CHAPTER I

PHYSICAL ASPECTS OF DYNAMICAL SYSTEMS

1. Introductory remarks. In dynamics we deal with physical systems whose state at a time t is completely specified by the values of n real variables

$$x_1, x_2, \cdots, x_n.$$

Accordingly the system is such that the rates of change of these variables, namely

$$dx_1/dt, \; dx_2/dt, \cdots, dx_n/dt,$$

merely depend upon the values of the variables themselves, so that the laws of motion can be expressed by means of n differential equations of the first order

$$(1) \qquad dx_i/dt = X_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n).$$

Thus, for a particle which falls in a vacuum at the surface of the earth, x_1 and x_2 may denote distance fallen and velocity respectively. In this case the equations of motion take the typical form

$$dx_1/dt = x_2, \qquad dx_2/dt = g,$$

where g is the gravitational acceleration.

2. An existence theorem. We proceed first to formulate an existence theorem for a set of differential equations of the general type (1).* The set of n functions X_i will be assumed to be real and uniformly continuous in some open

* In connection with the first five paragraphs the following general references may be given: E. Picard, *Traité d'Analyse*, vol. 2, chap. 11, and vol. 3, chap. 8; E. Goursat, *Cours d'Analyse mathématique*, vol. 2, chap. 19; G. A. Bliss, *Princeton Colloquium Lectures*, chap. 3.

finite n dimensional continuum R in the 'space' with rectangular coördinates $x_1, \cdots, x_n$. A 'solution' $x(t)$ of the equations (1) in the open interval $t' < t < t''$ is defined to be a set of n functions $x_i(t)$, all continuous together with their first derivatives and represented for any such t by a point x in R, such that the differential equations are satisfied by this set of functions.

EXISTENCE THEOREM. *If the point x^0 is in R at a distance at least D from the boundary of R, and if M is an upper bound for the functions $|X_i|$ in R, there exists a solution $x(t)$ of the equations* (1), *defined in the interval*

$$|t - t_0| < D/(\sqrt{n}\, M)$$

and for which $x(t_0) = x^0$.

To establish this theorem, we observe first that, for any solution of the type sought, the n equations

$$S_i \equiv x_i - x_i^0 - \int_{t_0}^{t} X_i(x_1, \cdots, x_n)\, dt = 0 \tag{2}$$

hold. Conversely, any set of continuous functions $x(t)$ in R, which make the expressions S_i vanish in an interval containing $t = t_0$ as an interior point, will obviously reduce to x^0 for $t = t_0$, and will satisfy the differential equations in question, as follows by direct differentiation.

Now define the set of infinitely multiple-valued functions $X_i^m(x_1, \cdots, x_n)$ as that given by *any* set $X_i(y_1, \cdots, y_n)$ taken at a point y whose various coördinates differ from those of the point x by not more than $1/m$ in numerical value. It is evident that with this definition the n components of X^m may be chosen as constant in any rectangular domain

$$|x_i - a_i| \leqq 1/m \qquad (i = 1, \cdots, n),$$

namely as the component parts of $X(a_1, \cdots, a_n)$.

If the functions X_i be replaced by X_i^m and the functions x_i by x_i^m, the expressions for S_i become

$$S_i^m \equiv x_i^m - x_i^0 - \int_{t_0}^{t} X_i^m(x_1^m, \cdots, x_n^m)\, dt.$$

We propose to show that these expressions can be made to vanish.

Choose X^m as $X(x_1^0, \cdots, x_n^0)$ in the rectangular domain

$$|x_i - x_i^0| < 1/m \qquad (i = 1, \cdots, n).$$

The integrals in the above expressions for S_i^m will then be linear functions of t, and hence x_i^m may be defined as

$$x_i^0 + X_i(x_1^0, \cdots, x_n^0)(t - t_0)$$

as long as the point x^m continues to be in this domain. In geometrical terms, the expressions for $x_i^m(t)$ yield the coördinates of a straight line with t as parameter, which passes through the center of the domain for $t = t_0$. If the n functions X_i^m happen to vanish, the line reduces to the point x^0.

In case the line emerges from the domain for $t = t_1 > t_0$ at a point y^0, we can take this point as the center of a second like rectangular domain of the same dimensions, and take x_i^m as

$$y_i^0 + X_i^m(y_1^0, \cdots, y_n^0)(t - t_1)$$

in this second domain. The expressions S_i^m will then continue to vanish for $t \geqq t_1$ until the point x^m leaves this second domain at a point z^0.

Thus, by a succession of steps, the expressions S_i^m can be made to vanish for $t > t_0$ and likewise for $t < t_0$. The process can only terminate in case the broken line representing $x^m(t)$ passes a boundary point of R.

Now, if t be taken as the time and x_i^m as the n coördinates of a particle, its velocity

$$[(X_1^m)^2 + \cdots + (X_n^m)^2]^{1/2}$$

is clearly not more than $\sqrt{n}\, M$. Hence the particle must remain inside of R at least in the interval

$$|t - t_0| < D/(\sqrt{n}\, M).$$

All of the functions x_i^m are defined in this fixed t interval whatever be the value of m.

As m takes on the values $1, 2, 3, \cdots,$ there arises an infinite sequence of sets $x_i^m(t)$ of functions defined in this interval. All of these sets lie in R, and so are uniformly bounded. Furthermore, since the S_i^m vanish for all i and m, the inequality

$$|x_i^m(t+h) - x_i^m(t)| = \left| \int_t^{t+h} X_i^m(x_1^m, \cdots, x_n^m)\, dt \right| \leqq Mh$$

obtains. Hence, by a special case of a well known theorem due to Ascoli,* there exists an infinite sequence of values of m for which every element of the set x_i^m approaches a function $\overline{x}_i$ of the set $\overline{x}$ uniformly, these functions being themselves continuous.

It is easy to prove that the functions $\overline{x}_i$ so obtained satisfy the integral form (2) of the differential equations. In fact, since the S_i^m vanish for every i and m, we have

$$\begin{aligned} \overline{S}_i &= \overline{S}_i - S_i^m \\ &= (\overline{x}_i - x_i^m) - \int_{t_0}^t [X_i(\overline{x}_1, \cdots, \overline{x}_n) - X_i^m(x_1^m, \cdots, x_n^m)]\, dt. \end{aligned}$$

For m sufficiently large, the first term on the right becomes uniformly small inasmuch as each $\overline{x}_i$ is approached uniformly by the corresponding x_i^m over the sequence under consideration. Also $X_i(\overline{x}_1, \cdots, \overline{x}_n)$ will differ from $X_i(x_1^m, \cdots, x_n^m)$ for any i by a uniformly small quantity, since X_i is uniformly continuous in R by hypothesis; and $X_i(x_1^m, \cdots, x_n^m)$ in turn will differ from $X_i^m(x_1^m, \cdots, x_n^m)$ by a uniformly small quantity, in virtue of the definition of the functions X_i^m. Hence the quantity under the integral sign on the right also becomes uniformly small as m increases and the expressions $\overline{S}_i$, which are independent of m, must vanish as stated, so that $\overline{x}(t)$ yields the required solution of (1).

By repeated use of the existence theorem, the given solution $x(t)$ may be extended beyond its interval of definition unless

* For a brief statement and proof see W. F. Osgood, Annals of Mathematics, vol. 14, series 2, pp. 152–153.

as t approaches either end of the interval, the corresponding point $x(t)$ approaches the boundary of R. Hence we infer the truth of the following statement:

COROLLARY. *The interval of definition for any solution $x(t)$ of the equations* (1) *may be extended so as to take one of the following four forms*:

$$-\infty < t < +\infty; \quad -\infty < t < t''; \quad t' < t < +\infty; \quad t' < t < t'',$$

where, as t approaches t' or t'', the point x approaches the boundary of R.

3. A uniqueness theorem. It may now be proved that there is only one solution of the type described in the existence theorem, in case the functions X_i possess continuous first partial derivatives. This last requirement may be lightened to a well known form given by Lipschitz.

UNIQUENESS THEOREM. *If for every i and for every pair of points x, y in R the functions X_i satisfy a Lipschitz condition,*

$$|X_i(x_1, \cdots, x_n) - X_i(y_1, \cdots, y_n)| \leqq \sum_{j=1}^{n} L_j |x_j - y_j|,$$

the quantities $L_1, \cdots, L_n$ being fixed positive quantities, then there is only one solution $x(t)$ of (1) *such that $x(t_0) = x^0$.*

For if two distinct solutions $x(t)$ and $y(t)$ have the same values x^0 for $t = t_0$, the corresponding integral forms of the differential equations give at once

$$x_i - y_i - \int_{t_0}^{t} [X_i(x_1, \cdots, x_n) - X_i(y_1, \cdots, y_n)]\, dt = 0$$

for all values of i, and thence by the Lipschitz condition imposed,

$$|x_i - y_i| \leqq \int_{t_0}^{t} \sum_{j=1}^{n} L_j |x_j - y_j| \, |dt|.$$

Let L be the maximum of the n positive constants L_i, and let Q be the maximum of any of the n quantities $|x_i - y_i|$ in any closed interval within the interval

$$|t - t_0| \leqq 1/(2nL).$$

The maximum Q must be attained for some value of t, say t^*, and for some i. If we insert the value t^* of t in the corresponding inequality above, and apply the mean value theorem to the right-hand member, there results

$$Q \leqq nLQ|t^* - t_0| \leqq Q/2.$$

This proves that Q must be 0. Hence the two solutions $x(t)$, $y(t)$ which coincide for $t = t_0$ will continue to do so in any such interval. The theorem follows by repeated application of this result.

The physical meaning of the existence and uniqueness theorems is evidently that the motion of a dynamical system is completely determined by the differential equations and the initial values of the variables determining the state of the system—a fact which is intuitively obvious.

Thus the treatment of a dynamical problem requires a formulation of the appropriate differential equations by means of physical principles, and a subsequent mathematical treatment of the properties of the motions on the basis of these equations.

4. Two continuity theorems. There are certain further continuity theorems which are closely allied to the two theorems established above.

FIRST CONTINUITY THEOREM. *If the functions X_i in* (1) *satisfy a Lipschitz condition in R, the unique solution $x(t)$ for which $x(t_0) = x^0$ is a set of continuous functions of the n parameters x_i^0 and of $t - t_0$.*

We observe first that, in changing the independent variable t to $t' = t - t_0$, the modified differential equations obtained differ from (1) only in that t is replaced by t', while in the initial conditions t_0 is replaced by 0. Hence the dependent variables x_i involve t and t_0 in the combination $t - t_0$ only, so that it will suffice to prove the functions x_i to be continuous in x_i^0 and t in the case $t_0 = 0$.

A slight extension of the method used in proving the uniqueness theorem may be employed. It is apparent that if x_i and y_i are two solutions of (1) which reduce to x_i^0 and

y_i^0 respectively for $t=0$, then by subtraction of the corresponding integral equations there is obtained

$$(x_i - y_i) = (x_i^0 - y_i^0) + \int_0^t [X_i(x_1, \cdots, x_n) - Y_i(y_1, \cdots, y_n)]\, dt,$$

provided that the value of t lies within the common interval of definition of x_i and y_i.

Suppose that x^0 lies at a distance at least D from the boundary of R, and then y^0 at a distance not more than $D/2$ from x^0. This requirement will be met if we take the maximum difference $|y_i^0 - x_i^0|$ not more than $D/(2\sqrt{n})$. Restrict t further to lie in the interval $|t| \leqq 1/(2nL)$.

Under these circumstances if Q^0 denotes the maximum difference $|x_i^0 - y_i^0|$ for any i, and Q the maximum difference $|x_i - y_i|$ in the t interval under consideration, we find for some value t^* of t by means of the above integral equations,

$$Q \leqq Q^0 + nLQ\,|t^*| \leqq Q^0 + Q/2.$$

Hence in the stated interval we have constantly $Q \leqq 2Q^0$, i. e., the difference $x_i - y_i$ cannot exceed twice the maximum initial difference $x_j^0 - y_j^0$ in numerical value. This means that if y^0 approaches x^0, then y approaches x uniformly throughout the stated interval. Since $|dx_i/dt| \leqq M$ everywhere, the functions x_i must be continuous in x_i^0 and t in the restricted t interval.

It remains only to remove the restriction upon the interval t.

In any closed interval of definition $0 \leqq t \leqq T$, the point $x(t)$ is throughout at a distance exceeding a positive D from the boundary of R. Consequently in a t interval of fixed length about any point t' of the selected interval, each function $x_i(t)$ and will vary continuously with $x_i(t')$ and $t - t'$. It will then be possible to select points

$$t_0 = 0,\ t_1,\ \cdots,\ t_k = T,$$

such that t_1 is in the interval about t_0, t_2 in the interval about t_1, and so on. Thus if we take $|y_i^0 - x_i^0| \leqq q$ we obtain successively

$$|x_i(t_1) - y_i(t_1)| \leqq 2q, \cdots, |x_i(t_k) - y_i(t_k)| \leqq 2^k q.$$

The truth of the theorem is now obvious for the unrestricted interval.

SECOND CONTINUITY THEOREM. *If the functions X_i admit continuous bounded first partial derivatives in R while these derivatives themselves satisfy a Lipschitz condition, then the unique solution $x(t)$ such that $x(t_0) = x^0$ has components with continuous first partial derivatives as to x_i^0 and $t - t_0$.*

To establish this theorem we resort to a consideration of the difference equality introduced at the beginning of the proof of the preceding theorem. We shall restrict y to be sufficiently near x in the interval $|t - t_0| \leqq T$, precisely as in that proof, and in addition to be such that the straight line from $x(t)$ to $y(t)$ lies in the region R for any t. The preceding theorem shows that this will be possible if $|y_i^0 - x_i^0|$ is sufficiently small.

The mean value theorem allows us then to write

$$X_i(y_1, \cdots, y_n) - X_i(x_1, \cdots, x_n) = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j}(y_j - x_j)$$

where the arguments of $\partial X_i / \partial x_j$ are $z_{i1}, \cdots, z_{in}$ with

$$z_{ij} = x_j + \theta_i (y_j - x_j) \qquad (0 < \theta_i < 1)$$

so that z lies in R. Hence the difference equality can be written

$$y_i - x_i = y_i^0 - x_i^0 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j}(y_j - x_j)\, dt.$$

Suppose that $y_2^0, \cdots, y_n^0$ are taken equal to $x_2^0, \cdots, x_n^0$ respectively while y_1^0 is allowed to approach x_1^0. If we write

$$\frac{y_1 - x_1}{y_1^0 - x_1^0} = \frac{\Delta x_1}{\Delta x_1^0}, \cdots, \frac{y_n - x_n}{y_1^0 - x_1^0} = \frac{\Delta x_n}{\Delta x_1^0},$$

the n equations above gives us at once

$$\frac{\Delta x_1}{\Delta x_1^0} = 1 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_1}{\partial x_j} \frac{\Delta x_j}{\Delta x_1^0} dt,$$

$$\frac{\Delta x_2}{\Delta x_1^0} = 0 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_2}{\partial x_j} \frac{\Delta x_j}{\Delta x_1^0} dt,$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\Delta x_n}{\Delta x_1^0} = 0 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_n}{\partial x_j} \frac{\Delta x_j}{\Delta x_1^0} dt.$$

For $|t - t_0|$ sufficiently small, in particular for

$$|t - t_0| \leqq 1/(2nL')$$

where L' is the upper bound of $|\partial X_i/\partial x_j|$ in R, we find that none of the integrals on the right can exceed $Q'/2$, where Q' is the maximum of any $|\Delta x_j/\Delta x_1^0|$ in this interval. By using the value of t and of the index i for the corresponding $|\Delta x_i/\Delta x_1^0|$ which yields this maximum, we establish the fact that Q' is at most 2 (compare with previous section).

Also by differentiation of the equalities above it appears that the derivatives of these ratios $\Delta x_i/\Delta x_1^0$ as to t do not exceed $2nL'$.

Consequently Ascoli's theorem can be applied to show that if Δx_1^0 be allowed to approach 0 suitably, the difference ratios will approach limits, which may be designated by y_i, and that these satisfy the integral equations

$$y_1 = 1 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_1}{\partial x_j} y_j \, dt,$$

$$y_2 = 0 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_2}{\partial x_j} y_j \, dt,$$

$$\cdots\cdots\cdots\cdots$$

$$y_n = 0 + \int_{t_0}^{t} \sum_{j=1}^{n} \frac{\partial X_n}{\partial x_j} y_j \, dt.$$

Evidently these conditions are equivalent to the n 'equations of variation'

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} y_j \qquad (i = 1, \cdots, n), \tag{3}$$

and the set of initial conditions

$$y_1(t_0) = 1, y_2(t_0) = 0, \cdots. y_n(t_0) = 0.$$

But these n equations and conditions, joined with the n equations (1) and attached conditions, form a system of $2n$ equations and $2n$ initial conditions in $x_1, \cdots, x_n, y_1, \cdots, y_n$, to which the existence and uniqueness theorems apply; we recall that $\partial X_i / \partial x_j$ as well as X_i satisfy Lipschitz conditions. Since the functions y_i are uniquely determined, the ratios $\Delta x_i / \Delta x_1^0$ approach the limits y_i uniformly no matter how Δx_1^0 approaches 0.*

In this way it is seen that for any i and j the partial derivatives $y_i = \partial x_i / \partial x_j^0$ exist and satisfy the equations of variation and initial conditions

$$y_1(t_0) = 0, \cdots. y_{j-1}(t_0) = 0,$$
$$y_j(t_0) = 1, \ y_{j+1}(t_0) = 0, \cdots, y_n(t_0) = 0.$$

By application of the first continuity theorem it follows that these functions $\partial x_i / \partial x_j^0$ not only exist, but are continuous in x_i^0 and $t - t_0$.

5. Some extensions. The above theorems may be extended and completed in various ways.

In the first place suppose that the functions X_i are uniformly continuous functions of $x_1, \cdots, x_n$ and a parameter c, for x in R and $c' < c < c''$, and furthermore satisfy a Lipschitz condition in the $n+1$ variables $x_1, \cdots. x_n. c$. Consider the system of $n+1$ differential equations

$$dx_i/dt = X_i(x_1, \cdots, x_n, x_{n+1}) \qquad (i = 1, \cdots, n),$$
$$dx_{n+1}/dt = 0,$$

* Otherwise, by Ascoli's theorem, another distinct set y_i could be found.

with initial conditions

$$x_i(t_0) = x_i^0 \quad (i = 1, \cdots, n), \qquad x_{n+1}(t_0) = c,$$

where x^0 is in R, and $c' < c < c''$. The existence, uniqueness and first continuity theorems apply to show that a unique solution $x_i(t)$ $(i = 1, \cdots, n+1)$ exists and is continuous in x_i^0 and c. But these functions clearly satisfy the equations and conditions

$$dx_i/dt = X_i(x_1, \cdots, x_n, c), \quad x_i(t_0) = x_0 \quad (i = 1, \cdots, n).$$

If X_i have in addition bounded first partial derivations in $x_1, \cdots, x_n, c$ satisfying Lipschitz conditions in these variables, then $\partial x_i / \partial c$ will also exist by the second continuity theorem.

Consequently the existence, uniqueness, and continuity theorems can be immediately extended to the case in which the right-hand members X_i in the differential equations (1) *contain one or more parameters.*

Again, suppose that X_i involve t as well as $x_1, \cdots, x_n$. A similar consideration of the $n+1$ differential equations

$$dx_i/dt = X_i(x_1, \cdots, x_n, x_{n+1}), \ (i = 1, \cdots, n); \ \ dx_{n+1}/dt = 1$$

with $n+1$ initial conditions

$$x_i(t_0) = x_i^0, \quad (i = 1, \cdots, n); \qquad x_{n+1}(t_0) = t_0$$

shows that there will exist a solution when t is suitably restricted, and that there is only one solution if the functions $X_i(x_1, \cdots, x_n, t)$ satisfy a Lipschitz condition in $x_1, \cdots, x_n, t$. Analogs of the first and second continuity theorems are easily formulated for such a system also.

Thus a similar extension is possible to the case in which the functions X_i involve the time t.

Again, let us suppose that X_i contain only the variables $x_1, \cdots, x_n$ but possess continuous partial derivatives up to those of order $\mu > 0$, while the partial derivatives of the μth order satisfy Lipschitz conditions. The method of proof of the second continuity theorem given above shows that the given system (1) of differential equations can be replaced by a system of the same type of order $2n$ with $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ where $y_i = \partial x_i / \partial x_1^0$ for instance, as dependent variables; this system of order $2n$ consists of course of the n given equations and the n equations of variation. Now if we apply the second continuity theorem to this augmented system, we conclude at once that the second partial derivatives $\partial^2 x_i / \partial x_j^0 \partial x_1^0$ and likewise $\partial^2 x_i / \partial x_j^0 \partial x_k^0$ will exist and be continuous. In the augmented system, however, the right-hand members will in general possess continuous first partial derivatives of order $\mu - 1$ which will satisfy a Lipschitz condition.

Repeating the above process we obtain the existence of partial derivatives of $x_1, \cdots, x_n$ up to those of order μ which respect to the variables $x_1^0, \cdots, x_n^0$.

In case the functions X_i of $x_1, \cdots, x_n$ possess continuous first partial derivatives of order μ while the partial derivatives of order μ satisfy Lipschitz conditions, the components $x_1, \cdots, x_n$ considered as functions of $x_1^0, \cdots, x_n^0, t - t_0$ will possess continuous partial derivatives in these variables of order μ.

An important case, and the only one entering subsequently into consideration, is that in which the functions X_i admit continuous partial derivatives of all orders in the variables concerned. The components $x_1, \cdots, x_n$ will then necessarily possess continuous partial derivatives of all orders in $x_1^0, \cdots, x_n^0, t - t_0$.

If, furthermore, the functions X_i are analytic in $x_1, \cdots, x_n$, the components $x_1, \cdots, x_n$ considered as functions of $x_1^0, \cdots, x_n^0, t - t_0$, will be analytic in these variables.

Let us indicate briefly a proof of this important fact.

We observe first that it suffices to show that the unique solution of (1) for which x reduces to x^0 for $t = 0$ has components analytic in $x_1^0, \cdots, x_n^0, t$; here the device used in the proof of the second continuity theorem, namely the introduction of $t' = t - t_0$ in the differential equations, is applicable.

Furthermore by writing

$$x_1' = x_1 - x_1^0, \cdots, x_n' = x_n - x_n^0,$$

in these equations, it becomes clear that we need only prove the components $x_1, \cdots, x_n$ to be analytic in $x_1^0, \cdots, x_n^0$ in the neighborhood of the origin.

Now since X_i are then analytic in the neighborhood of the origin, we may write

$$X_i \ll \frac{M}{1 - \frac{x_1 + \cdots + x_n}{r}} \qquad (i = 1, \cdots, n),$$

where M is a sufficiently large positive quantity and r is a sufficiently small positive quantity. The relations written mean that every coefficient in the series expansion of X_i in powers of $x_1, \cdots, x_n$ does not exceed the corresponding coefficient of the series on the right in numerical value.*

Now consider the comparison differential system

$$\frac{dx_i}{dt} = \frac{M}{1 - \frac{x_1 + \cdots + x_n}{r}} \qquad (i = 1, \cdots, n),$$

of which the unique solution which satisfies the conditions

$$x_1 = x_1^0, \cdots, x_n = x_n^0$$

for $t = 0$ is evidently given by

$$x_i = x_i^0 + u \qquad (i = 1, \cdots, n),$$

where u is defined by the implicit equation

$$\left(1 - \frac{x_1^0 + \cdots + x_n^0}{r}\right) u - \frac{nu^2}{2r} = Mt.$$

In this case $x_1, \cdots, x_n$ are clearly analytic in $x_1^0, \cdots, x_n^0, t$; furthermore the explicit formulas obtained for $x_1, \cdots, x_n$ on

* For a proof of this type of relation see, for instance, E. Picard, *Traité d'Analyse*, vol. 2, chap. 9.

successive differentiation of the comparison system and setting $x_1^0 = \cdots = x_n^0 = t = 0$ shows that the coefficients in the convergent power series for $x_1, \cdots, x_n$ in $x_1^0, \cdots, x_n^0, t$ are all positive.

But the inequality relations written above obviously imply similar relations between any partial derivatives of X_i and of the same partial derivative of the right-hand member of the comparison equations. Thus we see in succession that the formal series made out with coefficients obtained by successive differentiation of the equations (1) as to $x_1^0, \cdots, x_n^0, t$ and setting $x_1^0 \cdots = x_n^0 = t = 0$, must converge since the coefficients are less than the corresponding coefficients of known convergent series. Thus these formal series define analytic functions $x_1, \cdots, x_n$, while the mode of determination of these functions renders it certain that every difference function

$$dx_i/dt - X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

considered as functions of $x_1^0, \cdots, x_n^0, t$, vanishes together with all of its partial derivatives at the origin when these analytic functions are substituted in. Hence these difference functions must vanish identically. Thus $x_1, \cdots, x_n$ obtained in this formal manner will constitute the unique solution satisfying the prescribed conditions, and the stated analyticity is proved.

6. The principle of the conservation of energy.* Conservative systems. In the case of many dynamical systems the geometric configuration is determined by m 'coördinates' $q_1, \cdots, q_m$ having a spatial nature, while the state of the system is fixed by the coördinates and the velocities $q_1', \cdots, q_m'$, where $q_i' = dq_i/dt$. Such a system is said to have m 'degrees of freedom'. With these coördinates may be correlated 'generalized external forces' Q_i so that by definition

* For historical and critical remarks concerning this principle see the article by A. Voss in the *Encyklopädie der mathematischen Wissenschaften*, vol. 4 or in the French version by E. and F. Cosserat. I presented the results here obtained at the Chicago Colloquium in 1920. The following treatment of the principle differs essentially from any other which I have seen.

the 'work' W done on the system is given by

$$dW = \sum_{j=1}^{m} Q_j \, dq_j$$

in which the differential symbols have their ordinary significance.

We shall assume that the functions Q_i are real, single-valued, analytic functions of the coördinates, velocities, and accelerations; thus there is one and only one set of external forces Q_i which yields a prescribed set of accelerations for a given set of coördinate values and velocities. In this case the variables determining the state of the system are clearly the $2m$ coördinates and velocities.

As a concrete model of such a dynamical system, we may think of a concealed mechanism which is controlled by a set of m rods which project from a wall. If the rods project by distances $q_1, \cdots, q_m$, then $Q_1, \cdots, Q_m$ are the ordinary forces applied to these rods in an outward direction.

The fundamental hypothesis which embodies the principle of the conservation ot energy is that if, by any application of such external forces, the dynamical system is carried through a closed cycle, so that the set of $2m$ final values of q_i and q_i' coincides with the set of initial values, the total amount of work done on the system during the cycle vanishes. Any system of this type will be called 'conservative'.

Conservative dynamical systems can only be regarded as idealizations of the systems actually found in nature, but nevertheless they are of great importance.

Let us now consider the properties of such a conservative system. If it is carried through a cycle $ABCA$ and a modified cycle $AB'CA$, (which may be represented graphically by closed curves in the $2m$ dimensional space of the q_i and q_i'), the work done in parts ABC and $AB'C$ is the same, namely the negative of that done along the common part CA. Thus the work done along the part AC is independent of the path taken, and so depends only upon the values of $q_1, \cdots, q_m$, $q_1', \cdots, q_m'$ at C:

$$\int_A^C \sum_{j=1}^m Q_j \, dq_j = W(q_1, \cdots, q_m, q_1', \cdots, q_m')\Big|^C.$$

By differentiation with respect to t we obtain the following fundamental identity in the $3m$ variables q_i, q_i', q_i'':

$$\sum_{j=1}^m Q_j q_j' \equiv \sum_{j=1}^m \left(\frac{\partial W}{\partial q_j} q_j' + \frac{\partial W}{\partial q_j'} q_j''\right). \tag{4}$$

This relation must subsist if the principle of the conservation of energy is to hold, and conversely it is easily seen to ensure that the principle is valid.

It is possible to give the identity an interesting explicit form. Let us endeavor to determine a function L of the $2m$ variables q_i, q_i' so that the following identity holds:

$$\sum_{j=1}^m \left[\frac{d}{dt}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j}\right] q_j' \equiv \sum_{j=1}^m \left(\frac{\partial W}{\partial q_j} q_j' + \frac{\partial W}{\partial q_j'} q_j''\right).$$

Comparing the coefficients of q_i'' on both sides, we find m conditions

$$\sum_{j=1}^m \frac{\partial^2 L}{\partial q_i' \partial q_j'} q_j' \equiv \frac{\partial W}{\partial q_i'},$$

which hold if

$$\sum_{j=1}^m \left(q_j' \frac{\partial L}{\partial q_j'}\right) - L \equiv W, \tag{5}$$

as follows by differentiation with respect to q_i'. Comparing the remaining terms which are independent of q_i'', we get the further condition

$$\sum_{i,j=1}^m \frac{\partial^2 L}{\partial q_i \partial q_j'} q_i' q_j' - \sum_{j=1}^m \frac{\partial L}{\partial q_j} q_j' \equiv \sum_{j=1}^m \frac{\partial W}{\partial q_j} q_j'$$

which is obviously satisfied if L satisfies (5).

A value of L for which (5) holds can always be found. Observe first that if $Q_1, \cdots, Q_m$ can be expanded in ascending powers of the velocities $q_1', \cdots, q_m'$, then no first degree terms appear in W. That is, we have

$$W = W_0 + * + W_2 + W_3 + \cdots,$$

where the subscript indicates the degree of the term in the velocities. In fact the presence of a term W_1 would lead to a term not involving the velocities on the right-hand side of the fundamental identity whereas there is no such term on the left. If we substitute the above expansion of W and the corresponding expansion of L,

$$L = L_0 + L_1 + L_2 + \cdots$$

in the partial differential equation (5) while noting that by Euler's theorem concerning homogeneous functions,

$$\sum_{j=1}^{n} q_j' \frac{\partial V_n}{\partial q_j'} = n V_n \qquad (n = 0, 1, \cdots),$$

and if we equate terms of equal degree in the velocities, we find

$$L_0 = -W_0, \; L_2 = W_2, \; \cdots, \; L_n = \frac{W_n}{n-1}, \; \cdots,$$

while L_1 is unrestricted.

Any such function L may be called a 'principal function' associated with the arbitrary conservative system with which we started. When the linear terms in the velocities are lacking in L, a special function is obtained which has important properties.

On defining the functions R_i by means of the equations

$$(6) \qquad Q_i \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) - \frac{\partial L}{\partial q_i} + R_i \qquad (i = 1, \cdots, n),$$

we observe that by the definition of L we have

$$(7) \qquad \sum_{j=1}^{m} R_j q_j' \equiv 0.$$

Conversely, if $Q_1, \cdots, Q_m$ are of such a form that (7) obtains, the principle of the conservation of energy holds.

If W is the work function of a conservative dynamical system and if L is the associated principal function, then the generalized external forces Q_i may be written in the form (6), (7).

The italicized conclusion above may be expressed in a somewhat different fashion. As is customary, let us call a dynamical system for which $R_i \equiv 0$ $(i = 1, \cdots, n)$ a 'Lagrangian' system. Also let us call a system $W = 0$ a 'non-energic' system. The appropriateness of the latter new term lies in the fact that whatever external forces may be applied no work can be done. In this case we may take $L \equiv 0$ also. The alternative statement is the following:

Any conservative dynamical system has external forces which are the sum of the forces of a Lagrangian system and of a non-energic system.

Before leaving this topic we may note that for unconstrained motion we have $Q_1 = \cdots = Q_m = 0$ by definition. Here the equations of motion take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) - \frac{\partial L}{\partial q_i} = -R_i \qquad (i = 1. \cdots, m)$$

where the quantities R_i are subject to (7). Hence we may state the following conclusion:

An unconstrained, conservative, dynamical system undergoes the same motion as a Lagrangian system to which a set of non-energic external forces is applied.

An unconstrained conservative dynamical system clearly admits an energy integral $W =$ const., which by means of (5) can be written in the alternative form

$$\sum_{j=1}^{m}\left(q_j' \frac{\partial L}{\partial q_j'}\right) - L = \text{const.}$$

Lagrangian and non-energic systems have been defined by means of the types of the external forces. These definitions are not mutually exclusive. In fact let us inquire when a dynamical system will be both non-energic and Lagrangian. Since it is non-energic, we have

$$W = W_0 + W_2 + \cdots = 0,$$

and hence we find the most general function

$$L = L_1 = \sum_{j=1}^{m} \alpha_j q'_j.$$

Since the system is Lagrangian, we may take $R_i = 0$ for every i, and thus find directly

$$Q_i = \sum_{j=1}^{m} \left(\frac{\partial \alpha_i}{\partial q_j} - \frac{\partial \alpha_j}{\partial q_i}\right) q'_j \quad (i = 1, \cdots, n).$$

Hence for a system to be of both types the generalized external forces must have this specific form.

It is worthy of note that in the case $m = 1$, equation (7) implies that R_1 is zero, so that any conservative dynamical system with a single degree of freedom is Lagrangian. Similarly in the case $m = 2$ the external forces may be represented in the form

$$Q_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial q'_1}\right) - \frac{\partial L}{\partial q_1} + \lambda q'_2, \quad Q_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial q'_2}\right) - \frac{\partial L}{\partial q_2} - \lambda q'_1,$$

where λ is an arbitrary function of the coördinates, velocities, and accelerations.

7. Change of variables in conservative systems. In the first instance the coördinates q_i of a conservative dynamical system are actual distances, while the Q_i are forces which act in the direction of these coördinates. But for most physical purposes it is not desirable to adhere to a single set of coördinates.

Let us now *define* the modified external forces Q_i corresponding to the new coördinates $\bar{q}_i$ by means of the equations

$$\bar{Q}_i = \sum_{j=1}^{n} Q_j \frac{\partial q_j}{\partial \bar{q}_i} \qquad (i = 1, \cdots, n). \tag{8}$$

When this definition is adopted and a further change of variables from $\bar{q}_i$ is made, while the new external forces are defined in terms of $\bar{Q}_i$ by analogous formulas, it is found that the final functions Q_i obtained have the same expressions as if only a single direct change of variables is made. This group property is an immediate consequence of the above definition.

Hence the Q_i are uniquely defined for any coördinate system whatsoever.

It may be observed that for change from one rectangular system to another the above formula for the determination of the $\overline{Q}_i$ in terms of Q_i agrees with that obtained by the ordinary laws for the composition of forces. In more general cases the equations above define the generalized force components in the appropriate sense.

Now on account of the identity

$$d\,W = \sum_{j=1}^{m} Q_j\,dq_j = \sum_{j=1}^{m} \overline{Q}_j\,d\overline{q}_j,$$

it follows that the system will remain a conservative dynamical system according to our definition, in these new coördinates. Furthermore the modified work function will be the same as before (up to an additive constant), and since the formulas of transformation of the velocities

$$q_i' = \sum_{j=1}^{m} \frac{\partial q_i}{\partial \overline{q}_j}\,\overline{q}_j' \qquad (i = 1, \cdots, m)$$

are linear and homogeneous in the velocities, it follows that the various components $W_0, W_2, \cdots$ in W will be unaltered.

If we agree for the sake of definiteness always to choose the unique determination of L which lacks linear terms in the velocities, it follows that the principal function L is the same in both problems.

Suppose now that we define $\overline{R}_i$ by means of the equations

$$\overline{Q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \overline{q}_i'}\right) - \frac{\partial L}{\partial \overline{q}_i} + \overline{R}_i$$

where L is the first given principal function, but expressed in terms of the new variables $\overline{q}_i$, $\overline{q}_i'$.

It is easy to prove the formal identities

$$\sum_{j=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial \varphi}{\partial q_j'}\right)-\frac{\partial \varphi}{\partial q_j}\right]\frac{\partial q_j}{\partial \bar{q}_i}=\frac{d}{dt}\left(\frac{\partial \varphi}{\partial \bar{q}_i'}\right)-\frac{\partial \varphi}{\partial \bar{q}_i} \qquad (i=1,\cdots.n),$$

where on the left side φ is any function of q_i, q_i'. To see this, we note that from the linear relationship written between the variables q_i' and $\bar{q}_i'$ we have

$$\frac{\partial q_i'}{\partial \bar{q}_j'}=\frac{\partial q_i}{\partial \bar{q}_j},\quad \frac{\partial q_i'}{\partial \bar{q}_j}=\frac{d}{dt}\left(\frac{\partial q_i}{\partial \bar{q}_j}\right) \quad (i,j=1,\cdots,n).$$

Hence we deduce for any i

$$\sum_{j=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial \varphi}{\partial q_j'}\right)\right]\frac{\partial q_j}{\partial \bar{q}_i}=\sum_{j=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial \varphi}{\partial q_j'}\frac{\partial q_j}{\partial \bar{q}_i'}\right)-\frac{\partial \varphi}{\partial q_j'}\frac{d}{dt}\left(\frac{\partial q_j}{\partial \bar{q}_i}\right)\right]$$
$$=\frac{d}{dt}\left(\frac{\partial \varphi}{\partial \bar{q}_i'}\right)-\sum_{j=1}^{m}\frac{\partial \varphi}{\partial q_j'}\frac{d}{dt}\left(\frac{\partial q_j}{\partial \bar{q}_i}\right).$$

Moreover we have also for any i

$$\sum_{j=1}^{m}\frac{\partial \varphi}{\partial q_j}\frac{\partial q_j}{\partial \bar{q}_i}=\frac{\partial \varphi}{\partial \bar{q}_i}-\sum_{j=1}^{m}\frac{\partial \varphi}{\partial q_j'}\frac{\partial q_j'}{\partial \bar{q}_i}.$$

Subtracting the two identities thus obtained we obtain the specified identity.

This identity with $\varphi = L$ shows of course that the functions $\bar{R}_i$ as obtained from R_i by the defining formulas have the same structure as those which give $\bar{Q}_i$ in terms of Q_i. Hence we are led to the following general result:

If the variables $q_1, \cdots, q_m$ of a conservative dynamical system are transformed to $\bar{q}_1, \cdots, \bar{q}_m$, the system remains conservative in the new variables with L, W unaltered, while Q_i, R_i are both modified to corresponding new expressions obtained as in (8). *In particular then if the system is Lagrangian or non-energic in the first set of variables, it remains so in the modified variables.*

8. Geometrical constraints. We are now in a position to deal with the question of 'geometrical constraints.' Let us suppose that various geometrical points of the given conservative system are fixed or are constrained to lie in smooth curves or surfaces or to move subject to connections by various rigid bars without mass.

The effect of such constraints is to reduce the number of degrees of freedom. In fact, by properly taking coördinates $q_1, \cdots, q_m$, the k conditions of constraint may be made to take the form

$$q_{\mu+1} = \text{const.}, \cdots, q_m = \text{const.} \qquad (\mu = m - k).$$

Now denote by $\bar{L}$ that which L becomes when the k constrained coördinates $q_{\mu+1}, \cdots, q_m$ have these assigned constant values, while the corresponding q_i', q_i'' vanish of course. Then it is clear that

$$L = \bar{L}, \quad \frac{\partial \bar{L}}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial \bar{L}}{\partial q_i'} = \frac{\partial L}{\partial q_i'} \qquad (i = 1, \cdots, \mu).$$

Hence we have the relations

$$Q_i = \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial q_i'}\right) - \frac{\partial \bar{L}}{\partial q_i} + R_i \qquad (i = 1, \cdots, \mu),$$

where Q_i and R_i are defined as usual.

But the original external forces Q_i may be decomposed into a sum

$$\bar{Q}_i + P_i, \qquad (i = 1, \cdots, m),$$

in which the 'forces of constraint' P_i can do no work for any possible displacement subject to the constraints. It follows that the functions $P_1, \cdots, P_\mu$ must vanish when the coördinates are selected as above. Hence we may replace Q_i by $\bar{Q}_i$ in the formula above for $i = 1, \cdots, \mu$.

Thus we are led to the following conclusion:

If a conservative dynamical system with m degrees of freedom is subject to k geometrical constraints, it may be treated as such a conservative system with $m-k$ degrees of freedom.

9. Internal characterization of Lagrangian systems. In most of the dynamical applications, Lagrangian systems can be regarded as dealing with a system of particles subject to certain forces and geometrical constraints. This type of internal characterization formed the basis of Lagrange's derivation of these equations, and is considered briefly in the present section. In the next section an external characterization is developed.

We shall begin by considering three particular types of particles in ordinary space:

(a) *The inertial particle.*

Here if x, y, z are the rectangular coördinates of the particle, the external forces X, Y, Z in the directions of the corresponding axes are proportional to the accelerations in these directions:

$$X = mx'', \qquad Y = my''. \qquad Z = mz'',$$

where the constant of proportionality, m, is termed the 'mass' of the particle.

This is the case of an ordinary mass particle.

The particle is seen to be of Lagrangian type with Lagrangian function

$$L = \frac{1}{2} m (x'^2 + y'^2 + z'^2),$$

and L is its 'kinetic' energy.

(b) *The non-kinetic particle.*

Such a particle is subject to forces independent of the velocity and having the particular form:

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y, \qquad Z = -\partial V/\partial z$$

where V depends on the coördinates of the particle in space. Here the dynamical system is Lagrangian with $L = V$.

The function $-V$ is the 'potential' energy of the particle due to the field of force in which the particle moves.

An electrified particle of slight mass moving in a static electric field is nearly of this type.

(c) *The gyroscopic particle.*

By definition the gyroscopic particle is subject to forces with components of the type

$$X = (\partial \alpha / \partial y - \partial \beta / \partial x) y' + (\partial \alpha / \partial z - \partial \gamma / \partial x) z',$$

so that the force vector is perpendicular to the velocity vector and therefore can do no work. Nevertheless the system is Lagrangian with

$$L = \alpha x' + \beta y' + \gamma z'.$$

It will be noted that this is the type of special Lagrangian system which is also non-energic.

An electrically charged particle of negligible mass moving in a static magnetic field falls under this case.

(d) *The system of generalized particles.*

If a particle moves subject to a sum of forces of the inertial, non-kinetic, and gyroscopic types it may be termed a generalized particle. Such a situation is realized for example when an ordinary mass particle moves in a gravitational field. It is evident that the resultant system will then be Lagrangian with a principal function merely the sum of the principal functions associated with the component forces.

Consider further a set of such particles which do not at first interact in any way. If we add together the Lagrangian functions for the several particles, there is obtained a single function L which can serve as a single function from which the equations of motion of the system of particles may be derived.

It is necessary of course to use suitable variables (x_i, y_i, z_i) where $i = 1, 2, \cdots, m$ to differentiate between the coördinates of the various particles.

Clearly this yields a principal function L which will be quadratic in the velocities. It is another step in the way of generalization to take L to be any quadratic polynomial in the velocities, in which the homogeneous quadratic part is the kinetic energy T, in which the term U independent of the velocities is the potential energy, and in which the homogeneous term of first degree may be called the 'gyroscopic energy'.

Furthermore, as has been seen above, we may suppose the particles to be subject to certain types of geometrical constraints, thus diminishing the number of degrees of freedom, without affecting the Lagrangian character of the problem.

It is such a system of generalized particles which suffices for most of the applications.

(e) *The generalized particle in m-dimensional space.*

By an obvious extension into which we shall not enter here it appears that a single mass particle lying on an m-dimensional manifold defined by a quadratic differential form, subject to a field of force derived from a potential function in the surface, and furthermore to gyroscopic forces derived from some linear function of the velocities in the surface, will be of Lagrangian type. The function L is quadratic in the velocities. Conversely any Lagrangian system with m degrees of freedom for which L is quadratic in the velocities is representable by the motion of a mass particle in such an m-dimensional manifold.

Thus we may interpret the motion of any dynamical system with m degrees of freedom as isomorphic with the motion of a single generalized particle on a suitable m-dimensional surface.

10. External characterization of Lagrangian systems.* In this section we propose to characterize an important type of Lagrangian systems by means of certain simple properties ot the external forces.

In fact, we shall characterize those 'regular' dynamical systems for which the Lagrangian function L is a quadratic function of the velocities, without first degree terms. These form an important class of dynamical systems in which L has the form $T - U$ where T is homogeneous and quadratic

* The material of this section was presented before the Chicago Colloquium in 1920. For an analytic characterization of the Lagrangian system in the case when the external forces are linear in the velocities see E. T. Whittaker, *Analytical Dynamics*, p. 45.

in the velocities, while U is a function of the coördinates only. It will be observed that regular systems remain of this type under an arbitrary transformation of the coördinates $q_1, \cdots, q_m$.

The first of the characteristic properties which we shall state is the following:

I. *The external forces vary linearly with the coördinate accelerations.*

Evidently this means that we may write.

$$Q_i = \sum_{j=1}^{m} a_{ij} q_j'' + b_i$$

where a_{ij}, b_i do not involve the accelerations.

II. Principle of Reciprocity.* *The change in the acceleration q_j'' due to a change in the i-th force Q_i is the same as the change in the acceleration q_i'' due to an equal change in the j-th force Q_j, $(i, j = 1, \cdots, n)$.*

In order to see what this means we suppose that Q_k receives a certain acceleration increment Q, in which case the above equations give

$$Q \delta_{ik} = \sum_{j=1}^{m} a_{ij} \Delta_1 q_j'' \qquad (i = 1, \cdots . m)$$

where Δ denotes the increment as usual and where $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$.

Now suppose that Q_l receives the same increment. We find similarly

$$Q \delta_{il} = \sum_{j=1}^{m} a_{ij} \Delta_2 q_j''.$$

If we assume that the determinant $|a_{ij}|$ is not zero, we may solve these equations and obtain for all i, k, l

$$\Delta_1 q_i' = \sum_{j=1}^{m} \bar{a}_{ij} Q \delta_{jk} = \bar{a}_{ik} Q, \ \Delta_2 q_i' = \bar{a}_{il} Q,$$

where $\bar{a}_{ij}$ is the cofactor of the element in the j-th row and i-th column of $|a_{ij}|$, divided by this determinant. Putting

* Compare Rayleigh, *Theory of Sound,* vol. 1, chap. 4.

$i = l$ and $i = k$ respectively, we obtain from II, $\bar{a}_{lk} = \bar{a}_{kl}$. Thus the cofactors $\bar{a}_{ij}$ are symmetric in i and j, whence it follows that the elements a_{ij} are symmetric also, i. e. we must have $a_{ij} = a_{ji}$ for all values of i and j.

III. *For a family of similar motions, the forces are quadratic functions of the speed.*

In other words, let $q_i = q_i(t)$, $(i = 1, \cdots, m)$ be a motion of the system. Suppose that the motion is speeded up in the ratio λ to 1. The external forces become

$$Q_i = \sum_{j=1}^{m} a_{ij}(q_1, \cdots, q_m, \lambda q_1', \cdots, \lambda q_m')\lambda^2 q_j'' \\ + b_i(q_1, \cdots, q_m, \lambda q_1', \cdots, \lambda q_m'),$$

inasmuch as the coördinates q_i are unaltered while the velocities q_i' and the accelerations q_i'' are multiplied by λ and λ^2 respectively.

If these expressions Q_i are to be quadratic in λ $(q_i, q_i', q_i''$ being entirely independent variables of course), the functions a_{ij} cannot depend on the velocities, while b_i will be quadratic in them. Thus in virtue of III we obtain more precisely

$$Q_i = \sum_{j=1}^{m} a_{ij}\, q_j'' + \sum_{j,k=1}^{m} b_{ijk}\, q_j' q_k' + \sum_{j=1}^{m} b_{ij}\, q_j' + b_i$$

where the functions a_{ij}, b_{ijk}, b_{ij}, b_i depend only upon the coördinates $q_1, \cdots, q_m$.

This form is rendered still more specific by the following hypothesis.

IV. Reversibility. *Any motion under prescribed external forces may equally be described in the reversed order of time.*

The meaning here is that the above relations continue to hold when t is replaced by $-t$. But this changes the velocities to their negatives while leaving the coördinates q_i and the accelerations q_i'' unaltered. We infer that the functions b_{ij} must be lacking. Hence we may write

$$Q_i = \sum_{j=1}^{m} a_{ij}\, q_j'' + \sum_{j,k=1}^{m} b_{ijk}\, q_j'\, q_k' + b_i$$

where a_{ij}, $b_{ijk} = b_{ikj}$, b_i involve only the coördinates.

All of the properties I to IV so far employed are invariant under a change of coördinates q_i and have to do with the nature of the external forces in the neighborhood of a set of values $q_1^0, \cdots, q_m^0$.

By a suitable choice of coördinates at a point $q_1^0, \cdots, q_m^0$, *we can reduce the expressions for* $Q_1, \cdots, Q_m$ *to the simple form*

$$Q_i = q_i'' + b_i \qquad (i = 1, \cdots, m)$$

at that point.

To establish this fact we assume that $q_1^0, \cdots, q_m^0$ is at the origin, and make a first linear transformation

$$q_i = \sum_{j=1}^{m} \beta_{ij}\, \bar{q}_j \qquad (i = 1, \cdots, m),$$

where the β_{ij} are constants with $|\beta_{ij}| \neq 0$. For the functions $\bar{Q}_i$ we have then

$$\bar{Q}_i = \sum_{j=1}^{m} Q_j \frac{\partial q_j}{\partial \bar{q}_i} = \sum_{j=1}^{m} Q_j\, \beta_{ji}.$$

Hence we find by substitution for every i.

$$\bar{Q}_i = \sum_{j,k,l=1}^{m} a_{jk}\, \beta_{ji}\, \beta_{kl}\, \bar{q}_l'' + \text{terms independent of } q_1'', \cdots, q_m''.$$

It follows that if we choose our transformation so as to transform the quadratic form

$$\sum_{j,k=1}^{m} a_{jk}\, q_j\, q_k$$

into the sum of squares

$$q_1^2 + \cdots + q_m^2,$$

the quantities $\bar{a}_{ij}$ will be 0 for $i \neq j$ and 1 for $i = j$. Consequently it is at least legitimate to assume that a_{ij} has been reduced to δ_{ij} at the origin by this preliminary transformation, so that we have

$$Q_i = q_i'' + \sum_{j,k=1}^{m} b_{ijk}^0 q_j' q_k' + b_i^0 \qquad (i = 1, \cdots, m)$$

at the origin.

Now suppose that we write further

$$\bar{q}_i = q_i + \frac{1}{2} \sum_{j,k=1}^{m} b_{ijk}^0 q_j q_k$$

where the constants b_{ijk}^0 have the values specified in the equations above. We find by differentiation that the equations

$$\bar{q}_i' = q_i', \quad \bar{q}_i'' = q_i'' + \sum_{j,k=1}^{m} b_{ijk}^0 q_j' q_k' \qquad (i = 1, \cdots, m)$$

hold at the origin.

It is then found at once that in these variables $\bar{q}_i$, the formula for $\bar{Q}_i$ has the stated form at the origin.

V. Conservation of Energy. *The dynamical system is conservative.*

If W is the work function we have the fundamental relation

$$d W = \sum_{j=1}^{m} Q_j q_j' \, d t,$$

characteristic of conservative systems. But the sum on the right is linear in the accelerations, and comparing the coefficients of q_j'' obtained by employing the form for Q_i above we find

$$\partial W / \partial q_j' = \sum_{i=1}^{m} a_{ij} q_i' \qquad (j = 1, \cdots, m),$$

whence

$$W = T + U$$

where

$$T = \frac{1}{2} \sum_{j,k=1}^{m} a_{jk} q_j' q_k',$$

and where U is a function of $q_1, \cdots, q_m$ only.

Employing this more specific form for W, we have of course $L = T - U$ according to the earlier developments. and thence

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) + \frac{\partial U}{\partial q_i} + R_i \qquad (i = 1, \cdots, m)$$

where R_i satisfy (7). But the first two terms on the right yield expressions just like those for those derived above for Q_i, the terms in q_j'' being identical. It follows that we must have the differences R_i of the form

$$R_i = \sum_{j,k=1}^{m} c_{ijk}\, q_j' q_k' + c_i \qquad (i = 1, \cdots, m).$$

Applying now the above condition (7) we conclude further that for all i, j, k the relations

$$c_{ijk} + c_{jki} + c_{kij} = 0, \qquad c_i = 0$$

must obtain where $c_{ijk} = c_{ikj}$ also of course.

Hence principles I–V lead to the type of external forces,

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) + \frac{\partial U}{\partial q_i} + \sum_{j,k=1}^{m} c_{ijk}\, q_j' q_k',$$

where the $c_{ijk} = c_{ikj}$ are functions of the coördinates such that for all i, j, k

$$c_{ijk} + c_{jki} + c_{kij} = 0.$$

It remains to specify a final condition, as simple as possible, which will allow us to conclude $c_{ijk} = 0$ for all i, j, k.

VI. *If by a particular choice of coördinates, the kinetic energy T is made stationary in $q_1, \cdots, q_m$ at a certain point $q_1^0, \cdots, q_m^0$, then the forces Q_i yield accelerations which are independent of the velocities.*

Suppose for a moment that such a stationary T exists, so that we have at $q_1^0, \cdots, q_m^0$

$$\partial a_{ij}/\partial q_k = 0 \qquad (i, j, k = 1, \cdots, m).$$

The form for Q_i at this point becomes

$$Q_i = \sum_{j=1}^{m} a_{ij} q_j'' + \frac{\partial U}{\partial q_i} + \sum_{j,k=1}^{m} c_{ijk} q_j' q_k'.$$

It is to be observed that the form of Q_i employed holds for any coördinate system. Now if these forces Q_i are to be independent of the velocities, we must have $c_{ijk} = 0$ for all i, j, k in the special coördinate system and thus in the most general system by the known law of transformation of the terms R_i. Consequently the desired Lagrangian form of external forces is obtained.

The hypothesis that a stationary T exists is justified by the well-known fact that for any coördinates of geodesic type at $q_1^0, \cdots, q_m^0$, the surface element ds where

$$ds^2 = \sum_{i,j=1}^{m} a_{ij}\, dq_i\, dq_j$$

has coefficients a_{ij} stationary at this point.

Conversely, it is readily seen that a regular Lagrangian system has external forces Q_i which satisfy I–VI.

11. **Dissipative systems.** Conservative systems are often limiting cases of what is found in nature, since actual work is usually done on the system during a closed cycle. A system for which work is done may be called dissipative. More explicitly we shall define dissipative systems to be such that

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) - \frac{\partial L}{\partial q_i} + R_i \quad (i = 1, \cdots, m)$$

where

$$\sum_{j=1}^{m} R_j q_j' \geqq 0.$$

Furthermore we shall assume that the equality sign can only hold for motions in a manifold of dimensionality less than m in the m-dimensional coördinate space.

Suppose now that such a system is unconstrained, or at least is subject to external forces which do no work, so that

$$\sum_{j=1}^{m} Q_j q_j' = 0.$$

Because of the obvious relation

$$\frac{dW}{dt} + \sum_{j=1}^{m} R_j q'_j = 0,$$

where W denotes the work function associated with L, namely

$$\sum_{j=1}^{m} \left(q'_j \frac{\partial L}{\partial q'_j}\right) - L,$$

we infer that W constantly diminishes toward some limiting value W_0. It is assumed that the work function cannot diminish to $-\infty$.

Now consider the limiting motions of the given motion. Along these motions W has this limiting value W_0, and of course the sum

$$\sum_{j=1}^{m} R_j q'_j$$

vanishes.

A dissipative system of this type tends in its unconstrained motion either toward equilibrium or, more generally, toward the motion of a conservative system with fewer degrees of freedom.