X_i(F_1, \cdots, F_n, t) \qquad (i = 1, \cdots, n),$$

where now $c_1, \cdots, c_n$ replace $y_1, \cdots, y_n$ as the variables in the series F_i. And if we replace $c_1, \cdots, c_n$ by $y_1, \cdots, y_n$ respectively, as we may, and compare with the equations which precede, it appears that

$$\sum_{j=1}^{n} (\partial F_i/\partial y_j)\, Y_j \equiv 0,$$

whence of course $Y_i \equiv 0$, $i = 1, \cdots, n$. Consequently this special case is the case in which the given differential system can be formally transformed into the normal form

$$dx_i/dt = 0 \qquad (i = 1, \cdots, n)$$

by a transformation within the specified formal group. As we shall see, this is not in general possible of accomplishment.

If the variables in (3) *are transformed within the formal group, any general formal solution* (5) *is transformed into a general formal solution of the transformed equations.*

An easy way of proving this is to extend the formal group for the moment by allowing analytic coefficients not periodic of period τ. There will then be a transformation, obtained from (5) by replacing $c_1, \cdots, c_n$ by $y_1, \cdots, y_n$ respectively, which is in the extended group, and which takes the given equations into the system for which $Y_i = 0$ $(i = 1, \cdots, n)$ as before. But the transformed system in $\bar{x}_1, \cdots, \bar{x}_n$ can then be taken into this special system directly by means of the composite transformation taking $\bar{x}_i$ to x_i, and then x_i to y_i. But this means precisely that the general solution of the transformed system may be obtained in the specified manner. It is assumed in this reasoning that the formal laws remain valid within the extended formal group.

The most general formal solution $x_i = G_i(t, d_1, \cdots, d_n)$, $(i = 1, \cdots, n)$ *of* (3) *can be obtained from any particular formal solution* (5) *by substitution of n arbitrary real series in* $d_1, \cdots, d_n$ *for* $c_1, \cdots, c_n$, *say*

$$c_i = \varphi_i(d_1, \cdots, d_n) \qquad (i = 1, \cdots, n),$$

with the sole proviso that the determinant $|\partial \varphi_i/\partial d_j|$ *is not zero for* $d_1 = \cdots = d_n = 0$.

This almost obvious fact may also be established readily by use of the extended group. The two transformations

$$x_i = F_i(t, z_1, \cdots, z_n), \qquad x_i = G_i(t, w_1, \cdots, w_n) \qquad (i = 1, \cdots, n)$$

take the equations (3) into

$$dz_i/dt = 0, \quad dw_i/dt = 0 \qquad (i = 1, \cdots, n)$$

respectively. Hence we may write

$$z_i = \varphi_i(t, w_1, \cdots, w_n) \quad (i = 1, \cdots, n)$$

as the transformation taking directly the equations in $z_1, \cdots, z_n$ into the equations in $w_1, \cdots, w_n$. We infer at once that

$$\frac{dz_i}{dt} = \frac{\partial \varphi_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial w_j} \frac{dw_j}{dt} = \frac{\partial \varphi_i}{dt} = 0,$$

so that the variable t does not appear in $\varphi_1, \cdots, \varphi_n$, i. e.,

$$z_i = \varphi_i(w_1, \cdots, w_n) \qquad (i = 1, \cdots, n).$$

Thus we obtain the formal identities

$$F_i(t, \varphi_1, \cdots, \varphi_n) \equiv G_i(t, d_1, \cdots, d_n) \quad (i = 1, \cdots, n),$$

provided that we replace $w_1, \cdots, w_n$ by $d_1, \cdots, d_n$ respectively. This is the relationship which we set out to prove. Of course $|\partial \varphi_i / \partial d_j|$ is not 0 for $d_1 = \cdots = d_n = 0$ by virtue of the definition of the extended group.

The question of the existence of formal solutions is immediately disposed of. In fact let us take $c_1 = y_1^0, \cdots, c_n = y_n^0$ where y_i^0 stands for the value of y_i at $t = t_0$. Then it has been established that the general solution

$$y_i = F_i(t, c_1, \cdots, c_n) \qquad (i = 1, \cdots, n)$$

is analytic in $c_1, \cdots, c_n$ for $|c_i|$, $(i = 1, \cdots, n)$, small and for any t lying in the t interval for $c_1 = \cdots = c_n = 0$. But the solution is then $x_i = 0$ $(i = 1, \cdots, n)$ for *all* values of t, so that the solution is analytic in $c_1, \cdots, c_n$, t for t arbitrarily large, provided that $|c_i|$ are then sufficiently small. Thus the right-hand members may be expanded in power series in $c_1, \cdots, c_n$ with coefficients analytic in t for all values of t. Furthermore we have clearly for $t = t_0$

$$\partial \varphi_i / \partial c_j = \delta_{ij} \quad (\delta_{ii} = 1;\ \delta_{ij} = 0,\ i \neq j).$$

Hence $\partial \varphi_i / \partial c_j$ $(i = 1, \cdots, n)$ constitute n linearly independent solutions of the equations of variation for $j = 1, \cdots, n$, and the determinant $|\partial \varphi_i / \partial c_j|$ will not vanish for any t.

There exist formal solutions of any system (3).

It is obvious that if conjugate variables are introduced in the fashion of the preceding section, then the formal solutions will have a corresponding modified form.

The significance of the formal solutions will be specified further in the following chapter. Suffice it to say here that these are the type of solutions actually employed in astronomical problems when the perturbations of a periodic motion need to be calculated.

4. The equilibrium problem. As has been pointed out earlier (section 1), the simplest case for consideration is that presented by an ordinary equilibrium point. For this case the functions $X_1, \cdots, X_n$ do not involve t explicitly. In dealing with this special case we shall use only transformations of the formal group which do not involve the time t.

The equations of variation take the form

$$dy_i/dt = \sum_{j=1}^{n} c_{ij} y_j \qquad (i = 1, \cdots, n)$$

where the constants c_{ij} are the values of $\partial X_i/\partial x_j$ evaluated at the origin. In the case to which we limit attention at first the determinant equation

$$|c_{ij} - m\,\delta_{ij}| = 0 \tag{6}$$

has n roots $m_1, \cdots, m_n$, not subject to any commensurability relation

$$i_1 m_1 + \cdots + i_n m_n = 0 \tag{7}$$

for any set of integers $i_1, \cdots, i_n$ not all zero. These roots are real except possibly for certain conjugate imaginary pairs. It is to be noted that the assumed inequality excludes the possibility of a zero root so that the determinant $|c_{ij}|$ is not zero.

Now we can clearly determine a square array $l_{ij}, (i, j = 1, \cdots, n)$, with l_i not all zero for any particular j, such that the sets $l_{1k}, \cdots, l_{nk}$ satisfy the n homogeneous linear equations

$$\sum_{j=1}^{n} c_{ij}\, l_{jk} = l_{ik}\, m_k. \tag{8}$$

In fact the vanishing of the determinant of this system gives precisely the defining equation for the roots m_k. Moreover the determinant $|l_{ij}|$ is not zero. This fact and the reasoning which follows are of course well known in the ordinary theory of linear transformations; however, for the sake of completeness the reasoning is included. In fact, suppose the contrary to be true, so that multipliers $\varrho_1, \cdots, \varrho_n$ not all zero exist such that

$$\sum_{j=1}^{n} l_{ij} \varrho_j = 0 \qquad (i = 1, \cdots, n).$$

Multiplying the earlier equations by ϱ_k, $(k = 1, \cdots, n)$, and adding, there results at once

$$0 = \sum_{k=1}^{n} l_{ik} m_k \varrho_k$$

so that $m_i \varrho_i$, $(i = 1, \cdots, n)$, also yield a second set of such multipliers. Continuing in this way we infer that $m_i \varrho_i$, $m_i \varrho_i^2, \cdots$ must form further sets of multipliers. Hence by linear combination still more general sets of multipliers

$$(c_0 + c_1 m_i + c_2 m_i^2 + \cdots + c_{n-1} m_i^{n-1}) \varrho_i \quad (i = 1, \cdots, n),$$

are obtained. But the n quantities in parentheses here can be made to take on n arbitrarily assigned values, just because the roots m_i are distinct. In particular the coefficient of any $\varrho_k \neq 0$ can be replaced by 1 while all the others are made to vanish. But this would necessitate that $l_{ik} = 0$, $(i = 1, \cdots, n)$, contrary to hypothesis. Consequently $|l_{ij}|$ cannot vanish.

It is also obvious that we may take the quantities l_{ij} so that in the transformation

$$y_i = \sum_{j=1}^{n} l_{ij} z_j \qquad (i = 1, \cdots, n),$$

from y_i to z_i, the variables z_i and z_j corresponding to conjugate imaginary roots m_i and m_j will have conjugate imaginary values when $y_1, \cdots, y_n$ are real, and conversely.

Now let us introduce this change of variables in the special type of equations (1) under consideration. The equations obtained by substitution in (1) of the linear expressions in z_i for y_i will then assume the form

$$\sum_{j=1}^{n} l_{ij} \frac{dz_j}{dt} = \sum_{j=1}^{n} l_{ij}\, m_j\, z_j + \cdots \qquad (i = 1, \cdots, n),$$

where the terms of higher than the first degree in $z_1, \cdots, z_n$ in the right-hand members are only indicated; in thus writing the first terms on the right, the characteristic property of the quantities l_{ij} is of course employed. It follows from these relations that the equations in z_i have the form

$$dz_i/dt = m_i\, z_i + \cdots \qquad (i = 1, \cdots, n)$$

where only the linear terms are explicitly written.

Thus we may take the system to be of the prepared form

$$dx_i/dt = m_i\, x_i + F_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

in which we may write

$$F_i = F_{i2} + F_{i3} + \cdots \qquad (i = 1, \cdots, n),$$

with F_{ik} a homogeneous polynomial of degree k in $x_1, \cdots, x_n$.

We shall show next that we can obtain formal series

$$\varphi_i(x_1, \cdots, x_n) = \varphi_{i2} + \varphi_{i3} + \cdots \qquad (i = 1, \cdots, n),$$

such that the transformation

$$\bar{x}_i = x_i + \varphi_i \qquad (i = 1, \cdots, n)$$

reduces the differential equations to the form

$$d\bar{x}_i/dt = m_i\, \bar{x}_i \qquad (i = 1, \cdots, n).$$

This will be achieved, provided that the equations

$$dx_i/dt + d\varphi_i/dt = m_i(x_i + \varphi_i) \qquad (i = 1, \cdots, n)$$

follow in consequence of the differential equations in x_i. Using these equations to eliminate dx_i/dt, we obtain as the desired relations

$$F_i + \sum_{j=1}^{n} (\partial \varphi_i / \partial x_j)(m_j x_j + F_j) = m_i \varphi_i \qquad (i = 1, \cdots, n).$$

On expanding F_i and φ_i in series, these take the form for $i = 1, \cdots, n$,

$$F_{i2} + \sum_{j=1}^{n} \frac{\partial \varphi_{i2}}{\partial x_j} m_j x_j = m_i \varphi_{i2},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$F_{ik} + \sum_{j=1}^{n} \left(\frac{\partial \varphi_{ik}}{\partial x_j} m_j x_j + \sum_{p+q=k} \frac{\partial \varphi_{ip}}{\partial x_j} F_{jq} \right) = m_i \varphi_{ik},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Let us consider the first equation written for any i, which obviously constitutes a partial differential equation for φ_{i2}. The coefficient c_i of the term

$$c_i x_1^{l_1} \cdots x_n^{l_n} \qquad (l_1 + \cdots l_n = 2)$$

in φ_{i2} is then evidently determined in terms of the analogous coefficient d_i of F_{i2} be means of the n equations

$$d_i + [l_1 m_1 + \cdots + (l_i - 1) m_i + \cdots + l_n m_n] c_i = 0.$$

But the term in parenthesis is not zero by virtue of the hypothesis made concerning the quantities m_i, so that c_i can be determined as desired. Hence there is a unique set of homogeneous quadratic polynomials φ_{i2} satisfying the first set of the equations written above.

In the same way the second set of equations determines φ_{i3} uniquely since the equation for determination of the coefficients in φ_{i3} is of the same general type as above except that we have $l_1 + \cdots + l_n = 3$ in this case.

Ignoring then the questions of convergence of the series employed we arrive at the following conclusion:

By means of a formal transformation

$$x_i = f_i(z_1, \cdots, z_n) = \sum_{j=1}^{n} l_{ij} z_j + \frac{1}{2} \sum_{j,k=1}^{n} l_{ijk} z_j z_k + \cdots \qquad (i = 1, \cdots, n)$$

with $|l_{ij}| \neq 0$, *the differential equations* (1) *with an ordinary equilibrium point of general type at the origin, can be reduced to the normal form*

$$dz_i/dt = m_i z_i \qquad (i = 1, \cdots, n), \tag{9}$$

so that corresponding to conjugate roots m_i *and* m_j *there are conjugate variables* z_i *and* z_j.

Since the normal form just written is integrable with general solution

$$z_i = c_i e^{m_i t} \qquad (i = 1, \cdots, n)$$

we may state the following conclusion also:

The corresponding formal solution of (1) *may be written in the form*

$$x_i = f_i(c_1 e^{m_1 t}, \cdots, c_n e^{m_n t}) \qquad (i = 1, \cdots, n)$$

where the f_i *are the same formal power series as appear in the transformation to normal form.*

5. The generalized equilibrium problem. It is not difficult to extend the above method to the generalized equilibrium problem in which we start with equations of the form (3). Here the equations of variation form a system of n ordinary linear differential equations

$$dy_i/dt = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j}\bigg|_{t=0} y_j \qquad (i = 1, \cdots, n)$$

with coefficients $\partial X_i/\partial x_j\big|_{t=0}$ which are analytic periodic functions of t of period τ. Let $y_{1k}, \cdots, y_{nk}$, $(k = 1, \cdots, n)$, give for each k a solution such that the n solutions are linearly independent. Then the general solution is a linear combination of these particular solutions. When t is increased by τ, the equations of variation are unaltered. Hence we have

$$y_{ik}(t+\tau) = \sum_{l=1}^{n} y_{il}(t)\, c_{kl} \qquad (i, k = 1, \cdots, n).$$

Now if we define $m_1, \cdots, m_n$ as in (8) by means of the square array c_{ij} thus obtained, we may choose another linearly independent set of n solutions which will reduce the above relations to a normal form

$$y_{ik}(t+\tau) = m_k\, y_{ik} \qquad (i, k = 1, \cdots, n).$$

We confine ourselves as before to the general case, in that we exclude linear commensurability relations between $\mu_1, \cdots, \mu_m$ and $2\pi\sqrt{-1}/\tau$, where $\mu_k = (\log m_k)/\tau$. In this event $m_1, \cdots, m_n$ are all distinct.

Now let us write the elements appearing in the solution of the equations of variation in the form

$$y_{ik} = e^{\mu_k t} p_{ik}(t) \qquad (i, k = 1, \cdots, n),$$

when it is apparent that the functions p_{ik} will be periodic of period τ. Furthermore from a familiar theorem we know that the determinant

$$|\, y_{ij} \,| = |\, p_{ij} \,|\, e^{(\mu_1 + \cdots + \mu_n)t}$$

is nowhere 0. Consequently the particular linear change of variables

$$x_i = \sum_{j=1}^{n} p_{ij} z_j \qquad (i = 1, \cdots, n)$$

from $x_1, \cdots, x_n$ to $z_1, \cdots, z_n$ is within the admitted group. The equations of variation will have a solution

$$y_i = \delta_{ik} e^{\mu_k t} \qquad (i = 1, \cdots, n)$$

for $k = 1, \cdots, n$, so that the new equations must be

$$dz_i/dt = \mu_i z_i + \cdots \qquad (i = 1, \cdots, n).$$

Consequently we infer that it is no restriction to write the given equations in the prepared form

$$dx_i/dt = \mu_i x_i + F_i(x_1, \cdots, x_n, t) \qquad (i = 1, \cdots, n)$$

where F_i is periodic in t of period τ of course. It is obvious that if certain pairs of the quantities m_i are conjugate imaginaries the transformation of variables employed may be taken to be of the admitted conjugate type.

Suppose that we continue by effecting a further transformation of the same type as in the ordinary equilibrium problem save that the coefficients in the series φ_i need not be constants but may be periodic analytic functions of t of period τ. If we endeavor to choose this set of functions φ_i so as to normalize the transformed differential equations as in the special case of ordinary equilibrium we obtain analogous equations, namely

$$F_{i2} + \frac{\partial \varphi_{i2}}{\partial t} + \sum_{j=1}^{n} \frac{\partial \varphi_{i2}}{\partial x_j} \mu_j x_j = \mu_i \varphi_{i2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_{ik} + \frac{\partial \varphi_{ik}}{\partial t} + \sum_{j=1}^{n} \left(\frac{\partial \varphi_{ik}}{\partial x_j} \mu_j x_j + \sum_{p+q=k} \frac{\partial \varphi_{ip}}{\partial x_j} F_{jq} \right) = \mu_i \varphi_{ik},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

On considering a typical term in φ_{i2},

$$c_i(t) x_1^{l_1} \cdots x_n^{l_n} \qquad (l_1 + \cdots + l_n = 2),$$

we find as the required conditions

$$d_i(t) + \frac{dc_i}{dt} + \left[l_1 \mu_1 + \cdots + (l_i - 1) \mu_i + \cdots + \mu_n \right] c_i = 0$$
$$(i = 1, \cdots, n)$$

where d_i is the like coefficient in F_{i2}. Here the coefficient λ of c_i is not zero, and it is immediately possible to solve for c_i,

$$c_i(t) = k_i e^{-\lambda t} - e^{-\lambda t} \int_0^t d(t) e^{\lambda t} \, dt.$$

This solution will be periodic in t of period τ if and only if

$$k_i (1 - e^{\lambda \tau}) = \int_0^\tau d_i(t) e^{\lambda t} \, dt.$$

It is possible to determine k_i in one and only one way so that this equation holds, provided that λ is not an integral multiple of $2\pi\sqrt{-1}/\tau$. But this relation would require a commensurability relation between the 'multipliers' $\mu_1, \cdots, \mu_n$ and $2\pi\sqrt{-1}/\tau$ of the excluded type.

Thus, as before, no difficulty arises in determining $\varphi_{i2}, \varphi_{i3}, \cdots$, in succession in such fashion that the desired normal form is obtained.

By means of a formal transformation

$$x_i = f_i(z_1, \cdots, z_n, t) = \sum_{j=1}^{n} l_{ij}(t) z_j + \frac{1}{2} \sum_{j,k=1}^{n} l_{ijk}(t) z_j z_k + \cdots \qquad (i = 1, \cdots, n)$$

with l_{ij} analytic in t and periodic of period τ, and $|l_{ij}(t)| \neq 0$, the differential equations (3) *with a generalized equilibrium point at the origin of general type can also be reduced to the normal form*

$$dz_i/dt = \mu_i z_i \qquad (i = 1, \cdots, n). \tag{10}$$

The corresponding formal solution of (3) *is then evidently*

$$y_i = f_i(c_1 e^{\mu_1 t}, \cdots, c_n e^{\mu_n t}, t) \qquad (i = 1, \cdots, n).$$

6. On the Hamiltonian multipliers. As a first step toward obtaining an analogous normal form for a Hamiltonian system of equations at an equilibrium point, we demonstrate some fundamental well-known properties of the multipliers in this case.*

Here the equations occur in the particular form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, \cdots, m) \tag{11}$$

where H is a real analytic function of $n = 2m$ variables $p_1, \cdots, q_m$. If these equations have an equilibrium point at

* Cf. Poincaré, *Les Méthodes nouvelles de la Mécanique céleste*, vol. 1, chap. 4. His 'characteristic exponents' are our multipliers.

the origin, then evidently all of the first partial derivatives of H vanish at the origin and if we ignore an additive constant in H we may write

$$H = H_2 + H_3 + \cdots,$$

where H_k is a homogeneous polynomial of degree k in these dependent variables and where in particular we have

$$H_2 = \sum_{j,k=1}^{m} \left(\frac{1}{2} a_{jk} p_j p_k + b_{jk} p_j q_k + \frac{1}{2} c_{jk} q_j q_k \right).$$

Here we may take $a_{ij} = a_{ji}$, $c_{ij} = c_{ji}$, but b_{ij} are in general distinct from b_{ji}.

The equations of variation are obtained by replacing H by H_2 and p_i, q_i by P_i, Q_i, and may be written in the explicit form

$$\begin{aligned} dP_i/dt &= -\sum_{j=1}^{m} b_{ji} P_j - \sum_{j=1}^{m} c_{ij} Q_j, \\ dQ_i/dt &= \sum_{j=1}^{m} a_{ij} P_j + \sum_{j=1}^{m} b_{ij} Q_j \quad (i = 1, \cdots, m) \end{aligned}$$

which is a particular type of system of $2m$ linear differential equations of the first order with constant coefficients.

Our first remark is merely to the effect that *in general* the multipliers will be distinct. To verify this fact, it is merely necessary to exhibit the $2m$ exponential solutions in a single special case. If we take

$$H_2 = \sum_{j=1}^{m} \frac{1}{2} \mu_j (p_j^2 + q_j^2),$$

the equations of variation reduce to

$$dP_i/dt = -\mu_i Q_i, \quad dQ_i/dt = \mu_i P_i \qquad (i = 1, \cdots, m)$$

with $2m$ particular solutions

$$P_i = \delta_{ik} e^{\pm \mu_k \sqrt{-1}\, t}, \qquad Q_i = \mp \delta_{ik} e^{\pm \mu_k \sqrt{-1}\, t}$$

where δ_{ij} has its usual significance. Hence the $2m$ values of the multipliers are $\pm\mu_k\sqrt{-1}$, $(k=1,\cdots,m)$, and these will be distinct if $\mu_1,\cdots,\mu_m$ are distinct positive numbers, for instance.

Before leaving the special case just cited we note that if H_2 is of this special form, we can introduce conjugate variables

$$\xi_i = p_i+\sqrt{-1}\,q_i,\quad \eta_i = p_i-\sqrt{-1}\,q_i \quad (i=1,\cdots,m),$$

and then we find

$$\frac{d\xi_i}{dt} = -\frac{\partial \bar{H}}{\partial \eta_i},\quad \frac{d\eta_i}{dt} = \frac{\partial \bar{H}}{\partial \xi_i} \quad (i=1,\cdots,m)$$

where $\bar{H} = -2\sqrt{-1}\,H$ and $\bar{H}_2$ takes the normal form

$$\bar{H}_2 = -\sum_{j=1}^{m} \mu_j \sqrt{-1}\,\xi_j\eta_j.$$

Consequently under this type of change of variables the Hamiltonian form of the differential equations is maintained. In this event the equations of variation are still simpler, namely of the form

$$d\xi_i/dt = \mu_i\sqrt{-1}\,\xi_i,\quad d\eta_i/dt = -\mu_i\sqrt{-1}\,\eta_i \quad (i=1,\cdots,m).$$

This type of conjugate variables plays an important role later on.

Let us suppose then that we are confronted by the general case in which the $2m$ multipliers are distinct. We propose to show that these quantities occur in m distinct pairs, each one the negative of the multiplier paired with it. This has already been seen above to be true in the special case cited.

Since the multipliers are distinct by hypothesis, a complete set of solutions

$$P_{1k},\cdots,P_{mk},\ Q_{1k},\cdots,Q_{mk} \quad (k=1,\cdots,2m),$$

exists of the form

$$P_{ik} = C_{ik}e^{\lambda_k t},\ Q_{ik} = D_{ik}e^{\lambda_k t} \quad (i=1,\cdots,m,\ k=1,\cdots,2m),$$

where the corresponding determinant of constants of order $2m$ formed from C_{ik}, D_{ik} is not 0. Such a complete set of particular solutions has the property that the most general solution is expressible as a linear combination of these particular ones.

But if P_i, Q_i and P_i^*, Q_i^* are any two solutions of the equations of variation we have

$$\sum_{j=1}^{m}(Q_j P_j^* - P_j Q_j^*) = \text{const.}$$

This fact is readily verified by differentiation with respect to t, and use of the equations of variation, when it is seen that the derivative of the left-hand side reduces identically to zero.

If then we substitute in this integral relation, pairs of the above particular solutions we find

$$\sum_{j=1}^{m}(D_{ik} C_{il} - C_{ik} D_{il})\, e^{(\lambda_k+\lambda_l)t} = \text{const.}$$

for all k and l. This clearly implies at once that either $\lambda_k + \lambda_l$ is 0 or that the constant on the right-hand side is 0. A proper use of this fact will lead us easily to the desired conclusion.

If each λ_i has a corresponding λ_j such that $\lambda_i + \lambda_j = 0$, then there is clearly only one such root and the property under consideration is proved. But in the contrary case some root as λ_k has no value so paired with it. Hence, in the integral relations deduced above, the right-hand members must vanish for $l = 1, \cdots, 2m$ if k has this value, whence we find

$$D_{1k} C_{1l} + \cdots + D_{mk} C_{ml} - C_{1k} D_{1l} - \cdots - C_{mk} D_{ml} = 0 \qquad (l = 1, \cdots, 2m).$$

These $2m$ equations are linear and homogeneous in $D_{1k}, \cdots, D_{mk}, -C_{1k}, \cdots, -C_{mk}$, so that the determinant of their coefficients would necessarily vanish. But this determinant is precisely the determinant of order $2m$ referred to above which cannot be zero. Hence there is no such root.

In general at an equilibrium point for the Hamiltonian equations the multipliers can be grouped in m pairs λ_i, $-\lambda_i$, $(i = 1, \cdots, m)$, and are all distinct.

It is plain that in general the multipliers are real or are grouped in conjugate imaginary pairs with distinct moduli. Consequently the conjugate of an imaginary multiplier must coincide with its negative. By passing to the special cases by a limiting process we conclude further:

The multipliers λ_i are either real or pure imaginary quantities.

We define the general equilibrium point of this type as the one in which $\lambda_1, \cdots, \lambda_m$ are not subject to any linear commensurability relation of the type (7), and confine attention to this general case.

7. Normalization of H_2. Assuming then that the equilibrium point of the Hamiltonian system under consideration is of this general type, we can effect a linear transformation of variables

$$p_i = \sum_{j=1}^{m} (d_{ij}\bar{p}_j + e_{ij}\bar{q}_j), \qquad q_i = \sum_{j=1}^{m} (f_{ij}\bar{p}_j + g_{ij}\,q_j) \qquad (i = 1, \cdots, m),$$

which reduces the corresponding equations of variation to the normal form

$$dP_i/dt = \lambda_i P_i, \quad dQ_i/dt = -\lambda_i Q_i \quad (i = 1, \cdots, m).$$

In fact this reduction (see section 4) merely required that the roots of the characteristic equation (6) be distinct, as is here the case. Of course the associated pairs $\bar{p}_i$, $\bar{q}_i$ are taken as corresponding to associated roots λ_i, $-\lambda_i$. If λ_i is real, $\bar{p}_i$, $\bar{q}_i$ are real variables. If λ_i is a pure imaginary, $\bar{p}_i$, $\bar{q}_i$ are conjugate variables.

We propose to demonstrate that this linear transformation does not destroy the Hamiltonian form of the equations.

To begin with we observe that the equations of variation can be written in the variational Hamiltonian form

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} P_i Q_i' - H_2(P_1, \cdots, Q_m) \right] dt = 0,$$

in which H is replaced by its second degree terms. Under the above linear change of variables this evidently takes the form

$$\delta \int_{t_0}^{t_1} \Bigg[\sum_{j,k=1}^{m} (K_{jk} P_j P'_k + L_{jk} Q_j P'_k + M_{jk} P_j Q'_k + N_{jk} Q_j Q'_k) - H_2 \Bigg] dt = 0,$$

where we may write

$$H_2 = \sum_{j,k=1}^{m} (R_{jk} P_j P_k + S_{jk} P_j Q_k + T_{jk} Q_j Q_k).$$

Here the dashes over the letters have been omitted, and we may obviously assume

$$R_{ij} = R_{ji}, \qquad T_{ij} = T_{ji} \qquad (i, j = 1, \cdots, m).$$

Applying the ordinary Lagrangian rule this gives the equations of variation in the new variables,

$$\frac{d}{dt}\left[\sum_{j=1}^{m} (K_{ji} P_j + L_{ji} Q_j)\right] - \sum_{j=1}^{m} (K_{ij} P'_j + M_{ij} Q'_j) + \sum_{j=1}^{m} (2 R_{ij} P_j + S_{ij} Q_j) = 0, \qquad (i = 1, \cdots, m),$$

$$\frac{d}{dt}\left[\sum_{j=1}^{m} (M_{ji} P_j + N_{ji} Q_j)\right] - \sum_{j=1}^{m} (L_{ij} P'_j + N_{ij} Q'_j) + \sum_{j=1}^{m} (S_{ji} P_j + 2 T_{ij} Q_j) = 0, \qquad (i = 1, \cdots, m).$$

But the solutions of these equations are known. In particular we have a solution

$$P_i = \delta_{ik} e^{\lambda_k t}, \qquad Q_i = 0 \qquad (i = 1, \cdots, m)$$

which when substituted in the first of the above equations gives at once

$$\lambda_k (K_{ki} - K_{ik}) + 2 R_{ik} = 0.$$

Interchanging i and k, and noting that $R_{ik} = R_{ki}$ we infer further for any i and k

$$(\lambda_i - \lambda_k)(K_{ki} - K_{ik}) = 0,$$

so that $K_{ki} = K_{ik}$ for i distinct from k as well as for $i = k$. It follows also that R_{ik} vanishes for all i and k.

Similarly we can infer that $N_{ki} = N_{ik}$ and that T_{ik} vanishes from the second set of equations. Thus the terms

$$\sum_{j,k=1}^{m} K_{jk} P_j P'_k, \qquad \sum_{k=1}^{m} M_{jk} Q_j Q'_k$$

are exact derivatives, and may be omitted under the integral sign. The equations of variation are thus of the more special form

$$\frac{d}{dt}\left[\sum_{j=1}^{m} L_{ji} Q_j\right] - \sum_{j=1}^{m} M_{ij} Q'_j + \sum_{j=1}^{m} S_{ij} Q_j = 0 \qquad (i = 1, \cdots, m),$$

$$\frac{d}{dt}\left[\sum_{j=1}^{m} M_{ji} P_j\right] - \sum_{j=1}^{m} L_{ij} P'_j + \sum_{j=1}^{m} S_{ji} P_j = 0 \qquad (i = 1, \cdots, m).$$

To determine these equations still more completely we substitute

$$P_i = 0, \qquad Q_i = \delta_{ik}\, e^{-\lambda_k t} \qquad (i = 1, \cdots, m)$$

in the first set of these equations, and obtain immediately for all i and k

$$\lambda_k (M_{ik} - L_{ki}) + S_{ik} = 0.$$

Similarly from the second set we obtain for all i and k,

$$\lambda_k (M_{ki} - L_{ik}) + S_{ki} = 0.$$

Interchanging i and k in this equation, and comparing the equation obtained with the preceding one we infer that for $i \neq k$ we have

$$M_{ik} = L_{ki}, \qquad S_{ik} = 0.$$

In consequence the sum

$$\sum_{j,k=1}^{m} (L_{jk} Q_j P'_k + M_{jk} P_j Q'_k)$$

differs from the sum

$$\sum_{j=1}^{m} L_{jj} Q_j P_j' + M_{jj} P_j Q_j',$$

and so from

$$\sum_{j=1}^{m} (M_{jj} - L_{jj}) P_j Q_j'$$

by an exact differential. Thus it is legitimate to write the principle of variation in the specific form

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} (M_{jj} - L_{jj}) P_j Q_j' + \sum_{j=1}^{m} S_{jj} P_j Q_j \right] dt = 0$$

in such new variables with equations of variation*

$$(M_{ii} - L_{ii}) Q_j' + S_{ii} Q_i = 0, \qquad (M_{ii} - L_{ii}) P_i' - S_{ii} P_i = 0 \qquad (i = 1, \cdots, m),$$

so that we have necessarily

$$(M_{ii} - L_{ii}) \lambda_i = S_{ii} \qquad (i = 1, \cdots, m).$$

Consider now the simplest case when every root λ_i is real. In this case if we replace the real variable P_i by

$$\overline{P}_i = (M_{ii} - L_{ii}) P_i \qquad (i = 1, \cdots, m)$$

the variational principle is seen to have the form

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} P_i Q_i' - \sum_{j=1}^{m} \lambda_j P_j Q_j \right] dt = 0.$$

This further change of variables is legitimate inasmuch as p_i, q_i were not determined up to real multipliers in this case. It appears then that the term

$$\sum_{j=1}^{m} P_j Q_j'$$

* The constants $M_{ii} - L_{ii}$ are not zero since the equations of variation do not degenerate.

remains of essentially the same form after the linear transformation.

Another case is that in which every λ_i is a pure imaginary quantity. Here by taking the pairs p_i, q_i appropriately we can clearly write the pure imaginary quantities $M_{ii} - L_{ii}$ as $\varrho_i \sqrt{-1}$, $\varrho_i > 0$. Here we may replace P_i, Q_i by

$$\bar{P}_i = \sqrt{\varrho_i}\, P_i, \qquad \bar{Q}_i = \sqrt{\varrho_i}\, Q_i \qquad (i = 1, \cdots, m)$$

when a like variational form is obtained.

It is apparent that this same linear change of variables must preserve the original Hamiltonian form since

$$\sum_{j=1}^{m} p_j q_j'$$

is essentially unaltered by this transformation.

By a suitable preliminary linear transformation with constant coefficients any Hamiltonian system with equilibrium point of general type at the origin may be taken in a normalized form in which

$$H_2 = \sum_{j=1}^{m} \lambda_j p_j q_j.$$

8. The Hamiltonian equilibrium problem. In order to further normalize the Hamiltonian equations in the vicinity of an equilibrium point, we propose to apply a series of transformations

$$p_i = \partial K / \partial q_i, \qquad \bar{q}_i = \partial K / \partial \bar{p}_i \qquad (i = 1, \cdots, m)$$

with

$$K = \sum_{j=1}^{m} \bar{p}_j q_j + K_3 + K_4 + \cdots$$

where K_3, K_4, $\cdots$ are homogeneous functions of $\bar{p}_i$, q_i, $(i = 1, \cdots, m)$ of degree indicated by the subscripts. Such transformations have been seen to leave the Hamiltonian form undisturbed and to form a group. It will be observed that if $K_s = 0$ for $s > 2$, the transformation is the identity.

We begin by taking $K_s = 0$, $s > 3$, and by attempting to select K_3 so as to simplify H_3 as far as possible. Here we have

$$p_i = \bar{p}_i + \partial K_3/\partial q_i, \qquad \bar{q}_i = q_i + \partial K_3/\partial p_i \qquad (i = 1, \cdots, m).$$

If we solve explicitly for p_i, q_i, $(i = 1, \cdots, m)$ in terms of $\bar{p}_i, \bar{q}_i$, $(i = 1, \cdots, m)$, we will clearly obtain

$$p_i = \bar{p}_i + \partial K_3^*/\partial \bar{q}_i + \cdots, \qquad q_i = \bar{q}_i - \partial K_3^*/\partial \bar{p}_i + \cdots \qquad (i = 1, \cdots, m),$$

where K_3^* denotes the function obtained by replacing q_i by $\bar{q}_i$ in K_3. The terms explicitly written give the series expansion up to the terms of the third degree. The modified value of H, obtained by direct substitution is

$$H_2(\bar{p}_1 + \partial K_3^*/\partial \bar{q}_1 + \cdots, \cdots \bar{q}_m - \partial K_3^*/\partial p_m + \cdots) + H_3 + \cdots,$$

where the arguments of $H_3, H_4, \cdots$ are the same as those of H_2. To terms of the third degree inclusive we find then

$$\bar{H} = \sum_{j=1}^{m} \lambda_j \bar{p}_j \bar{q}_j + \sum_{j=1}^{m} \lambda_j \left(\bar{q}_j \frac{\partial K_3^*}{\partial \bar{q}_j} - \bar{p}_j \frac{\partial K_3^*}{\partial \bar{p}_j}\right) + H_3(\bar{p}_1, \cdots, \bar{q}_m) + \cdots.$$

Thus, as would be expected, the form of H_2 is unmodified while H_3 takes the form

$$\sum_{j=1}^{m} \lambda_j \left(q_j \frac{\partial K_3}{\partial q_j} - p_j \frac{\partial K_3}{\partial p_j}\right) + H_3(p_1, \cdots, q_m)$$

in which K_3 is at our disposal. Now any term in K_3 may be written

$$c p_1^{\alpha_1} \cdots p_m^{\alpha_m} q_1^{\beta_1} \cdots q_m^{\beta_m} \qquad (\alpha_1 + \cdots + \beta_m = 3).$$

The corresponding term in the modified H_3 has a coefficient

$$c[\lambda_1(\beta_1 - \alpha_1) + \cdots + \lambda_m(\beta_m - \alpha_m)] + h$$

in which h is the coefficient analogous to c in the original H_3. Moreover the coefficient of c cannot vanish unless

$$\beta_1 = \alpha_1, \cdots, \beta_m = \alpha_m,$$

which is clearly not possible inasmuch as the sum of all the α_i's and β_i's is 2. Thus by proper choice of each c we can make the new H_3 vanish.

If now we proceed to try to eliminate H_4 as far as possible by a further transformation of the same type in which $K_s = 0$ except for $s = 4$ we obtain a transformation

$$p_i = \bar{p}_i + \partial K_4^* / \partial \bar{q}_i + \cdots, \qquad q_i = \bar{q}_i - \partial K_4^* / \partial \bar{p}_i + \cdots \qquad (i = 1, \cdots, m)$$

which does not affect H_2 or $H_3 \equiv 0$, but alters H_4 to

$$\sum_{j=1}^{m} \lambda_j \left(q_j \frac{\partial K_4}{\partial q_j} - p_j \frac{\partial K_4}{\partial p_j} \right) + H_4(p_1, \cdots, q_m).$$

Here we can eliminate the terms of H_4 save those which contain each p_i, q_i to the same degree, namely those of the forms

$$c(p_i q_i)^2, \quad d p_i q_i p_j q_j \qquad (i, j = 1, \cdots, m),$$

by the same method. For we have

$$\alpha_1 + \cdots + \beta_m = 4$$

in this case, and all terms can be made to disappear except those for which $\alpha_i = \beta_i$, $(i = 1, \cdots, m)$, i. e., those of the stated type.

Thus it is readily seen that by an infinite series of steps we can eliminate from H all terms except the terms in the m products $p_i q_i$.

By suitable transformations of the above types, a Hamiltonian system with equilibrium point of general type at the origin may be taken into a normal Hamiltonian form in which *only the m products $p_1 q_1, \cdots, p_m q_m$ appear in H while H_2 has the special form*

$$\sum_{j=1}^{m} \lambda_j p_j q_j.$$

It may be noted that the linear transformation employed is also a contact transformation* so that in reality this normal form may be obtained through a single formal contact transformation.

In this normal form the general formal solution is at once obtainable. If we write $\pi_i = p_i q_i$, the normal Hamiltonian equations may be written

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial \pi_i} p_i, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial \pi_i} q_i \quad (i = 1, \cdots, m),$$

whence we find formally

$$p_i q_i = c_i \qquad (i = 1, \cdots, m).$$

Thus the series $\partial H/\partial \pi_i$ reduce to constants and we are led to the following conclusion in a purely formal manner:

The general formal solution of the normal Hamiltonian equations near such an equilibrium point has the form

$$p_i = \alpha_i e^{-\gamma_i t}, \qquad q_i = \beta_i e^{\gamma_i t} \quad (i = 1, \cdots, m),$$

where

$$\gamma_i = \partial H(\alpha_1 \beta_1, \cdots, \alpha_m \beta_m)/\partial \pi_i \quad (i = 1, \cdots, m).$$

In terms of the original variables the corresponding solution may be obtained with the aid of the contact transformation relating the given variables and the normal variables.

9. Generalization of the Hamiltonian problem. If

$$p_i = \varphi_i(t), \qquad q_i = \psi_i(t) \quad (i = 1, \cdots, m)$$

is a periodic motion of period τ, and if we write

$$p_i = \varphi_i + \bar{p}_i, \qquad q_i = \psi_i + \bar{q}_i \quad (i = 1, \cdots, m)$$

* See Whittaker, *Analytical Dynamics*, chap. 16.

then the differential equations for the modified variables are of Hamiltonian form with modified principal function

$$\bar{H} = H + \sum_{j=1}^{m} (\varphi_j' \bar{q}_j - \psi_j' \bar{p}_j)$$

where the accents denote differentiation. Furthermore, this transformation may be written as a contact transformation

$$p_i = \partial K / \partial q_i, \qquad \bar{q}_i = \partial K / \partial \bar{p}_i \quad (i = 1, \cdots, m)$$

with

$$K = \sum_{j=1}^{m} (\bar{p}_j q_j + \varphi_j q_j - \psi_j \bar{p}_j).$$

In these new variables H is a function of $\bar{p}_1, \cdots, \bar{q}_m$ and t, periodic of period τ in the last variable, with $\bar{p}_i = \bar{q}_i = 0$, $(i = 1, \cdots, m)$, a solution corresponding to the given periodic motion. Thus, at least in a formal sense (see chapter IV, section 1), the problem reduces to one of generalized equilibrium.

It will be our aim here to show that a reduction to normal form for such a generalized equilibrium problem can be made which is altogether analogous to that made above in the case of ordinary equilibrium.

We shall merely call attention to the modifications necessary in the argument in dealing with this more general problem.

The first difference to which attention needs to be called is to the obvious fact that in the equations of variation the constants a_{ij}, b_{ij}, c_{ij} are replaced by periodic functions of t with period τ. The second difference is that the constants C_{ij}, D_{ij} which appeared in the solutions of these equations are also such periodic functions.

These modifications do not, however, interfere with the argument made that the multipliers may be grouped in m pairs

$$\lambda_1, \; -\lambda_1, \cdots, \lambda_m, \; -\lambda_m$$

2 where $\lambda_1, \cdots, \lambda_m$ are real or pure imaginary.

The general equilibrium point may here be appropriately defined as that in which there are no linear commensurability relations between the $m+1$ quantities $\lambda_1, \cdots, \lambda_m$ and $2\pi\sqrt{-1}/\tau$.

In the corresponding linear transformation the coefficients $d_{ij}, e_{ij}, f_{ij}, g_{ij}$ are periodic in t of period τ. Likewise in the variational principle the quantities $K_{ij}, L_{ij}, M_{ij}, N_{ij}, R_{ij}, S_{ij}, T_{ij}$ are similar functions. In determining the form of these functions one finds modified conditions such as that

$$\lambda_k(K_{ki}-K_{ik})+\frac{dK_{ki}}{dt}+2R_{ik}=0.$$

If i and k be interchanged and the results subtracted, there is obtained

$$\frac{d}{dt}(K_{ki}-K_{ik})+(\lambda_k-\lambda_i)(K_{ki}-K_{ik})=0.$$

This differential equation in $(K_{ki}-K_{ik})$ has no periodic solution of period τ (other than 0), just because $\lambda_1, \cdots, \lambda_m$ are of the general type assumed. Hence we infer as before that K_{ik}, K_{ki} are equal while $2R_{ik}$ is $-dK_{ik}/dt$.

But in this case

$$\sum_{j,k=1}^{m} K_{jk}\, p_j\, p_k'$$

is an exact differential if

$$\frac{1}{2}\sum_{j,k=1}^{m}\frac{dK_{jk}}{dt}p_j p_k$$

be added while the negative of this last expression may be incorporated in H. Thus it is clear that we may assume

$$K_{ij}=N_{ij}=R_{ij}=T_{ij}=0$$

as before. In fact similar slight modifications show that the same normal form for H_2 is obtained by this linear transformation in the generalized equilibrium problem as in the ordinary equilibrium problem.

To make clear that the analogy is complete in dealing with $H_3, H_4, \cdots$, let us consider the new H_3 obtained by a transformation

$$p_i = \bar{p}_i + \partial K_3 / \partial q_i, \qquad \bar{q}_i = q_i + \partial K_3 / \partial \bar{p}_i \quad (i = 1, \cdots, m),$$

where K_3 is homogeneous of the third degree in $\bar{p}_i$, q_i, $(i = 1, \cdots, m)$ with coefficients which are periodic in t of period τ. The new form of H_3 is

$$\frac{\partial K_3^*}{\partial t} + \sum_{j=1}^{m} \lambda_j \left(q_j \frac{\partial K_3^*}{\partial q_j} - p_j \frac{\partial K_3^*}{\partial p_j} \right) + H_3(p_1, \cdots, q_m).$$

The terms

$$c(t) p_1^{\alpha_1} \cdots q_m^{\beta_m}, \qquad h(t) p_1^{\alpha_1} \cdots q_m^{\beta_m} \qquad (\alpha_1 + \cdots + \beta_m = 2)$$

in K_3^* and H_2 respectively lead to a total corresponding equation

$$\frac{dc}{dt} + c[\lambda_1(\beta_1 - \alpha_1) + \cdots + \lambda_m(\beta_m - \alpha_m)] + h = 0,$$

in the attempt to eliminate such a term. This ordinary non-homogeneous linear equation of the first order will have one and only one periodic solution inasmuch as the coefficient of c is incommensurable with $2\pi\sqrt{-1}/\tau$, since we are in the general case. Thus H_3 can be made to disappear.

Likewise all the terms of H_4 can be made to disappear save those in the products $p_i q_i$, $(i = 1, \cdots, m)$. The coefficients in these latter terms can be replaced by constants however; in fact this demand leads to an equation of the form

$$\frac{dc}{dt} + h(t) = C$$

where C is an arbitrary constant at our disposal; thus we find

$$c = \int (C - h(t))\, dt,$$

which is obviously periodic of period τ if C is chosen as the mean value of $h(t)$ over a period. Hence we are led to the same conclusion as before.

By means of such a series of transformations of the generalized Hamiltonian equilibrium problem, the Hamiltonian function *may be given the same normal form as was obtained in the case of ordinary equilibrium.**

10. On the Pfaffian multipliers. Suppose now that we take an extended Pfaffian variation problem

$$(12)\quad \delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{2m} X_j(x_1, \cdots, x_{2m}) x_j' + Z(x_1, \cdots, x_{2m}) \right] dt = 0$$

which leads at once to the system of ordinary equations of order $2m$

$$(13)\quad \sum_{j=1}^{2m} \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{d x_j}{d t} - \frac{\partial Z}{\partial x_i} = 0 \quad (i = 1, \cdots, 2m).$$

We propose to consider these equations in the case when there is an equilibrium point at the origin, under the assumption that the $2m$ analytic functions X_i are such that the skew-symmetric determinant

$$\left| \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right|$$

is not 0 at the origin. The constant terms in the series for the functions X_i may obviously be omitted throughout.

It is clear that the Hamiltonian equations appear as a special case of these Pfaffian equations (12).

As will be shown in the following chapter, this generalization of the Hamiltonian equations possesses the same property of automatically fulfilling all of the conditions for complete stability, once the obvious conditions for first order stability are satisfied. Hence from this point of view the Pfaffian equations seem as significant for dynamics as the Hamiltonian equations, although more general in type. Moreover they possess the additional advantage of maintaining their Pfaffian form under an arbitrary transformation of the formal group.

* The results of this chapter were announced in my Chicago Colloquium lectures of 1920. The material so far given is obviously in close relation with previous work, and, in particular, the normal form in the Hamiltonian case is in relation with the formal trigonometric series in dynamics treated for instance in Whittaker, *Analytical Dynamics*, chap. 15.

In fact it is only necessary to substitute the new variables under the integral sign in (12) to obtain the transformed functions X_i and Z.

As a first step in the direction of obtaining a normal form for the Pfaffian equations at an equilibrium point, we propose to prove that for these equations just as for the Hamiltonian equations the multipliers are associated in pairs $\lambda_i, -\lambda_i$.

To begin with we observe that in general these roots must be distinct since they are distinct in the Hamiltonian sub-case.

Now let us make the linear transformation with constant coefficients which takes the equations of variation into normal form. This does not affect the Pfaffian form of course. The corresponding equations of variation obtained from (13) are

$$\sum_{j=1}^{2m}\left[\left(\frac{\partial X_i}{\partial x_j}-\frac{\partial X_j}{\partial x_i}\right)\frac{dy_j}{dt}-\frac{\partial^2 Z}{\partial x_i\,\partial x_j}y_j\right]=0$$

and these must have the particular solutions

$$y_i = \delta_{ik}\,e^{\lambda_k t} \qquad (i = 1, \cdots, 2m)$$

for $k = 1, \cdots, 2m$. It is understood that the partial derivatives involved in the equations of variation are evaluated at the origin.

Substituting in these particular solutions we obtain readily

$$\left(\frac{\partial X_i}{\partial x_k}-\frac{\partial X_k}{\partial x_i}\right)\lambda_k-\frac{\partial^2 Z}{\partial x_i\,\partial x_k}=0.$$

If we interchange i and k here, and subtract the equation so obtained from the one last written, we find

$$\left(\frac{\partial X_i}{\partial x_k}-\frac{\partial X_k}{\partial x_i}\right)(\lambda_k+\lambda_i)=0 \qquad (i, k = 1, \cdots, m).$$

But if for each k we do not have $\lambda_k+\lambda_i = 0$ for some i, it would follow from these equations that the skew-symmetric determinant specified above would necessarily vanish. This is impossible since the equations of variation would then degenerate. Hence there is associated necessarily with the root λ a second root $\lambda_i = -\lambda_k$. This is precisely what we desired to prove.

In the Pfaffian equilibrium problem the multipliers also occur in pairs, one of each pair being the negative of the other. These multipliers may be designated by $\lambda_1, -\lambda_1, \cdots, \lambda_m, -\lambda_m$, and will be real or pure imaginary quantities. 3

It is clear that the general case is to be defined as that in which there are no linear relations of commensurability between $\lambda_1, \cdots, \lambda_m$ just as was done in the special Hamiltonian case. We shall restrict ourselves to this general case.

11. Preliminary normalization in Pfaffian problem. It is very easy to establish the fact that the normalization used in the preceding section makes the first degree terms in $X_1, \cdots, X_{2m}$ take essentially the Hamiltonian form. Indeed if we call the $2m$ dependent variables $p_1, \cdots, p_m, q_1, \cdots, q_m$, in such wise that p_i, q_i correspond to paired multipliers $\lambda_i, -\lambda_i$ and if we let $P_1, \cdots, Q_m$ denote the coefficients of $p_1', \cdots, q_m'$ respectively under the integral sign in (12), the previously obtained equations between the partial derivatives $\partial X_i/\partial x_j$ at the origin take the form

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial P_j}{\partial p_i}, \quad \frac{\partial Q_i}{\partial q_j} = \frac{\partial Q_j}{\partial q_i} \qquad (i, j = 1, \cdots, m),$$

$$\frac{\partial P_i}{\partial q_j} = \frac{\partial Q_i}{\partial p_j} \qquad (i, j = 1, \cdots, m;\ i \neq j).$$

The first sets of equations show that the linear terms in P_i involving $p_1, \cdots, p_m$ correspond to an exact differential, as do the linear terms of Q_i in $q_1, \cdots, q_m$. Similarly the second last set of equations shows that the term of P_i involving q_j $(i \neq j)$ together with the corresponding term of Q_j involving p_i combine in a like manner. All of these terms may be omitted, and there remains for consideration only terms

$$\sum_{j=1}^{m} (c_j p_j \, dq_j + d_j q_j \, dp_j),$$

which evidently may be replaced by

$$\sum_{j=1}^{m} (c_j - d_j)\, p_j\, dq_j.$$

In these terms no $c_i - d_i$ can vanish, because of the hypothesis that the fundamental skew-symmetric determinant does not vanish at the origin.

If then p_i, q_i are real variables we may make the further linear transformation

$$\bar{p}_i = p_i, \quad \bar{q}_i = (c_i - d_i)\, q_i \quad (i = 1, \cdots, m)$$

to obtain the desired linear Hamiltonian term. On the other hand if p_i, q_i are conjugate variables then c_i and d_i are conjugate imaginaries and $c_i - d_i$ is a pure imaginary quantity $\varrho\sqrt{-1}$. Here we may set

$$\bar{p}_i = \sqrt{\varrho}\, p_i, \quad \bar{q}_i = \sqrt{\varrho}\, q_i$$

if $\varrho > 0$. If $\varrho < 0$, we may interchange the roles of p_i and q_i.

Let us turn next to consider the function Z. Since we have an equilibrium point at the origin, it is plain that $\partial Z/\partial p_i$, $\partial Z/\partial q_i$, vanish there for $i = 1, 2, \cdots, m$, i. e., that there are no linear terms in Z. The lowest terms in Z are then of the second degree.

Thus it is apparent that the equations of variation, which depend only on these first degree terms in $X_1, \cdots, X_{2m}$, and upon the second degree terms in Z, are of the same type as in the Hamiltonian case. In consequence the same linear transformation employed to obtain a normal form for these lowest degree terms gives $-Z_2$ the form of H_2 in the Hamiltonian case.

We can summarize our results as follows:

By a preliminary linear transformation the Pfaffian equations with an equilibrium point of general type at the origin can be written

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} (P_j q_j' + Q_j p_j') - R \right] dt = 0$$

where

$$P_i = p_i + P_{i2} + \cdots, \quad Q_i = * + Q_{i2} + \cdots \quad (i = 1, \cdots, m),$$

$$R = \sum_{j=1}^{m} \lambda_j p_j q_j + R_3 + \cdots.$$

12. The Pfaffian equilibrium problem. After this preparatory work it is a simple matter to establish the general result, which is that by means of point transformations (independent of t), it is possible to reduce the Pfaffian type of equations to Hamiltonian form. More precisely we propose to show that it is possible to reduce Q_i, $(i = 1, \cdots, m)$ to 0 by a suitable succession of such transformations without interfering with the normal form of P_i. When this has been accomplished, it is merely necessary to write

$$\overline{p}_i = P_i, \quad \overline{q}_i = q_i \qquad (i = 1, \cdots, m)$$

to obtain complete Hamiltonian form in the case when $p_1, \cdots, q_m$ are real variables. A slight modification is necessary in case the variables p_i, q_i are not all real.

In the real case suppose that we write

$$p_i = \overline{p}_i, \qquad q_i = \overline{q}_i + \overline{G}_{i2}(\overline{p}_1, \cdots, \overline{q}_m) \qquad (i = 1, \cdots, m),$$

where, according to our usual notation, G_{i2} is a homogeneous quadratic polynomial in its arguments. The variational principle takes a corresponding form in which the new coefficient P_i has also an initial first degree term p_i as desired, while for the new Q_i we find readily expressions in series

$$* + Q_{i2} + \sum_{j=1}^{m} p_j \frac{\partial G_{j2}}{\partial p_i} + \cdots \qquad (i = 1, \cdots, m).$$

Here the linear terms are lacking, as desired, and only the quadratic terms are written explicitly.

Now we have

$$d\left(\sum_{j=1}^{m} p_j G_{j2}\right) = \sum_{j,k=1}^{m} \left(p_j \frac{\partial G_{j2}}{\partial p_k} + G_{k2}\right) dp_k$$
$$+ \sum_{j,k=1}^{m} \left(p_j \frac{\partial G_{j2}}{\partial q_k} dq_k\right).$$

This identity shows that, by subtracting an exact differential under the integral sign, we may modify the new Q_{i2} to the form $Q_{i2}-G_{i2}$ without introducing any first degree terms iñ P_i. Hence if we take $G_{i2}=Q_{i2}, (i=1,\cdots,m)$. the third degree terms in Q_{i2} will have been eliminated.

Next by a further transformation

$$p_i=\bar{p}_i, \qquad q_i=\bar{q}_i+\bar{G}_{i3} \quad (i=1,\cdots,m)$$

we can similarly eliminate the third degree term in Q_i. Proceding thus indefinitely we arrive at a variational form

$$\delta\int_{t_0}^{t_1}\left[\sum_{j=1}^{m}P_j q_j'+*-R\right]dt=0$$

where

$$P_i=p_i+P_{i2}+\cdots \qquad (i=1,\cdots,m),$$

which can evidently be given Hamiltonian form in the manner indicated.

In case some of the pairs of variables p_i, q_i are conjugate imaginaries we may first perform the simple linear transformations to corresponding real variables

$$p_i=(\bar{p}_i+\bar{q}_i\sqrt{-1})/\sqrt{2}, \qquad q_i=(\bar{p}_i-\bar{q}_i\sqrt{-1})/\sqrt{2}$$

so that the term $p_i q_i'$ is replaced by $\bar{p}_i\bar{q}_i'$ except for an exact differential. Thus the normal form for P_i, Q_i is maintained for these real variables. Operating with them as indicated we can reach the same conclusion in this case also.

By a suitable transformation of the formal group, say

$$p_i=\varphi_i(\bar{p}_1,\cdots,\bar{q}_m), \quad q_i=\psi_i(\bar{p}_1,\cdots,\bar{q}_m) \quad (i=1,\cdots,m),$$

the general Pfaffian equilibrium problem may be made to assume Hamiltionian form.

13. Generalization of the Pfaffian problem. Under the above circumstances it is natural to expect that Pfaffian equations containing the time t, with generalized equilibrium point at the origin, admit of formal reduction to Hamiltonian form. It is not difficult to establish the truth of this conjecture on the basis of certain slight modifications of the above discussion.

In the case of such equilibrium the equations are defined by

$$(14)\quad \delta \int_{t_0}^{t_1}\left[\sum_{j=1}^{2m} X_j(x_1, \cdots, x_{2m}, t)\, x_j' + Z(x_1, \cdots, x_{2m}, t)\right] dt = 0$$

with X_i, $(i = 1, \cdots, 2m)$, and Z periodic in t, that is by

$$(15)\quad \sum_{j=1}^{2m}\left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt} + \frac{\partial X_i}{\partial t} - \frac{\partial Z}{\partial x_i} = 0 \qquad (i = 1, \cdots, 2m).$$

In the first place it is obvious that the multipliers are in general distinct by consideration of the same special case as was taken up in the ordinary equilibrium problem.

Furthermore, the equations of variation may again be normalized by a linear transformation in which the coefficients involved are periodic analytic functions of t of period τ, so as to have the solutions

$$y_i = \delta_{ik}\, e^{\lambda_i t} \qquad (i = 1, \cdots, 2m)$$

for $k = 1, \cdots, 2m$. It follows that the multipliers λ_i occur in pairs, each one of a pair being the negative of the other, by essentially the same argument as was used in the equilibrium problem.

Moreover the same argument shows that the linear terms in P_i, Q_i lead to certain exact differentials and terms which may be absorbed in R, so that the same normal form for the first order terms in P_i, Q_i and for the second order terms in R is obtained as before.

Finally as in section 11 we write

$$p_i = \bar{p}_i, \quad q_i = \bar{q}_i + \bar{G}_{i2} \quad (i = 1, \cdots, m)$$

where now $\overline{G}_{i2}$ has coefficients which are periodic in t of period τ.

Then by an obvious modification of the argument there made, we can make $Q_{i2} = 0$ for $i = 1, \cdots, m$, and then in succession $Q_{i3} = 0, \cdots$.

By a suitable transformation of the formal group the generalized Pfaffian problem of periodic motion may be made to assume Hamiltonian form. Hence the normal form in the Hamiltonian case serves also in the Pfaffian case.