Young Diagrams and Tensors: The Particle Physics Dream Team

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Symmetries play a central role in particle physics, often described by groups like $SU(N)$, whose irreducible representations (irreps) are key to understanding particle interactions. To study the interplay of these representations, we use tensors and their tensor products. This essay examines the connection between the irreps of $SU(N)$ and the symmetric group S_k through Schur-Weyl duality. Leveraging this duality, we utilize Young Tableaux as a visual and computational framework to efficiently decompose tensor products into irreps. This approach not only provides insight into the mathematical structure of $SU(N)$ but also has profound applications in Quantum Chromodynamics (QCD), such as describing the color confinement of quarks and the formation of hadrons. The synergy between abstract group theory and particle physics exemplifies the power of representation theory in advancing our understanding of fundamental particles.

The study of irreducible representations (irreps) of Lie groups, such as $SU(N)$[1] is crucial in physics, because they provide a basis for the global and gauge symmetries, used to constrain models, such as the Standard Model.
This essay explores the irreducible representations (irreps) of $SU(N)$ within the framework of tensor representations, examining the group's action on tensor spaces. Representations of a group are associated with an action in a vector space: for $SU(N)$, we define $V = \mathbb{C}^N$ (standard vector space), which corresponds to the fundamental representation (N), the simplest irrep. While the fundamental representation describes how vectors transform under $SU(N)$ action, there is a conjugate representation describing how the covectors transform. These transformations influence elements of the vector space in the following ways:

Fundamental Representation [2] (N): $\psi'^a = U^a_b \psi^b$;

Conjugate Representation $(\overline{N})$: $\psi'_a = U^{*b}_a \psi_b$, $U \in SU(N)$.

Note that, a lower (higher) index refers to the components of an element of the vector space transforming under the conjugate (fundamental) representation. This is not coincidental since the fundamental representation describes how contravariant tensors transform under the group action, while the conjugate representation describes how covariant tensors transform. These two representations are the building blocks of higher irreps of $SU(N)$, where the representation acts on mixed tensors of type (p,q).These are constructed through tensor products,

$$N^p \times \overline{N}^q \text{ Representation}$$
$$\psi'^{a_1 \ldots a_p}_{b_1 \ldots b_q} = \Pi^p_{i=1} U^{a_i}_{c_i} \Pi^q_{j=1} U^{*\,d_j}_{b_j} \psi^{c_1 \ldots c_p}_{d_1 \ldots d_q}.$$

SCHUR-WEYL DUALITY

There is a deep relationship between $SU(N)$ and the permutation group S_N, reflected in the Schur-Weyl Duality [1]. Analysing the action of each on V^N [3] tensor space: while S_N permutes the indices in the tensor (changes the position of the vectors in the tensor product), $SU(N)$ acts on each individual copy of V[4]:

$$\sigma \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_N) = (v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \ldots \otimes v_{\sigma^{-1}(N)});$$
$$U \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_N) = (Uv_1) \otimes (Uv_2) \otimes \ldots \otimes (Uv_N).$$

The key property is that the actions of $SU(N)$ and S_N on the tensor space commute, which is clear since $SU(N)$ acts linearly on each factor of the tensor product, while S_N permutes the factors themselves. Hence, these operations affect disjoint aspects of the tensor product. Since, S_k, with $k \le N$, is a subgroup of S_N, acting on k-element subsets while fixing the remaining elements, its action also commutates with the $SU(N)$ action, when we now act on V^k.
Using this property, the Schur-Weyl Duality asserts that the commutativity means that V can be composed as a direct sum of $SU(N)$ and S_k irreps:

$$V^k \cong \bigoplus_D \pi^D_k \otimes \rho^D_N,$$

which then are paired under the commutating action. Here the index D runs over Young Diagrams with k boxes and at most N rows, π^D_k is an S_k irrep and ρ^D_N is an $SU(N)$ irrep. It provides a bridge between the representation theory of the two groups organizing the complex structure of V^k in terms of irreducible components associated with Young diagrams. The Young diagram D encodes the symmetrization of indices corresponding to the rows and the antisymmetrization of indices corresponding to the columns in the tensor product. Each diagram D corresponds to a unique irrep of $SU(N)$ (S_k) with the same symmetrization/antisymmetrization in the $SU(N)$ (k-factor) indices. The pairing between irreps is complete and exclusive, therefore, the Schur-Weyl Duality ensures a bijection between the irreps of S_k and $SU(N)$ appearing in the decomposition of V^k, where $k \le N$. This correspondence organizes the tensor product structure by mapping $SU(N)$ irreps to Young tableaux, with rules for permutating tensor indices.

YOUNG TABLEAUX FOR SU(N)

In the same way, as for S_N, irreducible representations of $SU(N)$ can be described using Young tableaux, which encode the symmetry properties of tensors under $SU(N)$ transformations.
Consider a tensor $\psi^{a_1 \cdots a_p}$ transforming under $SU(N)$. A partition $\lambda = (p_1, \ldots, p_s)$, with $p_1 \ge p_2 \ge \cdots \ge p_s$ and $\sum_{i=1}^s p_i = p$, determines how the tensor indices are symmetrized and antisymmetrized. Specifically, indices in the same row of the Young tableau (corresponding to p_i boxes) are symmetric, while indices in the same column are antisymmetric.

Young Tableau validity Rules:

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[1] $SU(N) = \{A \in GL_N(\mathbb{C}) : A^* A = I_N, \det(A) = 1\}$
[2] For the whole essay we assume Einstein convention, that is, repeated indices sum.
[3] $V^N \equiv \bigotimes_{i=1}^N \mathbb{C}^N$
[4] $\sigma \in S_N$ and $U \in SU(N)$

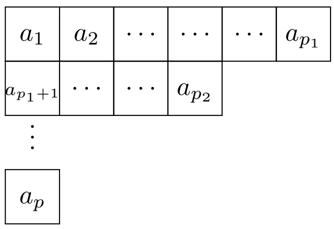
1. Shape: The number of boxes, which are left-justified, corresponds to the rank of the tensor (p indices), and the partition λ determines the symmetry/antisymmetry properties.

2. Row constraints: Symmetric combinations correspond to boxes in the same row.

3. Column constraints: Antisymmetric combinations correspond to boxes in the same column.

4. Maximum rows: The number of rows cannot exceed N because an antisymmetric tensor with $N+1$ indices in $SU(N)$ vanishes (determinant condition).

The corresponding Young tableau visually represents the partition, with each row containing p_i boxes, as follows:



In this visual way, the Young Diagram of the conjugate representation is obtained by: first replacing the j boxes on each column with $N-j$ boxes and then flipping the diagram around the vertical axis[5].

Compute an $SU(N)$ irrep dimension

In order to obtain the dimension of an irrep (d) based on a diagram, we apply the Hook Law, which claims that $d = F/h$. In this context, a hook is a line from the right that passes through the centre of some boxes before turning down, and exiting at the bottom, the box it turns down in is the box associated with that hook. From this definition, we see that there is one hook per box in any Young Tableau, so we assign to each box a the number of boxes the hook passes through. The product of these numbers is the hook factor, h. Then, to obtain F, we need to assign a number to each box: we assign N to the upper left box, and when going to the right increase by one, while when going down decrease by one. After this, F is just the product of these numbers. This process is represented in fig. 1. Note that, if the Young Diagram is a column with N boxes, then $h = N!$ and $f = N!$, so $d = 1$, therefore this Young Tableau always represents the column singlet in $SU(N)$, as seen in fig. 2

TENSOR PRODUCTS

The study of Young Tableaux is particularly useful in decomposing tensor products of $SU(N)$ representations into a sum of irreps. To do that, we need to follow some rules.

1. Choose one Young Tableaux (usually the smallest) and label the elements of each row by letters - 'a' for 1st row; 'b' for 2nd row and so on.

2. Stack the 'a' boxes of the labelled Young Tableaux on the other Young Tableaux, in all the ways that still result in a valid Young Tableaux.

3. Repeat the same procedure with 'b' boxes and so on.

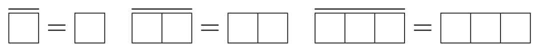
4. Read the letters on each diagram starting on the upper line from right to left. If at any point along the process, there are more 'b' than 'a', 'c' than 'b', or so on, delete the diagram.

5. Remove columns with N boxes (singlets).

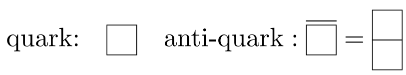
The diagrams with the columns that remain after this process are the irreps of the tensor product. Then, we use the hook law to get the dimensions of each one. An example of a tensor product computation can be seen in fig. 3.

APPLICATIONS OF THE TENSOR PRODUCT

The presented concepts are of extreme relevance in particle physics. In fact, the conjugate representation of $SU(2)$ is identical to the representation itself, as shown by the following visual equivalence:



In particle physics, the irreducible representations (irreps) of $SU(2)$ correspond to different spin states, labeled by half-integer quantum numbers j. This self-conjugacy arises because $SU(2)$ is a real (or pseudo-real) group, meaning a particle and its antiparticle (or conjugate) transform identically under $SU(2)$ rotations. For example, a fermion with spin $\frac{1}{2}$ and its antiparticle both transform according to the same spin-$\frac{1}{2}$ irrep of $SU(2)$. This symmetry reflects the fact that spin, as a fundamental property, does not distinguish between particles and antiparticles in terms of their rotational behavior.
Another very relevant group in the Quantum Chromodynamics(QCD) [2] is the $SU(3)$ group. This group appears when we add the colour charge, with 3 possible values, and impose colour charge conservation. According to $SU(3)$ action on the particles, quarks are associated with triplet irrep (fundamental), while anti-quarks are associated with the anti-triplet irrep (conjugate).



Knowing this we can combine quarks to get more complex structures, however, because of color confinement we can only observe color singlets. Therefore, we will look for combinations (tensor products) of $\mathbf{3}$ and $\overline{\mathbf{3}}$ that result in $\mathbf{1}$ in $SU(3)$. Combining a color triplet and anti-triplet: $\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1}\oplus\mathbf{8}$, results in a colour singlet (mesons) and an element of the adjoint representation, the gluon, which contains two colours. We can also combine three color triplets: $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}$, which results in a colour singlet (baryons). The details of these computations are in fig. 4.
More generally, in QCD we define flavour symmetries associated with the group $SU(N_f)$, where N_f is the number of quark flavours considered. In this, we can also combine irreps to get composite flavour hadrons and arrange them in multiples resulting from the tensor products, with now every combination being valid.
In summary, by establishing a relation between S_k and $SU(N)$ irreps, we computed tensor products that are of extreme relevance for particle physics.

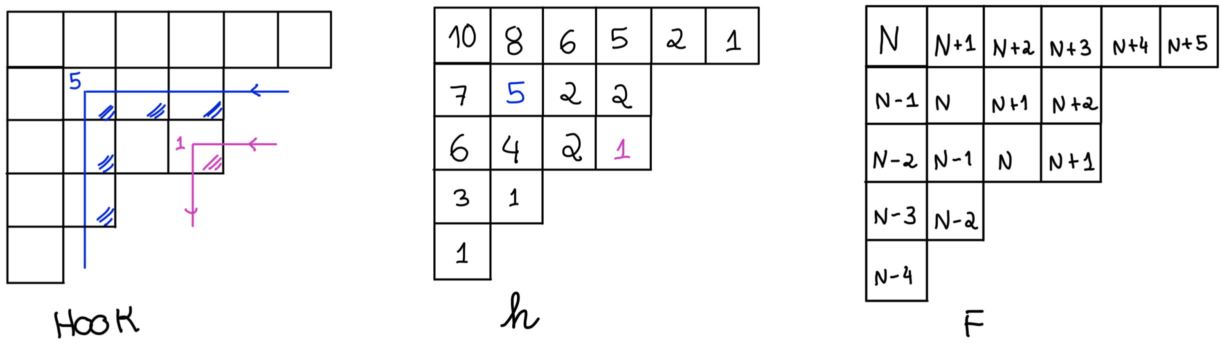


FIG. 1: Example of applying the hook law. On the left, we have an example of two hooks with the respective numbers inside each box. In the middle, we present these numbers for all boxes (h = $10*8*7*6^2*5^2*4*3*2^4*1^4$). On the right, we have the setup for F computation, whose value depends on N (F = $(N+5)(N+4)(N+3)(N+2)^2(N+1)^3N^3(N-1)^2(N-2)^2(N-3)(N-4)$).Note that N has a minimum value.

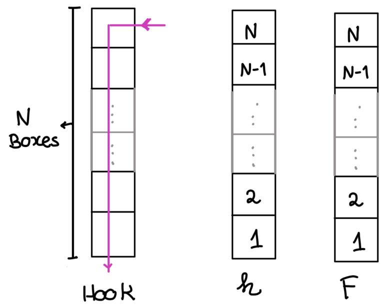


FIG. 2: Singlet Representation, and dimension computation setup, with h = F = $N!$

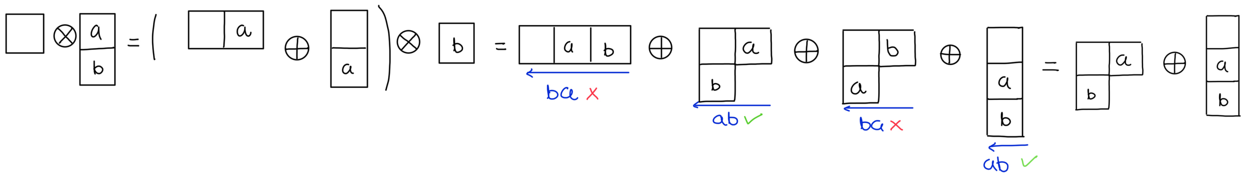


FIG. 3: Example of tensor product visual computation.

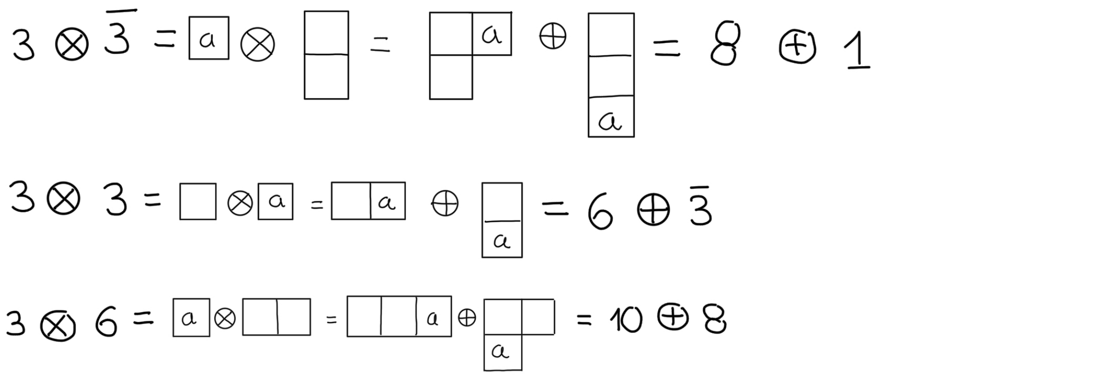


FIG. 4: Some tensor products relevant for QCD.

[5] The conjugate representation is different depending on the value of N.

REFERENCES

[1] Zhijie Dong and Haitao Ma. "Application of Schur-Weyl duality to Springer theory". In: *arXiv preprint arXiv:2108.12962* (2021).

[2] Gernot Eichmann. *Appendix A SU(N)*. http://cftp.ist.utl.pt/~gernot.eichmann/2020-QCDHP/App-SU(N).pdf. Aug. 2020.