Representations of $\mathfrak{sl}(2)$ and Clebsch-Gordan decomposition

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Abstract
This essay explores the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, with an emphasis on the Clebsch-Gordan decomposition. The structure of $\mathfrak{sl}(2,\mathbb{C})$ is introduced, highlighting the role of highest weight vectors and irreducible representations. The Clebsch-Gordan formula is derived, offering a method for decomposing tensor products of representations. The relevance of these concepts to quantum mechanics is presented, particularly in the context of spin-$\frac{1}{2}$ particle coupling, showcasing their importance in both mathematics and physics.

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1 Representations of $\mathfrak{sl}(2)$

$\mathfrak{sl}(2)$ is the Lie algebra of traceless 2x2 complex matrices.

1.1 Representation theory of $\mathfrak{sl}(2)$ and the highest weight vector

A convenient basis for $\mathfrak{sl}_2(\mathbb{C})$ is

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with the following relations,

$$[h,x] = 2x, \quad [h,y] = -2y, \quad [x,y] = h$$

Let V be a representation of $\mathfrak{sl}$(2,$\mathbb{C}$). Since $\mathbb{C}$ is algebraically closed, an element in V must have some eigenvector v with eigenvalue $\lambda \in \mathbb{C} \implies h \cdot (x \cdot v) = [h,x] \cdot v + x \cdot h \cdot v$ $= 2x \cdot v + x \cdot \lambda v = (\lambda + 2)(x \cdot v)$

We can see that xv will also be an eigenvector for h with eigenvalue $\lambda + 2$. Analogously for xy,
$h \cdot (y \cdot v) = [h,y] \cdot v + y \cdot h \cdot v = -2y \cdot v + y \cdot \lambda v = (\lambda - 2)(y \cdot v)$;

Starting with an eigenvector v and considering the span of the set $v, xv, ..., x^n v, ...$, given that V is finite dimensional, there is a maximum integer k such that $v, xv, ..., x^k v$ is linearly independent.
Eigenvectors with different eigenvalues are linearly independent $\implies x^{k+1} \cdot v = 0$. The vectors $x^k v$ will be important for the definitions to follow.

Definition 1.1 Let V be a representation of $\mathfrak{sl}$(2,$\mathbb{C}$). An element $v \in V$ is said to be a **highest weight vector** if v is an eigenvector for h and $xv = 0$. The weight of an eigenvector for h in a representation of $\mathfrak{sl}$(2,$\mathbb{C}$) is the corresponding eigenvalue. The eigenspaces for h will be called weight spaces.

Proposition 1.1 Let $v_0 \in V$ be a highest weight vector with weight λ. Define for $i > -1$

$$v_i = \frac{1}{i!} y^i \cdot v_0$$

where $v_{-1} = 0$. Then

$$\begin{aligned} (i)\ y \text{ raises the index:} \quad & y \cdot v_i = (i+1)v_{i+1}, \\ (ii)\ h \text{ acts diagonally:} \quad & h \cdot v_i = (\lambda - 2i)v_i, \\ (iii)\ x \text{ lowers the index:} \quad & x \cdot v_i = (\lambda - i + 1)v_{i-1}. \end{aligned}$$

The previous proposition suggests the highest weight λ must be a non-negative integer. The same way there is a highest weight vector, there must also be a lowest weight vector. If we take $l \geq 1$, we will have $v_l = 0$; Using (iii) of Proposition 1.1 we obtain $\lambda = l - 1$. The representation of $\mathfrak{sl}(2)$ defined by the equations in Proposition 1.1 is denoted by

$$V(l) = \langle v_0, ..., v_l \rangle$$

The action of the basis elements of $\mathfrak{sl}$(2,$\mathbb{C}$) on $V(l)$ is given by

$$h = \begin{bmatrix} l & & & \\ & l-2 & & \\ & & \ddots & \\ & & & -l \end{bmatrix}, \quad x = \begin{bmatrix} 0 & l & & & \\ & 0 & l-1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix},$$

$$y = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & l & 0 \end{bmatrix}$$

The representations $V(l)$ are irreducible, since, given any non-zero vector v in $V(l)$, we can write $x^k v$ as being the highest weight vector, for an appropriate k. If we act on $x^k v$ with y, which lowers indices, since $yv_k = (k+1)v_{k+1}$, for $k = 0, 1, ..., l-1$, we see that the action of $\mathfrak{sl}(2,\mathbb{C})$ on v generates the entire $V(l)$.

By the argument explained above, if V is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ it will contain a highest weight vector; Being irreducible means that it must equal the representation spanned by the highest weight vector, which proves the following:

Theorem 1.1 The irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are the representations $V(\ell)$ for each non-negative integer ℓ. An arbitrary finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to a unique representation of the form

$$V(n_1) \oplus V(n_2) \oplus \cdots \oplus V(n_k),$$

hence isomorphism classes of representations of $\mathfrak{sl}(2,\mathbb{C})$ are parametrized by tuples of non-negative integers.

2 The Clebsch-Gordan decomposition

2.1 Small Example

In order to apply the concepts presented in the previous section, we will start with the following example. It is safe to say that $V(0)$ is the trivial one-dimensional representation and $V(1)$ is clearly the defining representation of $\mathfrak{sl}(2,\mathbb{C})$, $\mathbb{C}^2$.

As a starting point, consider we want to decompose the tensor product $V(3) \otimes V(7)$ into the sum of irreducible representations: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$.

If we associate the basis u_0, u_1, u_2, u_3 with $V(3)$ and $v_0, ..., v_7$ with $V(7)$, then $M = V(3) \otimes V(7)$ has a basis $u_i \otimes v_j$ with $i = 0, ..., 3, j = 0, ..., 7$.

Now, recalling properties $(i), (ii)$ and (iii) from Proposition 1.1,

$$\begin{aligned} h \cdot (u_i \otimes v_j) &= (h \cdot u_i) \otimes v_j + u_i \otimes (h \cdot v_j) \\ &= (10 - 2(i+j))u_i \otimes v_j \end{aligned}$$

$$M_{10-2(i+j)} := \mathrm{span}\{u_i \otimes v_j\}$$

In M_{10}, $x \cdot (u_0 \otimes v_0) = 0$, so $u_0 \otimes v_0$ is a maximum weight vector with weight 10. Similarly for M_8, M_6 and M_4.

So, $V(4) \oplus V(6) \oplus V(8) \oplus V(10) \subseteq M$. But, since dim $V(l) = l + 1$, dim $M = 4 * 8 = 32$ and dim $V(4)$ + dim $V(6)$ + dim $V(8)$ + dim $V(10) = 5 + 7 + 9 + 11 = 32$. This means $M = V(4) \oplus V(6) \oplus V(8) \oplus V(10)$.

2.2 General Formula

We now want to generalize the previous example for $M = V(m) \otimes V(n)$. We suppose $m \leq n$. The basis for $V(m)$ is $u_i, i = 0, ..., m$ and $v_j, j = 0, ..., n$ for $V(n)$.

Just as for the example, we apply h to $(u_i \otimes v_j)$ which leads to $(m + n - 2(i+j))u_i \otimes v_j$

$$\text{So, } M_{m+n-2(i+j)} = \mathrm{span}\{u_i \otimes v_j\}$$

For $k = i + j = 0, \ldots, m$, suppose $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i} \in M_{m+n-2k}$ is a highest weight vector. Then

$$x \cdot w = \sum_{i=0}^{k} \left(\lambda_i (x \cdot u_i) \otimes v_{k-i} + u_i \otimes (x \cdot v_{k-i})\right)$$

$$= \sum_{i=0}^{k} \lambda_i (m-i+1) u_{i-1} \otimes v_{k-i} + \sum_{i=0}^{k} \lambda_i (n-k+i+1) u_i \otimes v_{k-i-1}$$

$$= \sum_{i=1}^{k} \left(\lambda_i (m-i+1) + \lambda_{i-1}(n-k+i)\right) u_{i-1} \otimes v_{k-i} = 0$$

Therefore

$$\lambda_i (m-i+1) + \lambda_{i-1}(n-k+i) = 0$$

We conclude that

$$\lambda_i = (-1)^i \frac{(n-k+i)!}{(n-k)!} \cdot \frac{(m-i)!}{m!} \lambda_0$$

Let $\lambda_0 = 1$, then $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i} = \sum_{i=0}^{k} (-1)^i \frac{(n-k+i)!}{(n-k)!} \cdot \frac{(m-i)!}{m!} u_i \otimes v_{k-i}$ is a highest weight vector with weight $m + n - 2k$.

So

$$\bigoplus_{k=0}^{m} V(m+n-2k) \subseteq V(m) \otimes V(n).$$

Comparing the dimension of the two sides,

$$\begin{aligned} \dim \left(\bigoplus_{k=0}^{m} V(m+n-2k) \right) &= \sum_{k=0}^{m} (m+n-2k+1) \\ &= (m+1)(m+n+1) - m(m+1) \\ &= (m+1)(n+1) = \dim V(m) \otimes V(n) \end{aligned}$$

Therefore,

$$V(m) \otimes V(n) = \bigoplus_{k=0}^{m} V(m+n-2k).$$

So we have,

$$V(m) \otimes V(n) \cong V(n-m) \oplus V(n-m+2) \oplus ... \oplus V(m+n)$$

which is the Clebsch-Gordan Formula.

3 Application to quantum mechanics

3.1 Coupling of Two Spin-1/2 Particles

Consider two particles, each with spin s=1/2. We want to determine the possible total spin states of the combined system.

Each particle can be in one of two spin states: spin up ($|\uparrow\rangle$) or spin down ($|\downarrow\rangle$). We want to figure out what spin combination the combined two-particle state can have. Let $m = n = 2s = 1$.

Applying the Clebsch-Gordan formula, we get:

$$V(1) \otimes V(1) \cong V(0) \oplus V(2)$$

In terms of spin, s, this means the total spin of the system can be either 0 or 1.

In general, if the two particles have spins $\frac{m}{2}$ and $\frac{n}{2}$ we can apply the Clebsch-Gordan formula and obtain all possible values for the total spin:

$V(m) \otimes V(n) \cong V(n-m) \oplus V(n-m+2) \oplus ... \oplus V(m+n)$

which means that the possible values for the total spin are $\frac{n-m}{2}$, ... , $\frac{n+m}{2}$, as expected from basic Quantum Mechanics.

4 Appendix

4.1 General form of λ_i

$$\lambda_i(m-i+1) + \lambda_{i-1}(n-k+i) = 0$$

$\implies$

$$\frac{\lambda_i}{\lambda_{i-1}} = \frac{k-n-i}{m-i+1}$$

$\Leftrightarrow$

$$\lambda_i = (-1)^i \frac{(n-k+i)!}{(n-k)!} \cdot \frac{1}{(m-i+1)(m-i+2)\cdots m} \lambda_0$$

$\Leftrightarrow$

$$\lambda_i = (-1)^i \frac{(n-k+i)!}{(n-k)!} \cdot \frac{(m-i)!}{m!} \lambda_0$$

References

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