Differential Geometry in General Relativity

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1 Introduction

General relativity is often regarded as one of the most elegant theories in physics. To formulate this theory of gravitation mathematically, we require tools from differential geometry. In particular, we rely on the theory of smooth manifolds, Lorentzian metrics, connections, and curvature. In this short essay, we introduce the mathematical structures underlying general relativity and illustrate their physical relevance by discussing important exact solutions of the Einstein Field Equations.

2 Spacetime as a Manifold

In general relativity, spacetime is modeled by a four-dimensional, connected, smooth pseudo-Riemannian manifold (M, g) with signature $(-, +, +, +)$. A metric with such a signature is called a Lorentzian metric.

Definition 2.1 (pseudo-Riemannian manifold)**.** A *pseudo-Riemannian manifold* is a connected and smooth manifold M of dimension n equipped with a smooth, symmetric and non-degenerate $(0, 2)$-tensor g, called the metric tensor. A *Lorentzian manifold* is a pseudo-Riemannian manifold with signature $(-, +, +, +)$.

Definition 2.2 (metric tensor)**.** Let M be a smooth manifold of dimension n. Then the function $g_p : T_pM \times T_pM \to \mathbb{R}$ defines a *metric tensor* at point $p \in M$ if it is bilinear, symmetric, and non-degenerate. The collection of all g_p defines a smooth $(0, 2)$-tensor field g on M.

The metric tensor is the fundamental object describing the local geometry of spacetime. It determines time, distances, and angles.

3 Connections and Geodesics

To differentiate vector fields on curved manifolds, we introduce affine connections.

Definition 3.1 (affine connection)**.** We call a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ with $(X, Y) \mapsto \nabla_X Y$ on a differentiable manifold M an *affine connection* if it satisfies the following properties:

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

where $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

Theorem 3.2 (Levi-Civita)**.** Let (M, g) be a (pseudo-)Riemannian manifold. Then there exists a unique affine connection ∇ on M such that:

1. $\nabla g = 0$, i.e. the connection preserves the metric.
2. It is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ for vector fields X and Y.

This connection is called the Levi-Civita connection. It is explicitly given by the Koszul formula

$$g(\nabla_X Y, Z) = \frac{1}{2}\{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)\}$$

[dC92].

The Levi-Civita connection allows us to define geodesics, which generalize the notion of straight lines to curved spacetime.

Definition 3.3 (geodesics)**.** A smooth curve $c(t)$ is called a *geodesic* if its tangent vector is parallel transported along itself, i.e. $\nabla_{\dot{c}}\dot{c} = 0$.

Remark. In local coordinates the geodesic equation can be written as

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}(x(t))\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0,$$

where $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$ are the Christoffel symbols.

In general relativity geodesics describe the motion of free falling particles.

4 Curvature and Einstein Field Equations

Gravitational phenomena are described by the curvature of spacetime. This curvature is described mathematically by the Riemann curvature tensor.

Definition 4.1 (Riemann Curvature tensor)**.** Let ∇ be the Levi Civita connection. Then the *curvature tensor* is a $(3,1)$ tensor defined as

$$R(X,Y)Z \coloneqq \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Definition 4.2 (Ricci and Scalar Curvature)**.** From the Riemann tensor, we obtain the *Ricci curvature* by taking a trace

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y).$$

The *scalar curvature* is defined as $R = g^{\mu\nu}R_{\mu\nu}$, where $R_{\mu\nu}$ are the components of the Ricci tensor in coordinate basis, i.e. $R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$.

These quantities appear in the following central equation of general relativity.

Theorem 4.3 (Einstein Field Equations)**.** The curvature of spacetime is related to the energy-momentum via the system of second-order partial differential equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, G the gravitational constant and $T_{\mu\nu}$ the stress-energy tensor describing matter and energy [Wal84].

The differential Bianchi Identity

$$\nabla^\mu G_{\mu\nu} = 0$$

together with (1) implies the local conservation of energy and momentum

$$\nabla^\mu T_{\mu\nu} = 0.$$

Thus, general relativity is consistent with the local conservation of energy and momentum.

4.1 Exact solutions to the Einstein Field Equations

The solutions to the Einstein Field Equations are metrics in spacetime. Since the equations are nonlinear it is very difficult to solve them completely without making approximations. Exact solutions are therefore rare but physically very significant.

4.1.1 Schwarzschild solution

The Schwarzschild solution describes the gravitational field outside a spherical mass. Assuming a static, spherically symmetric vacuum spacetime ($T_{\mu\nu} = 0$), the unique solution to the equations (1) with $G = 1$ is the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = (d\Theta^2 + \sin^2\Theta d\phi^2)$ [Wal84]. The Schwarzschild solution describes spacetime outside a non rotating, uncharged mass and is the mathematical foundation of the theory of black holes.

4.1.2 The Friedmann-Lemaître-Robertson-Walker metric

Assuming homogeneity and isotropy of space, we obtain the Friedmann-Lemaître-Robertson-Walker (FLRW) metric as a solution to the equations (1)

$$ds^2 = -dt^2 + a(t)^2(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2),$$

where $a(t)$ is a scalar factor and $k \in \{-1, 0, 1\}$ determines the curvature [Wal84]. The FLRW metric forms the geometric basis of modern cosmology. Substituting it into the Einstein Field Equations yields the Friedmann equations, which model an expanding universe and describe phenomena such as cosmic expansion and the Big Bang.

References

[dC92] Manfredo Perdigão do Carmo. *Riemannian Geometry.* Birkhäuser, 1992.

[Wal84] Robert M. Wald. *General Relativity.* The University of Chicago Press, 1984.