Braid Group and Anyons

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February 22, 2021

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If we denote by α the phase picked up after an exchange, then after two exchanges we must have $\alpha^2\Psi(x) = \Psi(x)$ and thus $\alpha = \pm 1$. This leads us to the case of bosons when $\alpha = 1$ and fermions when $\alpha = -1$.

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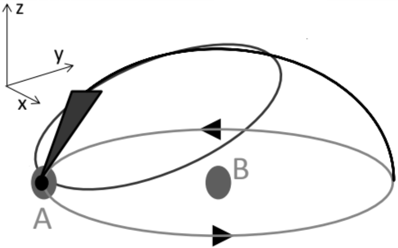

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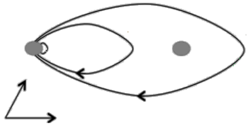
If we instead consider the particles living in 2-dimensions, then when we try taking one around the other, then we end up with two distinct non-equivalent paths



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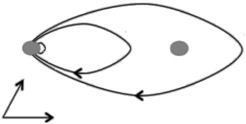

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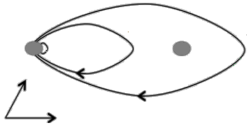

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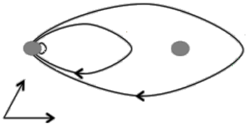

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Our goal will be to try to formally explain this.

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A geometric braid on n strings is a 1-manifold $b = a_1 \cup a_2 \cup \cdots \cup a_n$ embedded in $\mathbb{R}^2 \times I$ such that:

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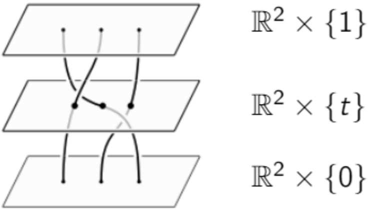

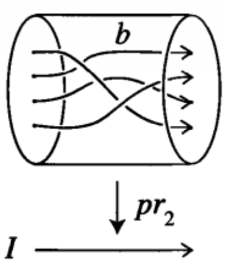

Geometric Braids

We say that two geometric n-braids b and b' are equivalent (or isotopic) if there is a continuous sequence of geometric n-braids b_s ($s \in [0, 1]$ with $b_0 = b$ and $b_1 = b'$).

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We can also define an operation on the set of geometric n-braids by concatenating the braids vertically. If we have two n-braids $b_1 \subset \mathbb{R}^2 \times I_1$ and $b_2 \subset \mathbb{R}^2 \times I_2$ and we identify I_1 with $[0, 1/2]$ and I_2 with $[1/2, 1]$ then we define $b_1 b_2 \subset \mathbb{R}^2 \times (I_1 \cup I_2)$ to be the product of b_1 and b_2.

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It should be easy to see that if $[b_1] = [b_1']$ and $[b_2] = [b_2']$, then $[b_1 b_2] = [b_1' b_2']$. So this product on geometric n-braids extends to a product on n-braids.

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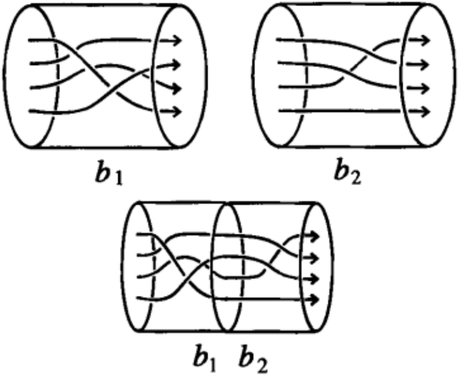

Braid Group

Theorem (Artin)

The group B_n admits a presentation with generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \textit{if} \quad 1 \leq i \leq n-2$$

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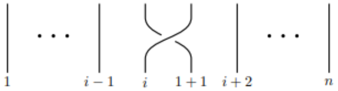

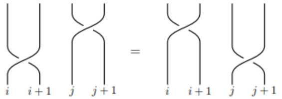

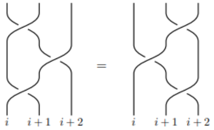

Symmetric Group

The Symmetric group S_n admits a presentation similar to the braid group. It is generated by $\{\tau_1, \tau_2, ..., \tau_{n-1}\}$, where each τ_i is the adjacent transposition $\tau_i = (i, i+1)$, and relations

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Configuration Space

Definition (Configuration Space)

Given a topological space M, the configuration space $F_n(M)$ of n ordered points is

$$F_n(M) = \{(p_1, p_2, ..., p_n) \in M \times M \times ... \times M : p_i \neq p_j \text{ if } i \neq j\}$$

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Consider now the natural action of S_n on $F_n(M)$, given by

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If we take the orbit space of this action, we end up with the configuration space of n unordered points $C_n(M) = F_n(M)/S_n$

Some topological concepts

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Definition (Path)

Let M be a topological space and x_0, x_1 points in M. A continuous function $f : I \to M$ is a path from x_0 to x_1 if $f(0) = x_0$ and $f(1) = x_1$. A loop at a point x_0 is a path where $f(0) = f(1) = x_0$.

Some topological concepts

Definition (Homotopy)

Let N, M be topological spaces and $f, g : N \to M$ be continuous maps. We say that f and g are homotopic, if there is a continuous function $H : N \times I \to M$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in N$. The map H is called an homotopy between f and g.

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If instead of taking generic continuous maps between topological spaces N and M, we took loops in some topological space M and considered their homotopy classes, we'd end up at the ideia of the fundamental group.

Some topological concepts

Definition (Fundamental Group)

Let M be a topological space and $x \in M$. The fundamental group $\pi_1(M, x)$ is the the set of homotopy classes of loops at x, with multiplication given by $[f] \star [g] = [fg]$ where fg is defined as the concatenation of loops

$$(fg)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2 \\ g(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

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This forms a group and in the case of a path connected space M, $\pi_1(M)$ is independent of the base point x, up to isomorphism

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Theorem

$B_n \cong \pi_1(C_n(\mathbb{R}^2, p))$ *where* $p = \{(1,0),(2,0),...(n,0)\}$

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Theorem

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Proof sketch.

Consider the map ϕ which sends a geometric braid b to the loop $\phi(b) : I \to C_n(\mathbb{R}^2)$ where $t \mapsto \{r_1(t), ..., r_n(t)\}$ and each $r_i(t)$ is the intersection of the i'th string b_i of b with $\mathbb{R}^2 \times \{t\}$. It is clear that ϕ is continuous and defines a loop. It can be checked that this map extends to braids, that is, equivalent braids map to the same homotopy class, and that it is bijective. $\square$

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Lemma

If X is a simply connected topological space and G a discrete group acting continuously on X, and $\forall x \in X\ \exists U \subset X$ open neighbourhood of x such that $U \cap g(U) = \varnothing$, $\forall g \in G$, then $\pi_1(X/G) \cong G$

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Proof.

For $m \geq 3$ $F_n(\mathbb{R}^m)$ is simply connected and the action of S_n, as defined before, is continuous. If we take $p = (p_1, ..., p_n) \in F_n(\mathbb{R}_m)$ then we can find open neighborhoods $U_1, ..., U_n \subset \mathbb{R}^m$ of $p_1, ..., p_n$ respectively, such that diam(U_k)$<$ $\frac{1}{2}$min(d(p_i,p_j)), then $U = U_1 \times ... \times U_n$ is an open neighborhood of p, and the previous lemma is satisfied (by construction of U). Thus $\pi_1(C_n(\mathbb{R}^m)) \cong S_n$ for $m \geq 3$. □

Representations

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We'll now see how different representation of $\pi_1(C_n)$, and thus either S_n or B_n, correspond to different kinds of particles.

One-dimensional Representations

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Thus ρ maps every element of G to the same complex number $e^{i\theta}$.
For the case of $G = S_n$ we have the extra relation $\sigma_i^2 = 1$, implying that $\rho(\sigma_i)^2 = e^{2\theta i} = 1$ and thus we must have that θ is either 0 or π, and thus $z = 1$ or $z = -1$.

Higher-dimensional Representations

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If we instead take $G = B_n$, then we don't have the previous condition that restricted the freedom of our assignment. In this case we can take θ to be any value in $[0, 2\pi)$. Particles realising such representations are called abelian anyons.

Higher-dimensional Representations

To achieve non-abelian anyons we must look at higher dimensional unitary representations of the braid group.

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where $z, w \in \mathbb{C}$ and $|z|^2 + |w|^2 = 1$.
The key ideia here is that we can identify $SU(2)$ with the real algebra generated by the quaternions.To see this write $z = a + bi$ and $w = c + di$.
Then an element M of $SU(2)$ has the form

$$M = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

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$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

we see that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$,$ki = j$, $ji = -k$, $kj = -i$, $ik = -j$, so any M, element of $SU(2)$, can be identified with a quaternion.

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Theorem

All representations of B_3 in $SU(2)$, except the one mapping each generator to plus or minus the identity, is of the following form

$$\rho(\sigma_1) = a + bu, \quad \rho(\sigma_2) = a + bv, \quad a^2 - \frac{1}{2} = (v \cdot u)b^2, \quad a^2 + b^2 = 1$$

where u and v are pure quaternions, that is, they have no real part.

Example: Majorana fermions

Take $g = a + bu$ and $h = a + bv$ and suppose that $u \cdot v = 0$. Then by the previous theorem we get $a^2 = \frac{1}{2}$ and $a^2 + b^2 = 1$, which means that $a = b = \frac{1}{\sqrt{2}}$.

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Since we assumed that $u \cdot v = 0$, and i, j and k are all orthogonal, then we have for the braiding generators the three operators

$$A = \frac{1}{\sqrt{2}}(1 + i), \quad , B = \frac{1}{\sqrt{2}}(1 + j), \quad C = \frac{1}{\sqrt{2}}(1 + k)$$

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$$A = \frac{1}{\sqrt{2}}(1 + i), \quad , B = \frac{1}{\sqrt{2}}(1 + j), \quad C = \frac{1}{\sqrt{2}}(1 + k)$$

They satisfy the braiding relation pairwise, $ABA = BAB$, $BCB = CBC$, $ACA = CAC$.

Example: Fibonacci anyons

Take $g = e^{i\theta} = a + bi$ and $h = a + b[(c^2 - s^2)i + 2csk]$, where $c^2 + s^2 = 1$ and $c^2 - s^2 = \frac{a^2 - b^2}{2b^2}$.

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The simplest example of this is given by $g = e^{7\pi i/10}$, $f = \tau i + \sqrt{\tau} k$ and $h = fgf^{-1}$, where $\tau^2 + \tau = 1$. They satisfy the braiding relation $ghg = hgh$ and generate a representation of B_3 that is dense in $SU(2)$.

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Thank you