Algebraic and Geometric Methods in Engineering and Physics 2020/2021

2^{nd} Exam - 2 February 2021 - 11:30
Duration: 2 hours

(8/20) **1.** Consider the finite group G formed by the matrices

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad R_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad R_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$
$$S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad S_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

under matrix multiplication.

(a) Show that G is not abelian, but admits subgroups isomorphic to $\mathbb{Z}_2$ and to $\mathbb{Z}_4$.

(b) Show that $\det : G \to \mathbb{R}^*$ is a group homomorphism, find a normal subgroup $N \subset G$ and identify the group G/N.

(c) The elements $g \in G$ can be naturally identified with linear maps $g : \mathbb{C}^2 \to \mathbb{C}^2$. Compute the character of the corresponding representation, and use it to prove that this representation does not contain a copy of the trivial representation.

(d) Show that this representation is irreducible, and determine the number of (equivalence classes of) irreducible representations of G.

(4/20) **2.** Recall that the Chinese remainder theorem states that if $m, n \in \mathbb{N}$ are coprime then the map $\mathbb{Z}_{mn} \ni [k] \mapsto ([k], [k]) \in \mathbb{Z}_m \times \mathbb{Z}_n$ is a group isomorphism. Use this theorem to show that if m and n are coprime then the system

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

has infinitely many solutions $x \in \mathbb{Z}$, differing by multiples of mn. What happens when m and n are not coprime?

(4/20) **3.** Show that the function $d : \mathbb{C} \times \mathbb{C} \to [0, +\infty)$ given by

$$d(z,w) = \begin{cases} |z| + |w| & \text{if } z \neq w \\ 0 & \text{if } z = w \end{cases}$$

is a distance function on $\mathbb{C}$, and sketch $B_{\frac{3}{2}}(i)$.

(4/20) **4.** Determine the weights of the representation of highest weight $(0, 1)$ of the complex semisimple Lie algebra $\mathfrak{g}_2$, whose Cartan matrix is

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Assuming that each weight space has multiplicity 1, what is the dimension of this representation?