Algebraic and Geometric Methods in Engineering and Physics

2020/2021

1$^{\text{st}}$ Exam - 13 January 2021 - 11:30
Duration: 2 hours

(8/20) **1.** Consider the set of matrices

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Show that $\{I, J, K\}$ forms a group under matrix multiplication.

(b) Prove that this group is isomorphic to $\mathbb{Z}_3$.

(c) Compute the character of the representation of $\mathbb{Z}_3$ induced by this isomorphism, and prove that it contains a copy of the trivial representation.

(d) Find the decomposition of this representation of $\mathbb{Z}_3$ into irreducible representations.

(4/20) **2.** Compute all integer solutions of the equation $21x + 30y = 9$.

(4/20) **3.** Consider the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

(a) Prove that S^1 is a Lie group under the complex multiplication.

(b) Find a surjective homomorphism $\Phi : \mathbb{R} \to S^1$, where $(\mathbb{R}, +)$ is the additive group, and determine its kernel.

(4/20) **4.** Show that the Lie algebra $(\mathbb{R}^3, \times)$ is simple (where $\times$ is the cross product).