Noether's Theorem

Ricardo Amadeu

Instituto Superior Técnico

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Euler-Lagrange Equations - Definitions

Let M be a differentiable manifold and C the set of differentiable curves $c : [t_i, t_f] \longrightarrow M$ such that $c(t_i) = p_i$ and $c(t_f) = p_f$.

Definition of Lagrangian Function L and Action S

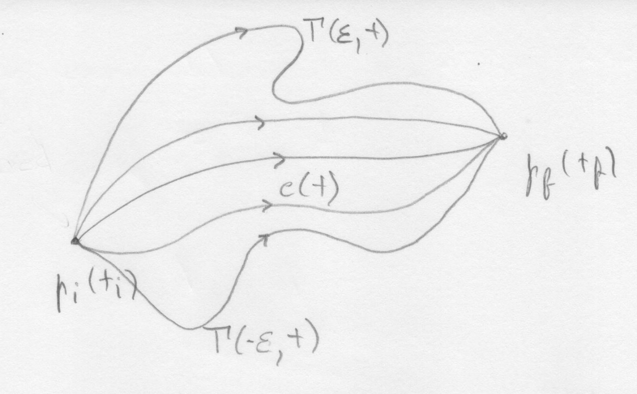
$L : TM \longrightarrow \mathbb{R}$
$S : C \longrightarrow \mathbb{R}$ such that:

$$S(c) := \int_{t_i}^{t_f} L(\dot{c}(t))dt$$

Euler-Lagrange Equations - Definitions

Definition of Variation γ of a curve $c(t) \in C$

$\gamma : (-\epsilon, \epsilon) \longrightarrow C$ such that $\gamma(0) = c$ and the map
$\Gamma : (-\epsilon, \epsilon) \times [t_i, t_f] \longrightarrow M$ defined as $\Gamma(s, t) := \gamma(s)(t)$ is differentiable.



Principle of Least Action

Definition of critical point of the Action S

The curve c is a critical point of the action if $\frac{d}{ds}S(\gamma(s))|_{s=0} = 0$ for any variation γ of c.

Principle of Least Action

The path taken by a physical system between configurations $q_i(t_i)$ and $q_f(t_f)$ is the one for which the action S is stationary. Specifically, the motions of a conservative system are the critical points of the action determined by the Lagrangian $L = K - U$, where K and U are the kinetic and potential energies of the system, respectively.

Euler-Lagrange Equations - Theorem

Theorem

A curve $c \in C$ is a critical point of the action S determined by the Lagrangian $L : TM \longrightarrow \mathbb{R}$ if and only if it satisfies the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^i}(x(t), \dot{x}(t))) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x(t), \dot{x}(t))\right) = 0$$

for any local chart $(x^1, ..., x^n)$ on M, where $(x^1, ..., x^n, v^1, ..., v^n)$ is the corresponding local chart induced on TM.

Euler-Lagrange Equations - Proof

Let's assume for simplicity that $c(t)$ is contained in the domain of a local chart $(x^1, ..., x^n)$.

$$x(s,t) := (x \circ \Gamma)(s,t)$$

$$S(\gamma(s)) := \int_{t_i}^{t_f} L\left(x(s,t), \frac{\partial x}{\partial t}(s,t)\right) dt$$

$$\frac{d}{ds} S(\gamma(s)) = \int_{t_i}^{t_f} \sum_{i=1}^{n} \frac{\partial x^i}{\partial s}(s,t) \frac{\partial L}{\partial x^i}\left(x(s,t), \frac{\partial x}{\partial t}(s,t)\right) dt$$

$$+ \int_{t_i}^{t_f} \sum_{i=1}^{n} \frac{\partial^2 x^i}{\partial s \partial t}(s,t) \frac{\partial L}{\partial v^i}\left(x(s,t), \frac{\partial x}{\partial t}(s,t)\right) dt$$

Euler-Lagrange Equations - Proof

Note

Remember that the end points of the curve $\gamma(u)$ are fixed for any $u \in (-\epsilon, \epsilon)$ since $\gamma : (-\epsilon, \epsilon) \longrightarrow C$. Then

$$\frac{\partial x}{\partial s}(s, t_i) = \frac{\partial x}{\partial s}(s, t_f) = 0$$

$$\int_{t_i}^{t_f} \sum_{i=1}^{n} \frac{\partial^2 x^i}{\partial s \partial t} \frac{\partial L}{\partial v^i} dt = -\int_{t_i}^{t_f} \sum_{i=1}^{n} \frac{\partial x^i}{\partial s} \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) dt$$

since

$$\left[\sum_{i=1}^{n} \frac{\partial x^i}{\partial s}(s, t) \frac{\partial L}{\partial v^i} \left(x(s, t), \frac{\partial x}{\partial t}(s, t) \right) \right]_{t_i}^{t_f} = 0$$

Euler-Lagrange Equations - Proof

We obtain that

$$\frac{d}{ds}S(\gamma(s))|_{s=0} = \int_{t_i}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\right) \frac{\partial x^i}{\partial s}(0,t)dt$$

If $c(t)$ satisfies the Euler-Lagrange Equations:

$$\frac{\partial L}{\partial x^i}\left(x(t),\dot{x}(t)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\left(x(t),\dot{x}(t)\right)\right) = 0$$

where $x(t) := x(0,t) = (x \circ c)(t)$.

Then $\frac{d}{ds}S(\gamma(s))|_{s=0} = 0 \implies c(t)$ is a critical point of S.

Euler-Lagrange Equations - Proof

Consider the variation γ of $c(t)$ given in local coordinates by $x(s,t) = x(t) + s\delta x(t)$, $\delta x(t)$ smooth and $\delta x(t_i) = \delta x(t_f) = 0$. Then $\frac{\partial x}{\partial s}(0,t) = \delta x(t)$. Let

$$\delta x^i(t) := \rho(t)\left(\frac{\partial L}{\partial x^i}(x(t),\dot{x}(t))) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x(t),\dot{x}(t))\right)\right)$$

where $\rho : [t_i, t_f] \longrightarrow \mathbb{R}$ is a smooth positive function and $\rho(t_i) = \rho(t_f) = 0$.
If $c(t)$ is a critical point:

$$\int_{t_i}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\right)^2 \rho(t)dt = 0 \implies \text{c(t) satisfies Euler-Lagrange Eqs.}$$

Noether's Theorem - Definitions

Definition of fiber derivative

The fiber derivative of a Lagrangian function $L : TM \longrightarrow \mathbb{R}$ at $v \in T_pM$ is the linear map $(\mathbb{F}L)_v : T_pM \longrightarrow \mathbb{R}$:

$$(\mathbb{F}L)_v(w) := \frac{dL}{dt}(v + tw)|_{t=0}$$

Definition of G-invariance

Let G be a Lie group acting on a manifold M. The Lagrangian L is said to be G-invariant if

$$L(g \cdot p, (dg)_p v) = L(p, v)$$

for all $v \in T_pM$, $p \in M$ and $g \in G$, where the map $g : M \longrightarrow M$ is the map $p \mapsto g \cdot p$.

Noether's Theorem - Definitions and Theorem

Definition of infinitesimal action

Let G be a Lie group acting on a manifold M. The infinitesimal action of $V \in \mathfrak{g}$ on M is the vector field $X^V \in \mathfrak{X}(M)$ defined as:

$$X_p^V := \frac{d}{dt}(exp(tV) \cdot p)|_{t=0} = (dA_p)_e V$$

where $A_p : G \longrightarrow M$ is the map $A_p(g) = g \cdot p$.

Noether's Theorem

Let G be a Lie group acting on a manifold M. If the Lagrangian L is G-invariant then $J^V : TM \longrightarrow \mathbb{R}$ defined as $J^V(u) := (\mathbb{F}L)_u(X^V)$ is constant along the solutions of the Euler-Lagrange Equations for any $V \in \mathfrak{g}$.

Noether's Theorem - Proof

Let $x = (x^1, ..., x^n)$ be local coordinates on M and $y = (y^1, ..., y^m)$ local coordinates centered at $e \in G$. Let the action of G on M be given in local coordinates by $A(x, y) = (A^1(x, y), ..., A^n(x, y))$.
The infinitesimal action of $U = \sum_{k=1}^m U^k \frac{\partial}{\partial y^k}$ in these coordinates is given by:

$$X^U = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n \left(\sum_{k=1}^m U^k \frac{\partial A^i}{\partial y^k}(x, e) \right) \frac{\partial}{\partial x^i}$$

For a G-invariant Lagrangian:

$$L(A(x, y), ((dA)v)(x, y)) = L(x, v)$$

$$\Leftrightarrow$$

$$L(A^1(x, y), ..., A^n(x, y), \sum_{i=1}^n \frac{\partial A^1}{\partial x^i}(x, y)v^i, ..., \sum_{i=1}^n \frac{\partial A^n}{\partial x^i}(x, y)v^i) = L(x, v)$$

Noether's Theorem - Proof

$$L\left(A^1(x,y),...,A^n(x,y),\sum_{i=1}^{n}\frac{\partial A^1}{\partial x^i}(x,y)v^i,...,\sum_{i=1}^{n}\frac{\partial A^n}{\partial x^i}(x,y)v^i\right)=L(x,v)$$

Writing $exp(tU)$ in local coordinates as $y(t)=(y^1(t),...,y^m(t))$ and differentiating with respect to t at $t=0$, we obtain (remember that $\frac{dy^k}{dt}=U^k$):

$$\sum_{i=1}^{n}\sum_{k=1}^{m}\left(\frac{\partial L}{\partial x^i}(x,v)\frac{\partial A^i}{\partial y^k}(x,e)+\sum_{j=1}^{n}\frac{\partial L}{\partial v^i}(x,v)\frac{\partial^2 A^i}{\partial y^k\partial x^j}(x,e)v^j\right)U^k=0$$

$$\Leftrightarrow$$

$$\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}(x,v)X^i(x)+\frac{\partial L}{\partial v^i}(x,v)\sum_{i,j=1}^{n}\frac{\partial X^i}{\partial x^j}(x)v^j=0$$

Noether's Theorem - Proof

Remember that:

$$J^U(v) := (\mathbb{F}L)_v(X^U) = \tfrac{dL}{ds}(v + sX^U)|_{s=0} \qquad X^U = \textstyle\sum_{i=1}^n X^i \tfrac{\partial}{\partial x^i}$$

$$J^U(x, v) = \sum_{k=1}^{n} \frac{\partial L}{\partial v^k}(x, v)\frac{d}{ds}(v^k + sX^k)|_{s=0} = \sum_{k=1}^{n} \frac{\partial L}{\partial v^k}(x, v)X^k(x)$$

Consider a curve $c(t)$ which satisfies the Euler-Lagrange equations described by $x(t) = (x^1(t), ..., x^n(t))$ in local coordinates. We shall now compute:

$$\frac{d}{dt}\left(J^U(x(t), \dot{x}(t))\right)$$

Noether's Theorem - Proof

$$\frac{d}{dt}\left(J^U(x(t),\dot{x}(t))\right) = \frac{d}{dt}\left(\sum_{k=1}^{n} \frac{\partial L}{\partial v^k}(x(t),\dot{x}(t))X^k(x(t))\right)$$

$$= \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial v^k}\right) X^k + \sum_{k,j=1}^{n} \frac{\partial L}{\partial v^k}\frac{\partial X^k}{\partial x^j}\dot{x}^j$$

Remember that from the G-invariance of the Lagrangian we obtain:

$$\sum_{k=1}^{n} \frac{\partial L}{\partial x^k}X^k + \sum_{k,j=1}^{n} \frac{\partial L}{\partial v^k}\frac{\partial X^k}{\partial x^j}\dot{x}^j = 0$$

simplifying the expression:

$$= \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i}\left(x(t),\dot{x}(t)\right)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\left(x(t),\dot{x}(t)\right)\right)\right) X^i(x) = 0$$

System of free particles

The Lagrangian $L : T\mathbb{R}^{3n} \longrightarrow \mathbb{R}$ of a system of n free particles moving in $\mathbb{R}^3$ with masses m_i is given by:

$$L(x_i, v_i) = \sum_{i=1}^{n} \frac{1}{2} m_i \langle v_i, v_i \rangle$$

Lagrangian does not depend on $(x^1, ..., x^n) \longrightarrow L$ is $\mathbb{R}^{3n}$-invariant, where the action A_x is defined as:

$$y \in \mathbb{R}^{3n} \longrightarrow A_x(y) = y \cdot x = (x^1 + y^1, ..., x^n + y^n)$$

Physically, we say that the system is invariant under spatial translations.

System of free particles

The infinitesimal action of $U \in T_e\mathbb{R}^{3n} = \mathbb{R}^{3n}$ is

$$X_x^U = (dA_x)_e(U) = \mathbb{I}_{\mathbb{R}^{3n}} U = U$$

Furthermore, from the definition $(\mathbb{F}L)_v(w) := \frac{dL}{dt}(v + tw)|_{t=0}$:

$$(\mathbb{F}L)_v(w) = \sum_{i=1}^{n} \frac{1}{2} m_i \frac{d}{dt} \left(\langle v_i + tw_i, v_i + tw_i\rangle\right)|_{t=0} = \sum_{i=1}^{n} m_i \langle v_i, w_i\rangle$$

$$\Longrightarrow J^U(v) = (\mathbb{F}L)_v(X^U) = \sum_{i=1}^{n} \langle m_i v_i, U_i\rangle$$

From Noether's Theorem:

$$\frac{dJ^U}{dt}(\dot{x}(t)) = 0 \implies \sum_{i=1}^{n} \langle \frac{d}{dt}(m_i v_i), U_i\rangle = 0 \implies \sum_{i=1}^{n} m_i \dot{x}_i \text{ conserved}$$

Invariance under space translations $\Leftrightarrow$ Conservation of Linear Momentum

System invariant under rotations

Consider a system of n particles moving in $\mathbb{R}^3$ with masses m_i under a potential energy $U : \mathbb{R}^{3n} \longrightarrow \mathbb{R}$ which depends only on distances between them. The Lagrangian $L : T\mathbb{R}^{3n} \longrightarrow \mathbb{R}$ is given by:

$$L(x_i, v_i) = \sum_{i=1}^{n} \frac{1}{2} m_i \langle v_i, v_i \rangle - U(x^1, ..., x^n)$$

Lagragian should be invariant to transformations that preserve distances (symmetry of the system) $\implies$ L is $SO(3)$-invariant, where the action R_M is defined as:

$$M \in SO(3) \longrightarrow R_M(x) = (Mx^1, ..., Mx^n)$$

since $SO(3)$ is the rotation group in $\mathbb{R}^3$ and rotations preserve lengths in $\mathbb{R}^n$.

System invariant under rotations

Lemma

There exists a linear isomorphism $\Omega : \mathfrak{so}(3) \longrightarrow \mathbb{R}^3$ such that

$$Mx = \Omega(M) \times x$$

for all $x \in \mathbb{R}^3$ and $M \in \mathfrak{so}(3)$.

The infinitesimal action of $W \in \mathfrak{so}(3)$ is

$$X_x^W = (dA_x)_e(W) = (Wx^1, ..., Wx^n) = (\Omega(W) \times x^1, ..., \Omega(W) \times x^n)$$

The fiber derivative of this Lagrangian is the same of the system of free particles, since $U(x)$ doesn't depend on v.

System invariant under rotations

We obtain:

$$J^W(v) = (\mathbb{F}L)_v(X^W) = \sum_{i=1}^{n} m_i \langle v_i, \Omega(W) \times x^i \rangle$$

From Noether's Theorem and $\langle a, b \times c \rangle = \langle b, c \times a \rangle$, $a, b, c \in \mathbb{R}^n$:

$$\frac{d}{dt} J^W(\dot{x}(t)) = \frac{d}{dt} \left(\sum_{k=1}^{n} m_i \langle \dot{x}_i, \Omega(W) \times x^i \rangle \right)$$

$$= \left\langle \Omega(W), \frac{d}{dt} \left(\sum_{k=1}^{n} x_i \times (m_i \dot{x}_i) \right) \right\rangle = 0 \implies \sum_{k=1}^{n} x_i \times (m_i \dot{x}_i) \text{ conserved}$$

Invariance under rotations $\Leftrightarrow$ Conservation of Angular Momentum

Final Remarks

Given the Lagrangian $L = K - U$ of a system, we can determine its the equations of motion from the Euler Lagrange equations:

$$\frac{\partial L}{\partial x^i}(x(t), \dot{x}(t))) - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x(t), \dot{x}(t))\right) = 0$$

Noether's Theorem:

- If a Lagrangian L is G-invariant then $J^V : TM \longrightarrow \mathbb{R}$ defined as $J^V(u) := (\mathbb{F}L)_u(X^V)$ is constant along the solutions of the Euler-Lagrange Equations for any $V \in \mathfrak{g}$
- Physically, if a system is invariant under a given transformation, i.e., if a system presents a given symmetry, there is an associated physical quantity which is conserved.

References

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