Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Lie Groups and Lie Algebras

2008/2009

1^{st} Make Up Test - 6 January 2008 - 10:00

Duration: 1 hour and 30 minutes.

1. What is the fundamental group of:

(a) $GL(n, \mathbb{R})$?

(b) $GL(n, \mathbb{C})$?

2. Show that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \notin \exp(\mathfrak{gl}(2, \mathbb{R}))$.

3. (a) Show that up to isomorphism there are only two connected, simply connected Lie groups of dimension 2.

(b) What are these groups?

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1. (a) Show that $\int_{SO(n)} \operatorname{tr}(g)dg = 0$ (where dg is the bi-invariant measure).

(b) Is this identity true for any compact group of matrices?

2. Recall that an **algebra** over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ equipped with a bilinear map $A \times A \ni (x, y) \mapsto xy \in A$. A linear map $D : A \to A$ is called a **derivation** if $D(xy) = (Dx)y + xDy$.

(a) Show that the set $\operatorname{Der}(A)$ of derivations on A is a Lie algebra over $\mathbb{F}$.

(b) Show that if A is itself a Lie algebra then $\operatorname{ad}_x \in \operatorname{Der}(A)$ for each $x \in A$.

(c) Assuming again that A is a Lie algebra, show that $\operatorname{ad}_A \subset \operatorname{Der}(A)$ is an ideal of $\operatorname{Der}(A)$ (the elements of $\operatorname{ad}(A)$ are called the **inner derivations**).