QUATERNIONS: theory and some applications

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Introduction

Quaternion Algebras are a very interesting and powerful tool for

◊ algebraic study

◊ modeling certain phenomena: e.g. it is significantly easier to compute 3-dimensional rotations with quaternions.

However, lacking of commutativity makes sometimes working with quaternion algebras complicated.

After a small introduction about the **algebraic and geometrical properties** of Quaternions, we will see some applications:

- the **Spacial rotations** via Quaternions
- the **Key-Frame Animation** via Quaternions
- the Quaternions as a subgroup of the **Rubik's Cube Group**

Quaternions

Problem: How to find a description of rotation in space corresponding to what happens on $\mathbb{C}$?

• a complex number is a vector on a plane

• scaling a vector by r and rotating by ϕ is equivalent to multiplying it by $w = r(\cos(\phi) + i\sin(\phi)) \in \mathbb{C}$. [12]

Sir William Rowan **Hamilton** originally tried (and failed) to **find a generalization to the complex numbers in dimension** 3.

In 1843 he invented the **Quaternions** $\mathcal{H}$ as an extension to $\mathbb{C}$ using $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ (commutative property is lost).

Four numbers are necessary: one describes the size of the scaling, one the number of degrees to be rotated and the last two give the plane in which the vector should be rotated.

In 1966, **May** proved that *The set of three-dimensional complex numbers is not closed under multiplication*.

Quaternions

Quaternions are used in pure and applied mathematics.

Their **general representation** (Hamilton, 1843 [8]) is given by
$Q = a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$
where $a, b, c, d \in \mathbb{R}$ and
$\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the fundamental quaternion units [7] with the relations
$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $-\mathbf{ik} = \mathbf{ki} = \mathbf{j}$ and $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$.

$\mathbb{R}, \mathbb{C} \subset \mathcal{H}$.
$Q = a$ is a real quaternion,
$Q = a + \mathbf{i}b$ or $Q = a + \mathbf{j}c$ or $Q = a + \mathbf{k}d$ is a complex number written using quaternions.

Other representations

1. Vectorial representation:
$Q = a{\cdot}\mathbf{1}+b{\cdot}\mathbf{i}+c{\cdot}\mathbf{j}+d{\cdot}\mathbf{k}= [q_0, \mathbf{q}]$ a quaternion of real part q_0 and vectorial part $\mathbf{q}$.
$\mathbf{q}$ is a vector $\in \mathbb{R}^3$.

2. Matricial representations:
• using the real 4×4 antisymmetric matrices

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{j} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{k} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \text{ we have}$$

Other representations

$$Q = a\cdot\mathbf{1}+b\cdot\mathbf{i}+c\cdot\mathbf{j}+d\cdot\mathbf{k}=\begin{bmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{bmatrix}.$$

Clearly to get Q^* we just need to take the transposed matrix.

- using the complex 2×2 matrices

$\mathbf{1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{i}=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\mathbf{j}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{k}=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ we have

$$Q = a\cdot\mathbf{1}+b\cdot\mathbf{i}+c\cdot\mathbf{j}+d\cdot\mathbf{k} = a\cdot\mathbf{1}+b\cdot\mathbf{i}+(c+d\cdot\mathbf{i})\cdot\mathbf{j} =$$

$$=\begin{bmatrix} a+b\mathbf{i} & c+d\mathbf{i} \\ -c+d\mathbf{i} & a-b\mathbf{i} \end{bmatrix}=\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}=aI_2+b(i\sigma_1)+c(i\sigma_2)+d(i\sigma_3)$$

where σ_J Pauli spin matrices and α, β Cayley-Klein parameters.

$$det\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} = ||Q||^2 \text{ and } \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}^\top = Q^*.$$

Quaternions Arithmetic

- component-wise addition and component-wise scalar multiplication:
  $Q + P = [q_0 + p_0, \mathbf{q} + \mathbf{p}] \ \forall P, Q \in \mathcal{H}$
  $rQ = [rq_0, r\mathbf{q}] \ \forall r \in \mathbb{R}, \forall Q \in \mathcal{H}$
- $PQ = [p_0 q_0 - \mathbf{p} \cdot \mathbf{q}, p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}]$.
  Hence,
  $\frac{PQ+QP}{2} = [p_0 q_0 - \mathbf{p} \cdot \mathbf{q}, p_0 \mathbf{q} + q_0 \mathbf{p}]$
  $\frac{PQ-QP}{2} = [0, \mathbf{p} \times \mathbf{q}]$ [11]
- not generally commutative quaternion multiplication:
  $PQ \neq QP$ generally, but true if they belong to the same complex plane and

Quaternions Arithmetic

- commutative scalar multiplication:
  $\forall Q \in \mathcal{H}, \forall r \in \mathbb{R}\ rQ = r[q_0, \mathbf{q}] = [rq_0, r\mathbf{q}] = [q_0, \mathbf{q}]r = Qr.$
- $Q^* = [q_0, -\mathbf{q}]$ the complex conjugate of Q.
  - conjugation $(^*)$ is an involution
  - $(PQ)^* = Q^* P^*$
- the multiplicative inverse is $Q^{-1} = \frac{Q^*}{||Q||^2}$
- the norm is
  $||Q|| = \sqrt{QQ^*} = \sqrt{q_0^2 + |\mathbf{q}|^2} = \sqrt{a^2 + b^2 + c^2 + d^2}.$
  Quaternions with norm 1 are called unit quaternions.

Hamilton's goal was the description of rotations in space, just as complex numbers describe rotations in the plane.

Quaternions Arithmetic

Proposition1: *The set* $\mathcal{H}_0 = \mathcal{H} - \{[0, \mathbf{0}]\} \cong \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, *under quaternion multiplication forms the non-abelian quaternion group* $\mathbb{Q}_8 =< \mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} >$. [9]
Its non-trivial subgroups are all normal.

• the identity is $I = [1, \mathbf{0}]$
• $\pm\mathbf{1}$ have inverses $\pm\mathbf{1}$ respectively and $\mathbf{i}^{-1} = \frac{\mathbf{i}^*}{||\mathbf{i}||^2} = \mathbf{i}^* = -\mathbf{i}$ and analogously for $\mathbf{j}$ and $\mathbf{k}$;
moreover $QQ^{-1} = Q\frac{Q^*}{||Q||^2} = \frac{||Q||^2}{||Q||^2} = I \ \forall Q \in \mathcal{H}_0$
• obvious because quaternion multiplication is associative and distributes across quaternion addition.

PROPOSITION2: *Let* $\mathcal{H}_1$ *the set of unit quaternions*;
if $Q = [q_0, \mathbf{q}] \in \mathcal{H}_1$ *then* $\exists \mathbf{v} \in \mathbb{R}^3, \theta \in]-\pi, \pi[$ *such that*
$Q = [\cos(\theta), \mathbf{v}\sin(\theta)]$.

COROLLARY TO PROPOSITION2:
1.*Euler's relation:* $e^Q = e^{q_0} e^{||\mathbf{q}||\hat{\mathbf{q}}} = e^{q_0}(\cos(||\mathbf{q}||) + \hat{\mathbf{q}} \sin(||\mathbf{q}||))$.
2.*If* $N = [0, \hat{\mathbf{n}}] (\in \mathcal{H}_1)$ *then* $N^2 = -1$.
Moreover, $< I, N > \cong \mathbb{C}$ *and* $e^N = \cos(\theta) + \hat{\mathbf{n}} \sin(\theta)$. [11]

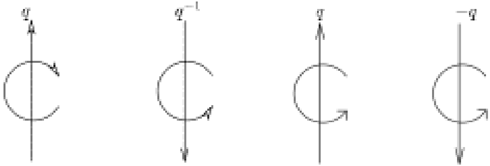
♯♯ Rotations in the plane $< I, N >$ correspond to perspective projections in 3-dimensions. [20]
♯♯ Therefore, rotations of quaternions that are the identity on the plane $< I, N >$ in 4-dimensions correspond to rotations about a line in 3-dimensions. [20]

PROPOSITION3: *Let* $Q = [cos(\theta), \mathbf{n} \sin(\theta)] \in \mathcal{H}_1$ *and* $\mathbf{r} = (x, y, z)^\top \in \mathbb{R}^3$ *and* $P = [0, \mathbf{r}] \in \mathcal{H}$.
Then QPQ^{-1} *is* P *rotated of* 2θ *about the axis* $\mathbf{n}$.

♯♯ QPQ^* acts as the identity on $< I, N >$ and QPQ acts as the identity on $< I, N >^\perp$. [20]

COROLLARY TO PROPOSITION3: *Any general three-dimensional rotation θ about $\mathbf{n}$ (of norm 1) can be obtained by a unit quaternion.*

- the inverse of Q, Q^{-1} rotates the same number of degrees as Q, but the axis points in the opposite direction
- the quaternion $-Q$ represents exactly the same rotation as Q
- all quaternions on the line rQ where $r \in \mathbb{R}$ represent the same rotation



Comparing Q and Q*, Q and -Q.

Quaternions Algebra $\mathcal{A}$

The quaternions $\mathcal{H}$ form a four dimensional centre simple algebra $\mathcal{A}$ over $\mathbb{R}$ [8]:

- as seen before $Qr = rQ$ $\forall Q \in \mathcal{H}$ and $\forall r \in \mathbb{R}$
- associative, non-commutative, finite-dimensional algebra.

Moreover, $\mathcal{A}$ is a Lie Algebra with Lie brackets given by
$[P, Q] = PQ - QP =$
$\mathrm{p}_0 q_0 - \mathbf{p} \cdot \mathbf{q}, p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} - [q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q} \times \mathbf{p}] =$
$[0, 2\mathbf{p} \times \mathbf{q}]$.
Hence, $\forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ we have $[\mathbf{p}, \mathbf{q}] = 2\mathbf{p} \times \mathbf{q}$.

We can generalize this concept to construct a **Quaternion Algebra over an arbitrary base field** F as a central simple algebra $\mathcal{A}$ over F [13] that has dimension 4 over F.

Quaternions Algebra $\mathcal{A}$

This generalization is quite useful because $\mathcal{A}$ is a 4-dimensional F-vector space with basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the multiplication rules: $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$ and $\mathbf{k}^2 = \mathbf{ijij} = -\mathbf{iijj} = -ab$ where a and b are nonzero elements of F and $char(F) \neq 2$. [14]

- By a theorem of **Frobenius** the 2×2 matrices over $\mathbb{R}$ and Hamilton's real quaternions $\mathcal{H}$ are the only two real quaternion algebras
- $\mathcal{A}$ is isomorphic to the quotient ring of $\mathbb{R}[\mathrm{Q}_8]$ by the ideal generated by $\mathbf{1} + (-\mathbf{1})$, $\mathbf{i} + (-\mathbf{i})$, $\mathbf{j} + (-\mathbf{j})$ and $\mathbf{k} + (-\mathbf{k})$
- The real group ring of Q_8 $\mathbb{R}[Q_8]$ is also an eight-dimensional vector space over $\mathbb{R}$.
  It has one basis vector for each element of Q_8.

Quaternions' Algebra Applications

- **Number Theory**, particularly to quadratic forms.
  The mathematician *Hurwitz* introduced the ring of integral quaternions, a subring of $\mathcal{H}$ consisting of all quaternions s.t. each of a, b, c, d either $\in \mathbb{Z}$ or else is congruent to $\frac{1}{2}$ mod $\mathbb{Z}$.
  This construction was used to prove *Lagrange's theorem*, that every positive integer is a sum of at most four squares. [15]

- calculations involving **Three-dimensional Rotations**.
  ♣ three-dimensional computer graphics [10]
  ♣ three-dimensional computer vision
  ♣ three-dimensional crystallographic texture analysis.

- to express the *Lorentz Transform* and to work on **Special** and **General Relativity**

Quaternions' Topology

$\mathcal{H}_1$ constitutes a subgroup $\tilde{Q}_8$ of Q_8, the quaternion group, that can be thought as a hypersphere in the quaternion space. [12]

- $\tilde{Q}_8$ can be identified with $\mathcal{S}^3$, the 3-dimensional sphere of radius 1
- $\tilde{Q}_8$ has the same Lie algebra of $\mathrm{SO}(3)$, the non-abelian group and closed three-dimensional manifold of the 3-dimensional rotations.

This equivalence is expressed by saying that $\tilde{Q}_8$ is a double cover of $\mathrm{SO}(3)$, whose topology is the same of $\mathbb{R}P^3$ not of $\mathcal{S}^3$. [16] Topologically stands a fundamental difference: **each element of $\mathrm{SO}(3)$ corresponds to two unit antipodal quaternions, Q and $-Q$.**

?How to visualize the topology of rotations, of unit quaternions and of $\mathrm{SU}(2)$?

Quaternions' Topology

Let us imagine $\mathcal{B}(O;\pi)$, a solid ball of radius π and origin O. [12]

$\Diamond$ The points $\in \mathcal{B}(O;\pi)$ correspond to a rotation that has axis given by the direction point-O and angle given by the distance point-O.

$\Diamond$ The **antipodes** of $\partial\mathcal{B}(O,\pi)$ must be identified.

$\Diamond$ Rotations are not depicted continuously $\Longrightarrow$ **source of the gimbal lock** of Euler angles and the axis-angle representation.

We consider a second ball superimposed upon the first with coordinates that are a mirror image of the first.

$\Diamond$ The **two surfaces are glued together**: rotating by π on one ball is connected to rotating by $-\pi$ on the other.

$\Diamond$ This object is equivalent to $\mathcal{S}^3 = \{\mathbf{x} \in \mathbb{R}^4 | \mathbf{x} \cdot \mathbf{x} = 1\}$, by a simply connected and continuous shape in four parameters with the **correspondence** 2 **points**-1 **rotation**.

Quaternions and $\mathrm{SU}(2)$

From [7] 1. $\mathrm{SU}(2)$ is identifiable with the quaternions of euclidean norm 1 and is therefore diffeomorphic to $\mathcal{S}^3$.

$\mathrm{SU}(2) = \{U | U^* U = \mathbb{I}_2, det(U) = 1\}$.

- $\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}$ s.t. $|a|^2 + |c|^2 = 1$ [7] general matrix form in $\mathrm{SU}(2)$.
- $\mathbb{R} \cdot \mathrm{SU}(2)$ is a 4-dimensional real linear subspace of $M_{2\times 2}(\mathbb{C})$
- $\mathbb{R} \cdot \mathrm{SU}(2)$ has basis given by the fundamental unit quaternions
- $\mathrm{SU}(2)$ is gained when $|a|^2 + |c|^2 = 1$, hence in $\mathbb{R}$ we take elements of euclidean norm 1.

  Indeed, a quaternion $Q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ squares to $-1 \iff$

$$\begin{cases} \mathrm{a}^2 - b^2 - c^2 - d^2 = -1 \\ \text{2ab=0} \\ \text{2ac=0} \\ \text{2ad=0} \end{cases} \iff \text{or} \begin{cases} \mathrm{a} = 0 \\ \mathrm{b}^2 + c^2 + d^2 = 1 \end{cases}$$

or $\begin{cases} \mathrm{b}=\mathrm{c}=\mathrm{d}=0 \\ \mathrm{a}^2=-1 \end{cases} \iff \begin{cases} \mathrm{a}=0 \\ \mathrm{b}^2+c^2+d^2=1 \end{cases}$ because a real can not have square $-1 \iff Q$ is a vector quaternion with norm 1. By definition, the set of all such vectors forms the unit sphere.

2. If $\mathbf{u}\in\mathbb{R}^3$ is a unit vector then $\mathbf{exp(u}\cdot\frac{\theta}{2})=\mathbf{1}\cdot cos(\frac{\theta}{2})+\mathbf{u}\cdot sin(\frac{\theta}{2})$ is also a unit quaternion.

- $\mathbf{u}=x\cdot\mathbf{1}+y\cdot\mathbf{i}+z\cdot\mathbf{j}+w\cdot\mathbf{k}$ where $x=0$ since $\mathbf{u}\in\mathbb{R}^3$
- $\mathrm{SU}(2)$ is the unit sphere in $\mathbb{R}\cdot\mathrm{SU}(2)$ by 1.
- a general element of $\mathrm{SU}(2)$ is $\mathbf{exp(u}\cdot\frac{\theta}{2})=\mathbf{1}\cdot cos(\frac{\theta}{2})+\mathbf{u}\cdot sin(\frac{\theta}{2})$

3. $\mathbb{R}^3$ is identified with the quaternions with zero real part.

Quaternions and $\mathrm{SU}(2)$

4. The map $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{v} \mapsto \mathbf{exp}(\mathbf{u} \cdot \frac{\theta}{2}) \cdot \mathbf{v} \cdot \mathbf{exp}(-\mathbf{u} \cdot \frac{\theta}{2})$ is a rotation of angle θ around the axis defined by **u**.

It is the conjugation of **v** by the unit quaternion $\mathbf{exp}(\mathbf{u} \cdot \frac{\theta}{2})$:
$f(\mathbf{v})$ is the new position vector of the point **v** after the rotation.

5. $\exists$ a surjective homeomorphism $\mathrm{SU}(2) \mapsto \mathrm{SO}(3)$ which is an embedding.

6. We conclude that $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$.

Example

- axis of rotation $v = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (1^{st} diagonal) of length $\sqrt{3}$
- rotation angle $\alpha = 120° = \frac{2\pi}{3} rad$
- $\mathbf{u} = \mathbf{1} \cdot cos(\frac{\alpha}{2}) + sin(\frac{\alpha}{2}) \cdot \frac{v}{||v||} = \frac{\mathbf{1+i+j+k}}{2}$ the union quaternion.

The conjugation of v is given by $f(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = \mathbf{u}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\mathbf{u}^{-1} = \frac{\mathbf{1+i+j+k}}{2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\frac{\mathbf{1-i-j-k}}{2} = c\mathbf{i} + a\mathbf{j} + b\mathbf{k}$.

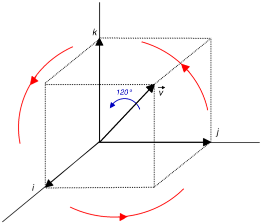


Figure: The rotation corresponds to keeping a cube held fixed at one point, and rotating it 120° about the long diagonal through the fixed point (the three axes are permuted cyclically).

Summary

- The rotation's formula for a generic $\mathbf{v} \in \mathbb{R}^3$ is $\mathbf{v}' = Q\mathbf{v}Q^{-1} = Q\mathbf{v}Q^*$.
  $\sharp\sharp$ *The multiplicative group of non-zero quaternions acts by conjugation on the copy of $\mathbb{R}^3$ consisting of quaternions with real part equal to zero.* [7]
- Conjugation by a unit quaternion $Q = [\cos(\phi), \mathbf{n}\sin(\phi)]$ is a rotation by an angle 2ϕ around the axis $\mathbf{n}$, given by the direction of the vectorial part of Q.
- Under the isomorphisms $\tilde{Q}_8 \cong \mathrm{SU}(2)$ the **quaternion multiplication operation corresponds to the composition operation of rotations**.
- 2 **unit quaternions correspond to each rotation**.

Rotations

Traditionally, rotation operators are given by 3×3 real matrixes obtained by the compositions of rotations by angles ψ,ϕ and θ around the Cartesian coordinate axes. [18]

An important theorem by **Euler** states that a general rotation of a rigid object can be described as a single rotation about some fixed vector. [19]
Given $\mathbf{v} = [l, m, n] \in \mathbb{R}^3$, then a rotation by an angle θ about $\mathbf{v}$ is given by $(\alpha = 1 - \cos(\theta))$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l^2\alpha + \cos(\theta) & lm\alpha - n\sin(\theta) & nl\alpha + m\sin(\theta) \\ lm\alpha + n\sin(\theta) & m^2\alpha + \cos(\theta) & mn\alpha - l\sin(\theta) \\ nl\alpha - m\sin(\theta) & mn\alpha + l\sin(\theta) & n^2\alpha + \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The most applicable property of quaternions is that they can easily describe rotations in 3-dimensional space. [19]

Rotation quaternions

We can define the quaternion rotation operator by $Q\mathbf{v}Q^*$.

From the previous transformation, using $\mathbf{w} = [0, x', y', z']^\top \in \mathbb{R}^4$, $\mathbf{v} = [0, x, y, z]^\top \in \mathbb{R}^4$ and $Q = [\cos(\frac{\theta}{2}), (l + m + n)\sin(\frac{\theta}{2})]$ we have $\mathbf{w} = Q\mathbf{v}Q^*$.
The rotation matrix given by $R = \frac{1}{||Q||^2}$
$$\begin{bmatrix} ||Q||^2 - 2(c^2 + d^2) & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & ||Q||^2 - 2(b^2 + d^2) & 2(cd - ad) \\ 2(bd - ca) & 2(cd + ab) & ||Q||^2 - 2(b^2 + c^2) \end{bmatrix},$$
the algebraic manipulation of Euler's angles into quaternions.

- Q must be a unit quaternion (in this case $Q^{-1} = Q^*$) *because rotations do not affect the magnitude of a vector*.
- $Q\mathbf{v}Q^*$ and $(-Q)\mathbf{v}(-Q^*)$ correspond to the same rotation matrix R *because $\tilde{Q}_8$ is a double cover of* $\mathrm{SO}(3)$. [12]

Rotation quaternions

- Quaternion Multiplication is non-commutative as in general are three-dimensional rotations.
- " $\times$ " defines the axis–angle representation and gives an orientation to the $3D$ vector space, as the quaternion multiplication is ordered, *for instance* $\mathbf{ij} = +\mathbf{k}$ but $\mathbf{ji} = -\mathbf{k}$. If one reverses the orientation, then $Q^{*}\mathbf{v}Q$.
- Only three of the quaternion components are independent. Likewise, angle/axis can be stored in a three-component vector by multiplying the unit direction by the angle, but this comes at additional computational cost.
- No Gimbal-Lock (one degree of freedom is instead lost in the Euler's angle/matrix representation).
  Therefore, it is less difficult to predict how successive rotations about the bases axis could affect each other. [12]

Visualising Quaternions

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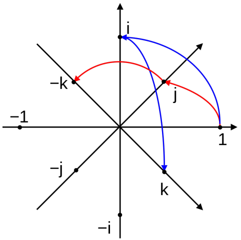


$ij = k$

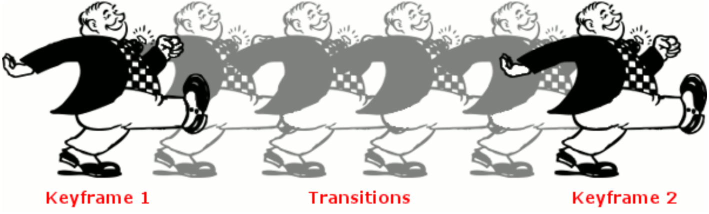
$ji = -k$

$ij = -ji$

Figure: Graphical representation of products of quaternion units as 90°-rotations in the planes of $4D$-space, spanned by two of $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The left factor can be taken as rotated by the right.
For example:
in blue: ($\mathbf{1}$, $\mathbf{i}$)-plane: $\mathbf{1}\cdot\mathbf{i}=\mathbf{i}$, ($\mathbf{i}$,$\mathbf{k}$)-plane: $\mathbf{i}\cdot\mathbf{j}=\mathbf{k}$
in red: ($\mathbf{1}$,$\mathbf{j}$)-plane: $\mathbf{1}\cdot\mathbf{j}=\mathbf{j}$, ($\mathbf{j}$,$\mathbf{k}$)-plane: $\mathbf{j}\cdot\mathbf{i}$ $=$-$\mathbf{k}$.

Key-Frame Animation

Key-frame techniques, meaning key-drawings, were used in traditional animation (e.g *Donald Duck*'s cartoon).
Computer animation of 3D objects works by using key-positions in space.



In 1971, computers were firstly used for Key-Frame Animation: the frames in-between can be generated by interpolation.

Key-Frame Animation via Quaternions

Problems [20]:

- necessary to use more advanced curves (for example splines) to produce a smooth movement across key-frames
- objects will change shape as they move
- complicated issues concerning light, sound, colors, camera angles, physical properties of the objects being modelled etc.

Our starting point is that a rigid body has general position given by the combination of a translation with a rotation.
The idea is to develop a smooth interpolation between unit quaternions, instead of using linear interpolation between the corresponding Euler angles that model a rotation. [12]

- Main advantages: lower computational cost and avoiding gimbal-lock/imaging distortions, which would severely affect the smoothness of the animation.

- Since $\mathcal{H}_1$ is not a Euclidean space, we cannot use standard usual interpolation methods such as splines; we need an equivalent interpolation curve on the surface of the four-dimensional unit sphere.

The optimal interpolation curve is the equivalent of a straight line in the space of rotations.

Shoemake [18] in 1985 proposed the alghoritm **SLERP** (spherical linear interpolation): its interpolation curve forms a great arc on the quaternion unit sphere, actually the shortest possible interpolation path between the two quaternions on $\mathcal{S}^3$. [10]

THEOREM: Let ϕ be the angle between Q_1 and $Q_2 \in \mathcal{H}_1$. Then **SLERP**$(Q_1, Q_2, t) = \frac{\sin((1-t)\phi)}{\sin(\phi)} Q_1 + \frac{\sin(t\phi)}{\sin(\phi)} Q_2$ is a unit quaternion in the plane $< Q_1, Q_2 >$ that makes the angle $t\phi$ with Q_1 and $(1-t)\phi$ with Q_2. [20]

Key-Frame Animation via Quaternions

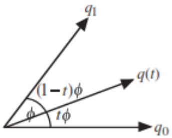


FIGURE 9.1: Spherical linear interpolation (SLERP): The quaternion $q(t)$ lies in between the two unit quaternions q_0, q_1 along a circular arc and makes an angle $t\phi$ with q_0 and $(1-t)\phi$ with q_1, where ϕ is the angle betwen q_0 and q_1.

By representing the quaternions of key-frames as points on $\mathcal{S}^3$, **SLERP** defines the intermediate sequence of rotations as a path along the great circle between the two points on the sphere. [20]

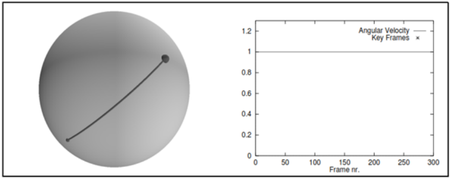
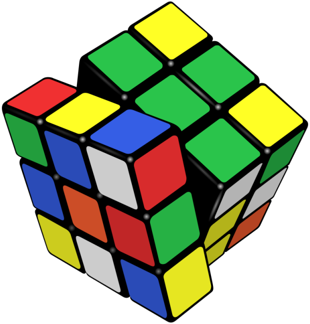


Figure 6.6: *Interpolation curve and velocity graph for spherical linear quaternion interpolation – Slerp. Between the two key frames there are 300 interpolated frames.*

The Rubik's Cube



The 54 facelets on **Rubik's Cube** [1] are rearranged by twisting one of its 6 faces.
Obviously, there are constraints on what rearrangements are possible.

Rubik's Cube group

- a permutation group $(G, \circ)$:
  G is the set of cube moves (sequence of rotations of the cube's faces)
  $\circ$ is the composition of cube moves (result of performing one cube move after another).

Any legal position of the cube can be represented as an element of G [2] because if we consider the solved cube as a starting point then each cube move is a rotation of the solved cube into that position.

- non-abelian, because this composition is not commutative.

Rubik's Cube group structure

Using the notation by **Bandelow** in [17], the cube moves are:

• 9 outer layer moves corresponding to quarter-turn clockwise ($+90°$) twists about the up, left, front, right, back and down faces, identified with the letters **U**, **L**, **F**, **R**, **B** and **D**, respectively

• 9 inverses outer layer moves corresponding to quarter-turn counter-clockwise ($-90°$) twists, identified with **U**', **L**', **F**', **R**', **B**' and **D**', respectively

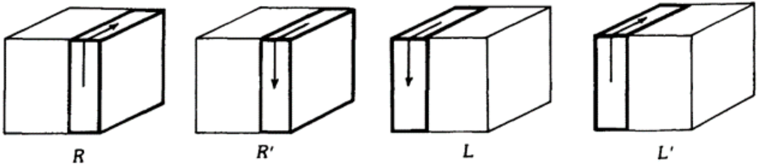


FIGURE 1. Four of the eighteen outer layer moves

- the empty move (identity permutation) is **E**.
- 9 middle layer moves corresponding to quarter-turn counter-clockwise and clockwise twists of one of the 3 middle layers.

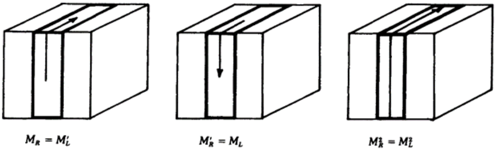


FIGURE 2. Three of the nine middle layer moves

Using *Singmaster* notation [3], we could just consider the 12 cube moves that rotate a layer 90° by their respective permutations. These leave fixed the 6 centered facets.

Rubik's Cube group structure

Hence, $\exists$ $54 - 6 = 48$ non-centered facetes.
Hence, $(G, \circ)$ is a subgroup of the symmetric group $(\mathrm{S}_{48}, \circ)$, generated by the six permutations corresponding to the six clockwise cube moves **U**, **L**, **F**, **R**, **B** and **D**.
According to how each move permutes the various facets.

The cardinality of G [5] is given by
$|G| = 43252003274489856000! = 12! \cdot 2^{10} \cdot 8! \cdot 3^{7} = 2^{27}3^{14}5^{3}7^{2}11$
and the largest order of an element in G is 1260 (e.g. **RU**^{2}**D'BD'**).

G is extremely large and complicated: its subgroups are isomorphic to any abelian group of order less than 13 (since at most it contains $\mathcal{C}_{13}$) and to every non-abelian group of order less than 26 (since at most it contains $\mathcal{D}_{13}$, the rotation group of a regular polygon of 13 vertices in three-dimensional space). [3]

Quaternions as a subgroup on the Cube

By **Lagrange**'s theorem, we have that G contains Q_8. [17]
Let's check it.

The maneuvers

◊ $\mathbf{1} = I_G = \mathbf{E}$

◊ $-\mathbf{1} = \mathbf{F}^2\mathbf{M}'_R\mathbf{F}^2\mathbf{M}^2_R \cdot \mathbf{U}' \cdot \mathbf{F}^2\mathbf{M}^2_R\mathbf{F}^2\mathbf{M}'_R \cdot \mathbf{U}$

◊ $\mathbf{i} = \mathbf{F}^2\mathbf{M}_R\mathbf{U}'\mathbf{M}'_R\mathbf{U}'\mathbf{M}_R\mathbf{U}\mathbf{M}'_R\mathbf{U}\mathbf{F}^2$

◊ $-\mathbf{i} = \mathbf{i}'$

◊ $\mathbf{j} = \mathbf{B}'\mathbf{F}^2 \cdot \mathbf{R}'\mathbf{U}'\mathbf{M}_R\mathbf{U}\mathbf{R}\mathbf{U}\mathbf{M}'_R\mathbf{U}'\mathbf{F}^2\mathbf{B}$

◊ $-\mathbf{j} = \mathbf{j}'$

◊ $\mathbf{k} = \mathbf{F}\mathbf{U}^2\mathbf{F}'\mathbf{U}'\mathbf{L}'\mathbf{B}'\mathbf{U}^2\mathbf{B}\mathbf{U}\mathbf{L}$

◊ $-\mathbf{k} = \mathbf{k}'$

form a subgroup of G (algebraically easy to prove); the computing rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ hold. We got again the general description given by **Hamilton** in 1843.

The End

Thank you for your attention!

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