Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Lie Groups and Lie Algebras

2008/2009

1^{st} Test - 5 November 2008 - 12:30

Duration: 1 hour and 30 minutes.

1. Consider the map $f : \mathbb{R} \times SU(n) \to U(n)$ given by

$$f(t,g) = e^{it}g.$$

(a) Show that f is a Lie group homomorphism.

(b) Compute the kernel of f.

(c) What is the fundamental group of $U(n)$? (You may use the fact that $SU(n)$ is simply connected).

2. (a) Show that any matrix $g \in GL(n, \mathbb{C})$ can be written uniquely as

$$g = pu,$$

where p is hermitian positive definite and $u \in U(n)$.

(b) What is the maximal compact subgroup of $GL(n, \mathbb{C})$? And of $SL(n, \mathbb{C})$?

(c) Show that every Lie algebra homomorphism $F : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is induced by a unique Lie group homomorphism $f : SL(2, \mathbb{R}) \to GL(n, \mathbb{R})$. (You may use the fact that if G is a connected Lie group and $U \ni 1$ is an open set then $G = \cup_{n \in \mathbb{N}} U^n$).

(d) Show that the universal covering group $\widetilde{SL(2, \mathbb{R})}$ is not a matrix group.