Lie Groups and Lie Algebras

2012/2013

1^{st} Test - 15 November 2012 - 12:00

(4/20) **1.** Show that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2,\mathbb{R})^+$ is not in the image of $\exp : \mathfrak{gl}(2,\mathbb{R}) \to GL(2,\mathbb{R})^+$.

2. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$.

(3/20) (a) Show that for each Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ there exists a Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to G$ such that $\exp \circ \phi = \Phi \circ \exp$.
Hint: Use the fact that $SL(2,\mathbb{C})$ is simply connected.

(3/20) (b) Use (a) to prove that the universal covering $\widetilde{SL(2,\mathbb{R})}$ is not a matrix Lie group.

(2/20) (c) Use (b) to show that (a) is not true if G is not a matrix Lie group.

3. Let $V = \mathbb{C}^2$ denote the standard representation of $\mathfrak{sl}(2,\mathbb{C})$.

(2/20) (a) Show that V is irreducible.

(2/20) (b) Is the fundamental representation of a matrix Lie group always irreducible?

(2/20) (c) Show that V is equivalent to the dual representation V^*.

(2/20) (d) Find a decomposition of $V \otimes V \otimes V$ as a direct sum of irreducible representations.