Representations of SU(3) and Quarks

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  - **Observables** - hermitian operators $\hat{A} = \hat{A}^\dagger$;

$$\left\langle\hat{A}\right\rangle = \langle\psi|\hat{A}|\psi\rangle \qquad , \qquad \frac{d}{dt}\left\langle\hat{A}\right\rangle = i\left\langle\left[\hat{H},\hat{A}\right]\right\rangle$$

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$$V = \sum_{j=1}^{n^2-1} \alpha_j B_j = \sum_{j=1}^{n^2-1} \alpha_j (i\tilde{B}_j) = i\alpha \cdot \tilde{B} \quad , \quad \tilde{B}_j^\dagger = \tilde{B}_j$$

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- $[\hat{H}, U] = 0 \implies [\hat{H}, \tilde{B}_j] = 0 \longrightarrow \tilde{B}_j$ are **observables** and **conserved quantities**.

Motivation

- **Heisenberg** - $m_p \approx m_n \quad , \quad V_{pp} \approx V_{pn} \approx V_{nn}$

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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- Extension to **Quarks** - $\hat{H} = \hat{H}_{kyn} + \hat{H}_{strong} + \hat{H}_{em}$
  - First SU(2) symmetry for up (u) and down (d) quarks
  - Later extended to SU(3) with introduction of strange (s) quark

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow |\psi'\rangle = \begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = U \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad , \quad U \in SU(3)$$

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- **Physics choice** - $U = e^{i\alpha\cdot T}$, $\alpha \in \mathbb{R}^8$, $T_i = \frac{1}{2}\lambda_i$
  - λ_i - **Gell-Mann matrices**:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} \quad \boldsymbol{\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}}$$

$$\lambda_4 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & & -i \\ & 0 & \\ i & & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} \quad \boldsymbol{\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}}$$

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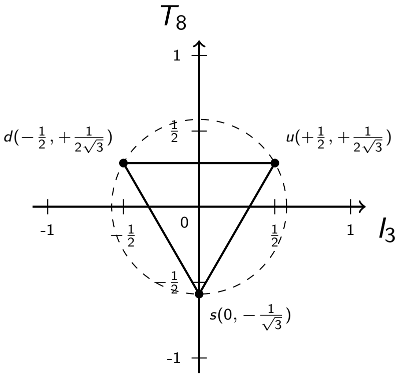

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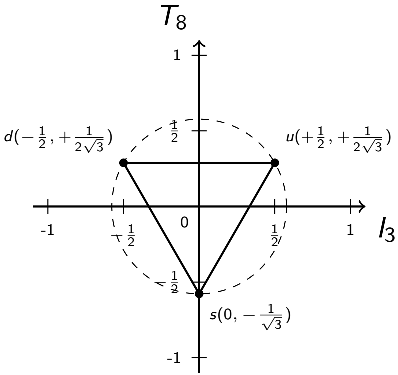

We now focus on the study of **representations of $\mathfrak{sl}_3(\mathbb{C})$**

$\mathfrak{sl}_3(\mathbb{C})$ and its Cartan sublagebra $\mathfrak{h}$

- $\mathfrak{sl}_3(\mathbb{C}) = \left\{A \in Mat_{3x3}(\mathbb{C}) : tr(A) = 0\right\}$
- Basis for $\mathfrak{sl}_3(\mathbb{C}) \to \{H_1, H_2, E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}\}$

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$$ad : \mathfrak{sl}_3(\mathbb{C}) \rightarrow L(\mathfrak{sl}_3(\mathbb{C}))$$
$$X \mapsto ad_X$$
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- Basis for $\mathfrak{h} \to \{H_1, H_2\}$

$$H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$[H, E_{ij}] = \alpha_{ij}(H) E_{ij} \quad , \quad i, j = 1, 2, 3 \quad i \neq j \quad , \quad \forall H \in \mathfrak{h}$$

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root	(H_1, H_2)	(I_3, Y)	(I_3, T_8)
$\boldsymbol{\alpha_{12}}$	$(2, -1)$	$(1, 0)$	$(1, 0)$
α_{13}	$(1, 1)$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
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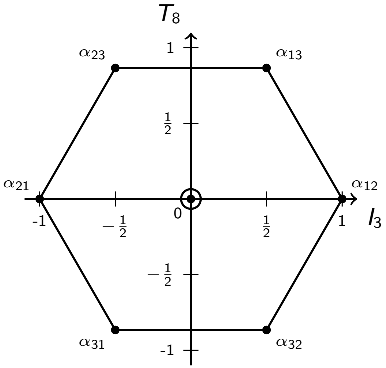

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Fundamental weights

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- Weight lattice : $\Lambda^w = span_{\mathbb{Z}}\{\theta_1, \theta_2\} \subset \mathfrak{h}^*$

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  - π is dominant weight representation
    1. Name highest weight;
    2. Obtain all other weights using the metod studied in this course.

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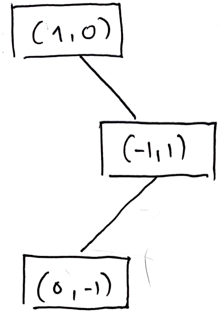

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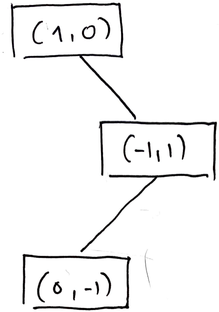
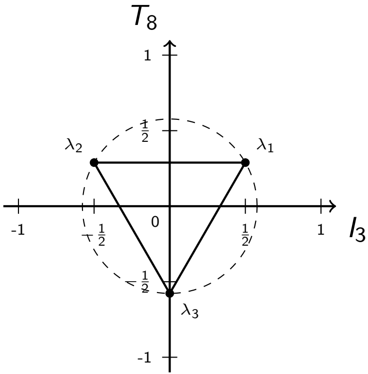


Representation **3**



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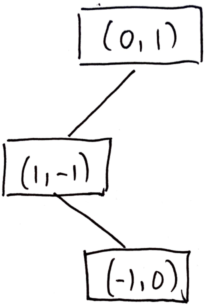

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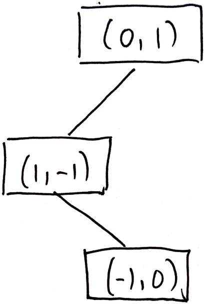
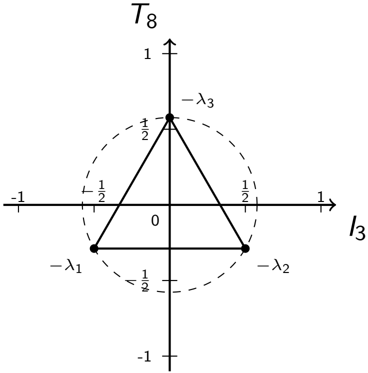


Representation $\bar{\mathbf{3}}$



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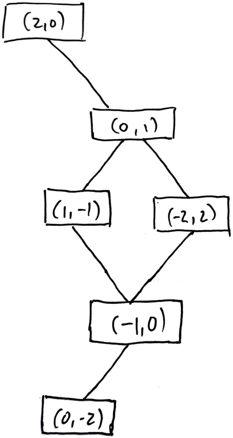

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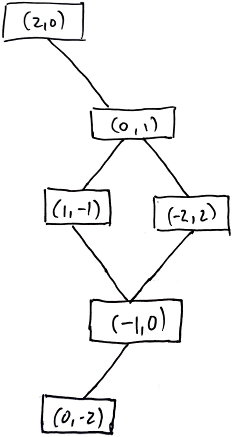
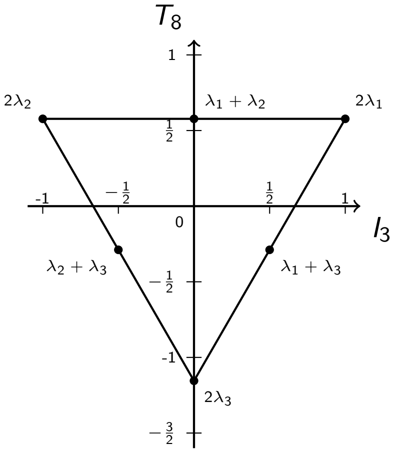


Representation **6**



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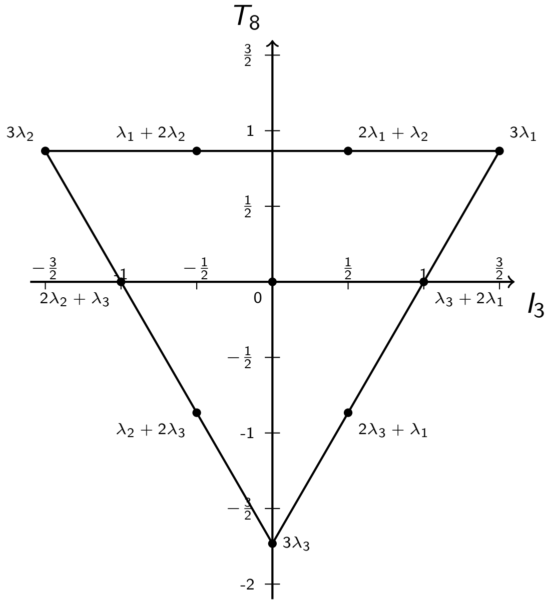

Combining Quarks and Antiquarks

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Proposition

If v_1 is a weight vector of a representation R_1 of $\mathfrak{sl}_3(\mathbb{C})$ for a weight w_1 and v_2 is a weight vector of a representation R_2 of $\mathfrak{sl}_3(\mathbb{C})$ for a weight w_2, then $v_1 \otimes v_2$ is a weight vector of $R_1 \otimes R_2$ for the weight $w_1 + w_2$.

$$\begin{aligned} R_i(H)v_i &= w_i(H)v_i \quad , \quad i = 1,2 \\ (R_1 \otimes R_2)(H)(v_1 \otimes v_2) &= (w_1 + w_2)(H)(v_1 \otimes v_2) \end{aligned}$$

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Two particles in 1D box $\mathbf{[0,1]}$

$$\psi_{1,2} \in L^2([0,1])$$

Combining Hilbert Spaces

- System composed of **two parts**, each living in different Hilbert spaces $\mathcal{V}$ **and** $\mathcal{W}$:

$$\boldsymbol{\mathcal{V} \otimes \mathcal{W}}$$

Example

Two particles in 1D box $\mathbf{[0,1]}$

$$\psi_{1,2} \in L^2([0,1])$$

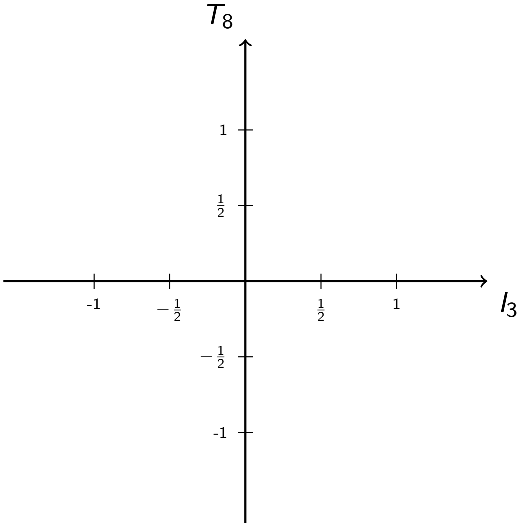
$$\psi \in L^2([0,1]x[0,1]) \cong L^2([0,1]) \otimes L^2([0,1]) \to \psi = \psi_1 \otimes \psi_2$$

Light mesons

Combining a quark ($\mathbf{3}$) and an antiquark ($\bar{\mathbf{3}}$)

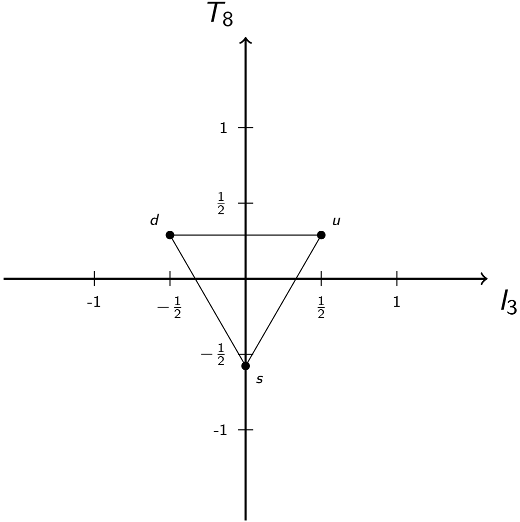
Light mesons

Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)



Light mesons

Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)



Light mesons

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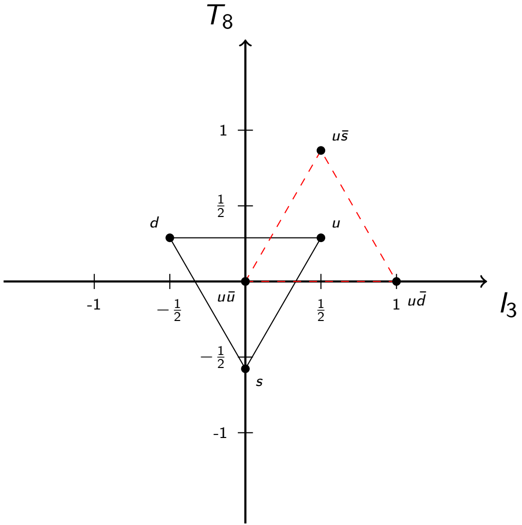

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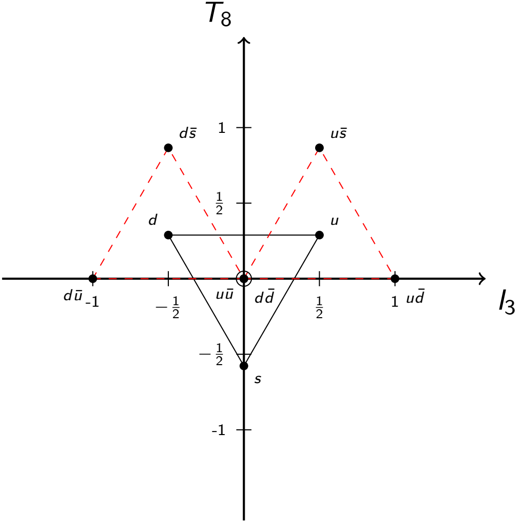
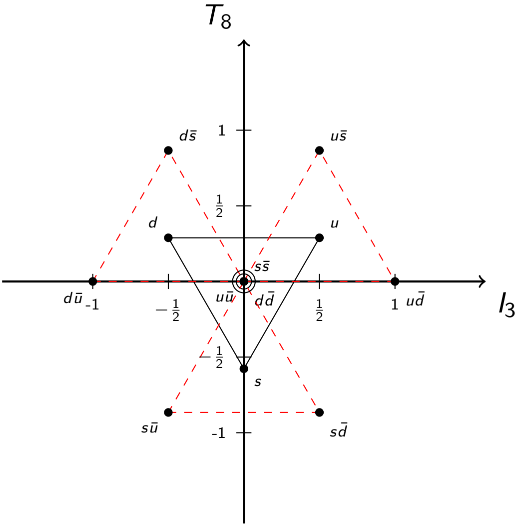
Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)



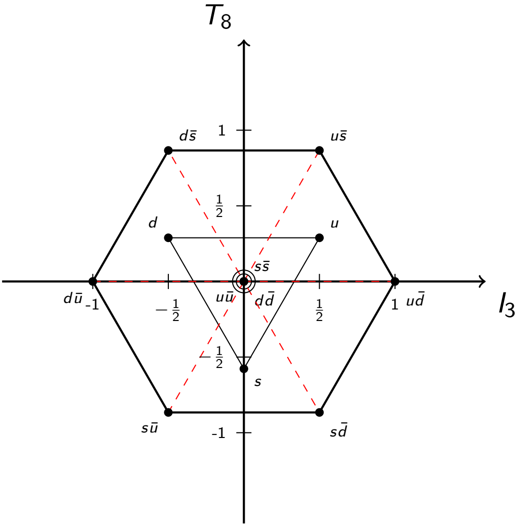
Light mesons

Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)



Light mesons

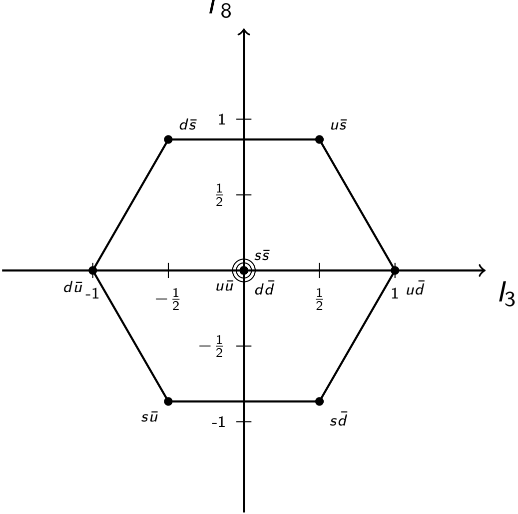
Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)



Light mesons

Combining a quark (**3**) and an antiquark ($\bar{\mathbf{3}}$)

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$$



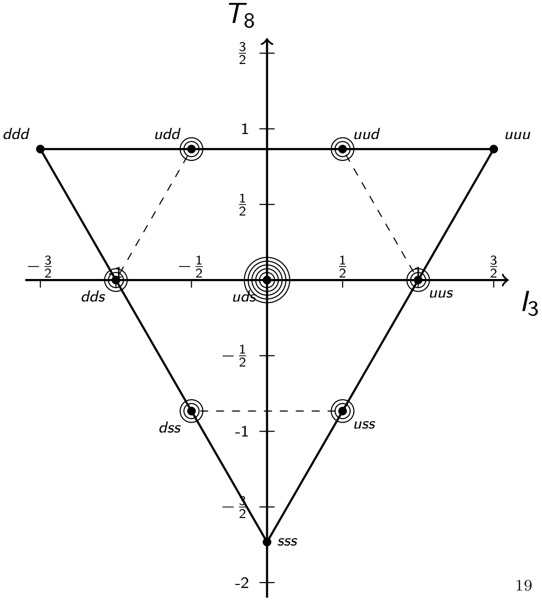
uds Baryons

Combining three quarks (**3**):

uds Baryons

Combining three quarks (**3**):

$$\begin{aligned} &\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \\ =&\mathbf{3} \otimes (\mathbf{6} \oplus \bar{\mathbf{3}}) \\ =&\mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \end{aligned}$$



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- Lecture notes