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Dynamical Systems

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INTRODUCTION TO THE 1966 EDITION

Many mathematicians will welcome the new edition of G. D. Birkhoff's book on Dynamical Systems. It represents essentially a continuation of Poincaré's profound and extensive work on Celestial Mechanics. Altogether Birkhoff was strongly influenced by Poincaré and devoted a major part of his mathematical work to subjects arising from Poincaré's tradition. The present book contains Birkhoff's views and ideas of his earlier period of life—it appeared when Birkhoff was 43.

To the modern reader the style of this book may appear less formal and rigorous than it is now customary. But just the informal and lively manner of writing has been inspiring to many mathematicians. The effect of this inspiration is visible in a number of later papers. For example, Morse's theory on geodesics on a closed manifold originated directly in Birkhoff's ideas in dynamical systems. The recent work by Anosov on *U*-Systems answers the question of ergodicity and density of periodic solutions for a wide class of differential equations—a problem which in Birkhoff's book was studied for a single model system. These and other examples (given below) justify the hope that the reprinting of this book again will stimulate further progress.

Of course, after nearly 40 years a number of statements are outdated. For this reason I have selected a list of references to pertinent literature after 1927. Naturally, the list cannot be complete and further references can be found in the books and survey articles quoted. In the Addendum section at the back of this book I added some general remarks to various chapters. More specific comments are supplied as footnotes at the end of the book. References to these footnotes are given as small numbers in the margin of the text.*

JURGEN MOSER

*I am indebted to Dr. R. Sacker for his assistance.

A PREFACE TO THE 1966 EDITION

I met George Birkhoff in 1914, the year after he published his "Proof of Poincaré's Geometric Theorem" [1]. In a paper [2] in 1912 Poincaré had enunciated a theorem of great importance for the restricted problem of three bodies, but had succeeded in treating only a variety of special cases after long efforts. Poincaré had also referred to this theorem in lectures in Göttingen. Birkhoffff formulated this theorem in [1] as follows.

"Let us suppose that a continuous one-to-one transformation T takes the ring R formed by concentric circles C_a and C_b of radii a and b respectively ($a > b > 0$) into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in a negative sense and at the same time preserve areas. Then there *are at least two invariant points*."

Birkhoff's proof of this theorem in 1913 was one of the most exciting mathematical events of the era and was widely acclaimed.

In 1912 in [3] Birkhoff outlined the conjecture of Poincaré that the "general" motion in dynamics in phase space was of the so-called "discontinuous type". It was in this paper that Birkhoff introduced his novel and beautiful conceptions of "minimal" or equivalently "recurrent" sets of motions. A minimal non-periodic set of motions is of "discontinuous" type if in phase space it possesses no subsets which are continua except arcs of motions. Birkhoff asked me in 1915 to examine the conjecture of Poincaré. My answer was by way of a "symbolic dynamics". A non-periodic recurrent symbol introduced at that time was discovered independently by a Russian in 1934 and used by Novikov in his disproof of the Frobenius-Burnside conjecture in group theory.

The problem of the generality of "non-degenerate" periodic orbits, that is orbits whose equations of variation admit no nontrivial periodic solutions was often emphasized by Birkhoff. It is now of major interest to differential topologists of whom Birkhoff and Poincaré were among the first, although not so called. The work of Moser, Arnol′d and others on stability belongs to a related field, one close to the center of Birkhoff's interest.

The above merely samples the many aspects of dynamics in which Birkhoff introduced new ideas, new theorems and new questions. One of Birkhoff's theorems which has aroused the greatest interest was his so-called "Ergodic Theorem" with its subsequent variations, interpretations and consequences in measure theory and probability.

Birkhoff once remarked that "it is fortunate that the world of mathematics is as large as it is". Only by bringing to bear the genius and imagination of many lands and intellectual origins can one be sure of adequate appreciation of new mathematical ideas and significant responses.

History has responded to these pages on Dynamical Systems in an unmistakable way. For this we are more than content.

References

1. Birkhoff, George, *Proof of Poincaré's geometric theorem*, Trans. Amer. Math. Soc. **14**(1913), 14-22.

2. Poincaré, H., *Sur un théorème de géométrie*, Rend. del Cir. Mat. di Palermo **33**(1912), 375-407.

3. Birkhoff, George, *Quelques théorèmes sur le mouvement des systèmes dynamiques*, Bull. Soc. Math. France **40**(1912), 305-323.

MARSTON MORSE

PREFACE TO THE 1927 EDITION

The Colloquium Lectures which I had the privilege of delivering at the University of Chicago before the American Mathematical Society, September 5-8, 1920, contained a large part of the material presented in the following pages. The delay in publication has been due to several causes, one of which has been my desire to wait until some of my own ideas had developed further. I have taken advantage of a well-established tradition of our Colloquia by giving particular emphasis to my own researches on dynamical systems. It is my earnest hope that the lectures may serve to stimulate others to investigate the outstanding problems in this most fascinating field.

It is only necessary to recall the work of Galileo, Newton, Laplace, Clausius, Rayleigh in the physical applications of dynamics, of Lagrange, W. R. Hamilton, Jacobi in its formal development, and of Hill and Poincaré in the qualitative treatment of dynamical questions, in order to realize the remarkable significance of dynamics in the past for scientific thought. At a time when no physical theory can properly be termed fundamental—the known theories appear to be merely more or less fundamental in certain directions—it may be asserted with confidence that ordinary differential equations in the real domain, and particularly equations of dynamical origin, will continue to hold a position of the highest importance.

In looking back over my own dynamical work, of which a certain period is finished with the publication of this book, I cannot but express my feeling of deep admiration and gratitude to Hadamard, Levi-Civita, Sundman and Whittaker, to whom many important recent advances in theoretical dynamics are due, and in whose work I have found especial inspiration. It is with much regret that I have been unable to give adequate space to their achievements.

Professor Philip Franklin coöperated with me in a first rewriting of part of my notes on these lectures. I owe him cordial thanks for his help.

November 18, 1927.

GEORGE D. BIRKHOFF.

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