Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Lie Groups and Lie Algebras

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2^{nd} Test - 17 December 2008 - 12:30

Duration: 1 hour and 30 minutes.

1. Let H_k be the space of spherical harmonics of degree k on S^{n-1} (that is restrictions to S^{n-1} of harmonic polynomials on $\mathbb{C}[x_1, \ldots, x_n]$ which are homogeneous of degree k).

(a) Show that
$$(g \cdot f)(x) = f(g^{-1}x)$$
defines a representation of $O(n)$ on H_k.

(b) Compute the character of the representation of $O(n)$ on H_1.

(c) Show that the representation of $O(n)$ on H_1 is equivalent to the fundamental representation of $O(n)$ on $\mathbb{C}^n$.

2. Recall the classification of simple Lie algebras:

$A_l = \mathfrak{sl}(l+1, \mathbb{C})$, $\quad \dim A_l = l(l+2) \quad (l \geq 1)$;

$B_l = \mathfrak{so}(2l+1, \mathbb{C})$, $\quad \dim B_l = l(2l+1) \quad (l \geq 2)$;

$C_l = \mathfrak{sp}(2l, \mathbb{C})$, $\quad \dim C_l = l(2l+1) \quad (l \geq 3)$;

$D_l = \mathfrak{so}(2l, \mathbb{C})$, $\quad \dim D_l = l(2l-1) \quad (l \geq 4)$;

$\dim G_2 = 14$;

$\dim F_4 = 52$;

$\dim E_6 = 78$;

$\dim E_7 = 133$;

$\dim E_8 = 248$.

Explain why the following Lie algebras are missing from this list:

(a) $\mathfrak{so}(3)$;

(b) $\mathfrak{so}(4)$;

(c) $\mathfrak{so}(6)$ (knowing that it is simple);

(d) $\mathfrak{sp}(2)$;

(e) $\mathfrak{sp}(4)$ (knowing that it is simple).