Application of Differential Geometry to study Mechanical Systems

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Some remembers

Remember your high school courses :
A mechanical system of mass m submitted to a force f can be modeled by :

$$x(t) \in \mathbb{R}^n, m\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt})$$

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Mechanical Energy : $E_m = K + U$

Some remembers

Mechanical energy theorem : If all forces are conservative, then on the motions of the system :

$$\frac{dE_m}{dt}(x(t), \dot{x}(t)) = 0$$

Very useful to find the motions of conservative systems !

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We could maybe find systems that don't belong to $\mathbb{R}^n$
Or having some forces hard to describe with our classical geometry.
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We could maybe find systems that don't belong to $\mathbb{R}^n$
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But something can save us : Differential geometry !

Requirements

We will need some mathematical tools for this :

- Differentiable manifolds
- Vector Fields
- Differentiable Maps

We still need :

- Tensors, Tensor Fields, Riemannian metrics

Tensors

Let V a vector space, $k \in \mathbb{N}$
A k-tensor T is a k-multinear map : $V^k \to \mathbb{R}$
Example : The euclidean scalar product $(a, b) = x_a x_b + y_a y_b$ is a 2-tensor

Tensors on tangent vectors

Let M a manifold and $(x^1, ..., x^n)$ a coordinate chart, $p \in M$.
Let $v = \sum_{i=1}^{n} v^i (\frac{\partial}{\partial x^i})_p \in T_p M$

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dx^i is a 1-tensor on T_pM, $dx^i(v) = v^i$
Each 1-tensor can be written as $\sum_{i=1}^{n} T_i dx^i$

Tensor product

Let T a k-tensor and S a p-tensor.
Tensor product :

$$(T \otimes S(v_1, .., v_k, w_1, ..., w_p)) = T(v_1, ..., v_k)S(w_1, ..., w_p)$$

Is a k+p tensor.

2-tensors on T_pM

Every 2-tensor on T_pM can be written as $T = \sum_{i,j=1}^{n} T_{i,j} dx^i \otimes dx^j$
Thus for $v, w \in T_pM$, $T(v, w) = \sum_{i,j=1}^{n} T_{i,j} v^i w^j$

Tensor Fields

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For example, a 1-tensor field has this form :

$$T(p) = \sum_{i=1}^{n} T_i(p) dx^i$$

Differential of a function

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is a 1-tensor field on TM.

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Positive definite : $g(v, v) > 0 \ \forall v \in T_p M \backslash 0$
Then g is a Riemannian metric on M
(M, g) : Riemannian Manifold.

Generalisation of Mechanical Systems

Let (M, g) a Riemannian Manifold and f a 1-tensor field :

$$\forall p \in M, f = \sum_{i=1}^{n} f_i(p) dx^i$$

. The triple (M, g, f) is called a Mechanical System.

Conservative Mechanical Systems

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$\Rightarrow$: The generalization of Kinetic Energy to all Manifolds.
Let $c : I \in \mathbb{R} \to M$ a smooth curve.

$$\forall t \in I, \dot{c} \in T_{c(t)}M$$

Kinetical energy of a curve : $K(\dot{c}(t)) = \frac{1}{2}g(\dot{c}, \dot{c})$

Fundamental theorem : Newton Equation

Let $(M, g, \sum_{i=1}^{n} f^i dx^i)$ a mechanical system, $(x^1, .., x^n)$ local coordinates on M and $(x^1, ..., x^n, v^1, ..., v^n)$ local coordinates induced on TM. Then :

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$$\frac{d}{dt}(\frac{\partial K}{\partial v^i}(c(t), \dot{c}(t))) - \frac{\partial K}{\partial x^i}(c(t), \dot{c}(t)) = f^i(c(t), \dot{c}(t))$$

$$\forall i \in [1, n]$$

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If the system is conservative (f = -dU), we have :

$$\frac{d}{dt}\left(\frac{\partial K}{\partial v^i}(c(t), \dot{c}(t))\right) - \frac{\partial K}{\partial x^i}(c(t), \dot{c}(t)) = -\frac{\partial U}{\partial x^i}(c(t))$$

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Allows us to study the motion of loads of mechanical systems !

Examples

We are now going to look at examples of conservative systems, to study their motions.

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Mechanical system : $(\mathbb{R}^2, m(dx \otimes dx + dy \otimes dy), -d(mgy))$.
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Potential energy : Classic gravitational potiential energy $U = mgy$.
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We have $K(x, y, v^x, v^y) = \frac{1}{2}m((v^x)^2 + (v^y)^2)$

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We have

$$\frac{\partial K}{\partial x} = \frac{\partial K}{\partial y} = 0$$

$$\frac{\partial K}{\partial v^x} = mv^x$$

$$\frac{\partial K}{\partial v^y} = mv^y$$

Example 1 : Free falling particle

The newton equation gives :

On x:

$$\frac{d}{dt}(\frac{\partial K}{\partial v^x}(x, y, \dot{x}, \dot{y})) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} = -\frac{\partial U}{\partial x} = 0$$

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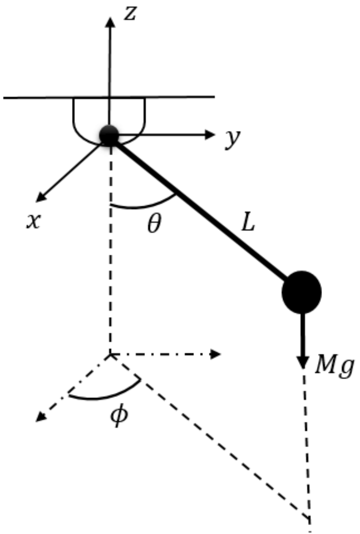
Two ODE's giving with initial conditions

$$x(t) = v_0^x t + x_0$$

$$y(t) = \frac{1}{2}gt^2 + v_0^y t + y_0$$

Example 2 : Spherical Pendulum



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Mechanical system : $(\mathbb{R}^3, m(dx \otimes dx + dy \otimes dy + dz \otimes dz), -d(mgz) + R)$.

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Mechanical system : $(\mathbb{R}^3, m(dx \otimes dx + dy \otimes dy + dz \otimes dz), -d(mgz) + R)$.
Two forces on p : His weight $(U = mgz)$, and the reaction force of the rod.

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How to express the reaction force ?

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Theorem : If the rod makes no dampling (amortecimento (pt), amortissement (fr),smorzamento (it)), by computing the motions of $(N, g, -dU)$

$$N = S_l^2 = \{p \in \mathbb{R}^3, x^2 + y^2 + z^2 = l^2\}$$

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How to do this ?
With spherical coordinates :

$$\Omega : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$$

$$\Omega(\theta, \varphi) = (lsin(\theta)cos(\varphi), lsin(\theta)sin(\varphi), lcos(\theta))$$

Example 2 : Spherical Pendulum

How to express the riemannian metric in spherical coordinates ?

Example 2 : Spherical Pendulum

In spherical coordinates :

$$g = g_{\theta\theta} d\theta \otimes d\theta + g_{\theta\varphi} d\theta \otimes d\varphi + g_{\varphi,\theta} d\varphi \otimes d\theta + g_{\varphi,\varphi} d\varphi \otimes d\varphi$$

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And we have :

$$g_{\theta\theta} = g(\frac{\partial\Omega}{\partial\theta}, \frac{\partial\Omega}{\partial\theta})$$

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With

$$\frac{\partial \Omega}{\partial \theta} = (lcos(\theta)cos(\varphi), lcos(\theta)sin(\varphi), -lsin(\theta))$$

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Example :

$$g_{cartesian} = m(dx \otimes dx + dy \otimes dy + dz \otimes dz)$$

$$g_{\theta\theta} = m(l^2cos^2(\theta)cos^2(\varphi) + l^2cos^2(\theta)sin^2(\varphi) + l^2sin^2(\theta))$$

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We obtain :

$$g_{\theta\theta} = ml^2$$

$$g_{\theta\varphi} = g_{\varphi,\theta} = 0$$

$$g_{\varphi\varphi} = ml^2sin^2(\theta)$$

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$$g = ml^2(d\theta \otimes d\theta + sin^2(\theta)d\varphi \otimes d\varphi).$$

Newton Equations :

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$$\frac{d}{dt}(\frac{\partial K}{\partial v^{\varphi}}(\theta, \varphi, \dot{\theta}, \dot{\varphi})) = ml^2(sin^2(\theta)\ddot{\varphi} + 2cos(\theta)sin(\theta)\dot{\varphi}\dot{\theta})$$

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$$U = mgz = mglcos(\theta)$$

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On θ

$$ml^2(\ddot{\theta} - sin(\theta)cos(\theta)(\dot{\varphi})^2) = -\frac{\partial U}{\partial \theta} = mglsin(\theta)$$

On φ

$$ml^2(sin^2(\theta)\ddot{\varphi} + 2cos(\theta)sin(\theta)\dot{\varphi}\dot{\theta}) = -\frac{\partial U}{\partial \varphi} = 0$$

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One solution : Numerical Methods

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How to solve these equations ?
One solution : Numerical Methods

- Euler Method
- Runge Kutta Methods

Gives good approximation of the solutions !
Runge Kutta example (Channel : Good Vibrations with Freeball)