Lie Groups and Lie Algebras

Summaries

1 Lie Groups

1.1 Preliminaries

1. A **matrix group** is a closed subgroup of $GL(n, \mathbb{R})$.

2. Two Lie groups are said to be **locally isomorphic** if there exist neighborhoods of the identity $U \subset G$ and $V \subset H$ and a homeomorphism $f : U \to V$ such that $f(xy) = f(x)f(y)$ whenever $x, y, xy \in U$.

3. Any matrix group is a Lie group. Any Lie group is locally isomorphic to a matrix group.

4. A **homogeneous space** is a differentiable manifold X where a Lie group G acts transitively. If
$$H = \{g \in G \mid gx_0 = x_0\}$$
is the **isotropy subgroup** of $x_0 \in X$ then $X \cong G/H$ as differentiable manifolds with a G-action.

5. The isometry group of a Riemannian manifold is a Lie group.

6. A **symmetric space** is a Riemannian manifold X such that for each point $x \in X$ there exists an isometry $f_x : X \to X$ such that $f_x(x) = x$ and $(df_x)_x = -\operatorname{id}$. Any symmetric space is complete and homogeneous.

7. Any invertible matrix $g \in GL(n, \mathbb{R})$ has a unique **polar decomposition**, i.e. a unique factorization $g = pu$, where p is a symmetric positive-definite matrix and $u \in O(n)$.

8. The space of **flags** in $\mathbb{C}^n$ is the homogeneous space
$$GL(n, \mathbb{C})/B \cong U(n)/T$$
(where $B \subset GL(n, \mathbb{C})$ is the subgroup of upper triangular invertible matrices and $T \subset U(n)$ is the subgroup of diagonal unitary matrices), and can be identified with the set of sequences of subspaces
$$E_1 \subset \ldots \subset E_n = \mathbb{C}^n$$
with $\dim E_k = k$.

9. Any element $g \in GL(n, \mathbb{C})$ can be factorized $g = n\pi b$, where n belongs to the subgroup $N \subset GL(n, \mathbb{C})$ of upper triangular matrices with 1's on the diagonal, π is a permutation matrix and $b \in B$. This decomposition is unique if n is chosen in $N_\pi = \pi \widetilde{N} \pi^{-1}$, where $\widetilde{N} \subset GL(n, \mathbb{C})$ is the subgroup of lower triangular matrices with 1's on the diagonal.

10. The orbits of N on the flag manifold $GL(n, \mathbb{C})/B$ decompose it into $n!$ cells C_π, with $C_\pi \cong N_\pi \cong C^{l_\pi}$ (where l_π is the length of the permutation π).

11. Almost all $g \in GL(n, \mathbb{C})$ have a unique factorization $g = \widetilde{n}b$, where $\widetilde{n} \in \widetilde{N}$ and $b \in B$.

12. If G is a compact Lie group then one can always choose a **maximal torus**, i.e. a subgroup $T \subset G$ isomorphic to a torus $S^1 \times \ldots \times S^1$ which is maximal. If G is connected then any element of G is conjugate to an element of T, and more generally any connected abelian subgroup of G is conjugate to a subgroup of T. In particular any two maximal tori are conjugate.

13. **Lefschetz fixed point theorem:** If X is a CW-complex with $\chi(X) \neq 0$ and $f : X \to X$ is a continuous map which is homotopic to the identity then f has a fixed point.

1.2 Lie Theory

1. A **Lie group** is a smooth manifold G together with a smooth map $G \times G \to G$ which makes it a group.

2. If G is a Lie group then the map $x \mapsto x^{-1}$ is smooth.

3. If G is a Lie group and $g \in G$ then the maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(x) = gx$ and $R_g(x) = xg$ are called the **left** and **right translations**. They are both diffeomorphisms.

4. Any **closed** subgroup of a Lie group is a Lie group.

5. The **Lie algebra** of a Lie group G is $\mathfrak{g} = T_1 G$.

6. A **Lie group homomorphism** is a smooth homomorphism between Lie groups.

7. A 1-**parameter subgroup** of a Lie group G is a Lie group homomorphism $f : \mathbb{R} \to G$.

8. For any Lie group G there is a $1-1$ correspondence between its Lie algebra $\mathfrak{g}$ and the homomorphisms $f : \mathbb{R} \to G$.

9. The **exponential map** on a Lie group G is the map $\exp : \mathfrak{g} \to G$ such that $\exp(A) = f(1)$, where $f : \mathbb{R} \to G$ is the unique 1-parameter subgroup such that $\dot{f}(0) = A$.

10. The exponential map is surjective on any compact Lie group.

11. The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at $0 \in \mathfrak{g}$.

12. The **Lie bracket** on the Lie algebra $\mathfrak{g}$ of a Lie group G is the antisymmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$\exp(A)\exp(B) = \exp(A + B + \frac{1}{2}[A, B] + \ldots)$$

for $A, B \in \mathfrak{g}$ sufficiently small.

13. The **Campbell-Baker-Hausdorff series**

$$\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[B, [B, A]] + \dots$$

converges in a neighbourhood of the origin, and can be expressed entirely in terms of $[\cdot, \cdot]$.

14. For matrix groups, $[A, B] = AB - BA$.

15. An abstract Lie algebra is a vector space V together with an antisymmetric bilinear map $[\cdot, \cdot] : V \times V \to V$ satisfying the **Jacobi identity**

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

16. If G is a Lie group then $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra.

17. A **Lie algebra homomorphism** between Lie algebras $(V, [\cdot, \cdot])$ and $(W, [\cdot, \cdot])$ is a linear map $F : V \to W$ such that $F([A, B]) = [F(A), F(B)]$ for all $A, B \in V$.

18. The functor taking G to $\mathfrak{g} = T_1 G$ is an equivalence of categories between the category of connected simply connected Lie groups and the category of Lie algebras.

1.3 Representations

1. A **representation** of a Lie group G is a continuous action of G on a complex Banach space V by linear isomorphisms.

2. If R is a representation of S^1 on V then $V = \hat{\bigoplus}_{n \in \mathbb{Z}} V_n$, where

$$V_n = \{\xi \in V \mid R_\alpha \xi = e^{-in\alpha}\xi \text{ for all } \alpha \in S^1\}$$

and $\hat{\bigoplus}$ means that each V_n is a closed subspace of V and each $\xi \in V$ has a unique convergent expansion $\xi = \sum_{n \in \mathbb{Z}} \xi_n$ with $\xi_n \in V_n$.

3. A representation is called **irreducible** if it has no closed G-invariant subspaces except 0 and V.

4. A representation is called **unitary** if V is a Hilbert space and

$$\langle g\xi, g\eta \rangle = \langle \xi, \eta \rangle$$

for all $g \in G$ and $\xi, \eta \in V$.

5. Finite-dimensional unitary representations are always direct sums of irreducibes.

6. **Schur's Lemma:** If V_1 and V_2 are finite-dimensional representations of G then any G-**map** $f : V_1 \to V_2$ (i.e. linear map such that $f(g\xi) = gf(\xi)$ for all $g \in G$ and $\xi \in V_1$) is either zero or an isomorphism. Moreover, if $V_1 = V_2$ then $f = \lambda 1$ for some $\lambda \in \mathbb{C}$.

7. If G is a compact Lie group and V is a Banach space then there exists a unique continuous linear map

$$\int_G : C(G, V) \to V$$

(where $C(G, V) = \{\text{continuous maps } f : G \to V\}$) such that

(a) $\int_G f(g)dg = v$ if $f(g) = v$ for all $g \in G$;

(b) $\int_G f(gh)dg = \int_G f(hg)dg = \int_G f(g)dg$ for all $h \in G$.

8. In a Lie group G with a finite number of connected components there always exist maximal compact subgroups. If K is one of them then any compact subgroup of G is conjugate to a subgroup of K (in particular any two maximal compact subgroups are conjugate). Moreover, G is homeomorphic to $K \times \mathbb{R}^m$ for some m.

9. **Peter-Weyl Theorem - Version 1:** If G is a compact Lie group and V is a representation then the isotypical part V_P (i.e. the sum of all copies of P inside V) is a closed subspace of V and
$$V = \hat{\bigoplus}_P V_P$$
where P runs through the finite dimensional irreducible representations of G.

10. All irreducible representations of a compact Lie group are finite-dimensional.

11. **Peter-Weyl Theorem - Version 2:** Any compact Lie group is isomorphic to a subgroup of $U(n)$. In particular it is a matrix group.

12. A **representative function** on a compact Lie group G is a function $f_{M;\xi,\eta} : G \to \mathbb{C}$ of the form
$$f_{M;\xi,\eta}(g) = \langle \xi, g\eta \rangle,$$
where M is a finite-dimensional unitary representation of G and $\xi, \eta \in M$. The representative functions form a subalgebra $C_{\mathsf{alg}}(G)$ of the algebra $C(G)$ of continuous functions on G.

13. **Peter-Weyl Theorem - Version 3:** if G is a compact Lie group then $C_{\mathsf{alg}}(G)$ is a dense subring of $C(G)$ for the topology of uniform convergence.

14. (a) If G is a compact Lie group then there is an isomorphism of representations of $G \times G$
$$\bigoplus \overline{P} \otimes P \to C_{\mathsf{alg}}(G)$$
given by
$$\eta \otimes \xi \mapsto f_{P;\xi,\eta}$$
where P runs through the irreducible representations of G and $G \times G$ acts on $C_{\mathsf{alg}}(G)$ by left and right translation.

    (b) Each $\overline{P} \otimes P$ is an irreducible representation of $G \times G$.

    (c) The isomorphism above takes the inner product defined by
$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \frac{1}{\dim P} \overline{\langle \eta_1, \eta_2 \rangle} \langle \xi_1, \xi_2 \rangle$$
on $\overline{P} \otimes P$ to the usual L^2 inner product on $C_{\mathsf{alg}}(G)$.

15. The **character** of a finite-dimensional representation V of a Lie group G is the function $\chi_V : G \to \mathbb{C}$ given by
$$\chi_V(g) = \mathrm{tr}(g_V)$$
where $g_V : V \to V$ is the action of g on V.

16. (a) A finite-dimensional representation of a compact Lie group G is determined up to isomorphism by its character.

    (b) If P and Q are irreducible representations then

$$\langle \chi_P, \chi_Q \rangle = \begin{cases} 1 \text{ if } P \cong Q \\ 0 \text{ otherwise} \end{cases}$$

    (c) The characters of the irreducible representations form an orthonormal basis for the Hilbert space of **class functions** on G (i.e. functions $f : G \to \mathbb{C}$ such that $f(ghg^{-1}) = f(h)$).

17. If $H \subset G$ is a Lie subgroup of a compact Lie group G then

$$C(G/H) \cong \hat{\bigoplus} \overline{P} \otimes P^H$$

as G-spaces, where P runs through the irreducible representations of G and P^H is the subspace of P which is fixed by the action of H.

18. $C(S^{n-1}) = \hat{\bigoplus}_{k \in \mathbb{N}_0} H_k$, where H_k is the space of **spherical harmonics of degree** k on S^{n-1}, i.e. restrictions of harmonic polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which are homogeneous of degree k. The spaces H_k give the irreducible representations of $O(n)$, and are eigenspaces of the Laplacian corresponding to the eigenvalue $-k(k+n-1)$.

19. The Lie algebra $M_{n \times n}(\mathbb{C})$ of $GL(n, \mathbb{C})$ is the complexification of the Lie algebra of $U(n)$.

20. The algebra $C_{\mathsf{alg}}(U(n))$ of representative functions on $U(n)$ is precisely the algebra $\mathbb{C}[a_{ij}, \Delta^{-1}]$ of polynomial functions on $GL(n, \mathbb{C})$, where $\Delta = \det(a_{ij})$.

21. Every representation of $U(n)$ is the restriction of a unique holomorphic representation of $GL(n, \mathbb{C})$.

22. If $V = \mathbb{C}^n$ is the fundamental representation of $U(n)$ then $V^{\otimes k} = \bigoplus_Q Q \otimes V_Q$, where Q runs through the irreducible representations of the symmetric group S_k and $V_Q = \mathsf{Hom}_{S_k}(Q; V^{\otimes k})$ are the homomorphisms equivariant under S_k. In fact this decomposition is an isomorphism of representations of $S_k \times U(n)$.

23. **Weyl's Theorem:** V_Q is an irreducible representation of $U(n)$ and, up to multiplication by a power of the determinant, all irreducible representations of $U(n)$ arise in this way for some $k \in \mathbb{N}$. Moreover, all irreducible representations of S_k occur in $V^{\otimes k}$ if $n \geq k$.

2 Lie Algebras

2.1 Introduction

1. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ on which a multiplication $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined satisfying:

    (a) $[x, y]$ is linear in x and y;

    (b) $[x, x] = 0$ for all $x \in \mathfrak{g}$;

(c) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$ (**Jacobi identity**).

2. The multiplication $[\cdot, \cdot]$ is anticommutative but not associative (hence the brackets).

3. If A is an associative algebra (i.e. a vector space with a bilinear associative multiplication $(x, y) \mapsto xy$) then $[A]$ is the Lie algebra $(A, [\cdot, \cdot])$, where $[x, y] = xy - yx$.

4. A **homomorphism** of Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ over $\mathbb{F}$ is linear map $\theta : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\theta([x, y]) = [\theta(x), \theta(y)]$. An **isomorphism** is a bijective homomorphism.

5. If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$ are subspaces then $[\mathfrak{h}, \mathfrak{k}] = \mathrm{span}\{[x, y] \mid x \in \mathfrak{h}, y \in \mathfrak{k}\}$. We have $[\mathfrak{h}, \mathfrak{k}] = [\mathfrak{k}, \mathfrak{h}]$.

6. A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An **ideal** of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

7. If $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$ then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra for the product $[x + \mathfrak{h}, y + \mathfrak{h}] = [x, y] + \mathfrak{h}$, and the quotient map is a Lie algebra homomorphism.

8. If $\theta : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism then $\mathfrak{k} = \ker(\theta)$ is an ideal of $\mathfrak{g}_1$. If θ is surjective then $\mathfrak{g}_1/\mathfrak{k}$ is idomorphic to $\mathfrak{g}_2$.

9. We define $\mathfrak{gl}(n, \mathbb{F}) = [M_{n\times n}(\mathbb{F})]$.

10. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. A **representation** of $\mathfrak{g}$ is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{F})$ for some $n \in \mathbb{N}$. Two representations ρ and ρ' of degree n are called **equivalent** if there is a nonsingular matrix $T \in M_{n\times n}(\mathbb{F})$ such that $\rho'(x) = T^{-1}\rho(x)T$ for all $x \in \mathfrak{g}$.

11. A **left** $\mathfrak{g}$**-module** is a vector space V over $\mathbb{F}$ with a multiplication $\mathfrak{g} \times V \ni (x, v) \mapsto xv \in V$ satisfying:

    (a) xv is linear in x and in v;

    (b) $[x, y]v = x(yv) - y(xv)$ for all $x, y \in \mathfrak{g}$ and $v \in V$.

    A choice of basis on a finite-dimensional $\mathfrak{g}$-module gives a representation of $\mathfrak{g}$; a different choice of basis gives an equivalent representation.

12. If $U \subset V$ is a subspace and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra then $\mathfrak{h}U = \mathrm{span}\{xu \mid x \in \mathfrak{h}, u \in U\}$. U is a **submodule** of V if $\mathfrak{g}U \subset U$. A $\mathfrak{g}$-module is called **irreducible** if V has no submodules other than 0 and V.

13. $\mathfrak{g}$ is a $\mathfrak{g}$-module under the multiplication $(x, y) \mapsto [x, y]$ (called the **adjoint representation**).

14. A Lie algebra $\mathfrak{g}$ is called **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$.

15. If $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$ are ideals then so is $[\mathfrak{h}, \mathfrak{k}]$.

16. A Lie algebra $\mathfrak{g}$ is called **nilpotent** if the descending series of ideals

$$\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \mathfrak{g}^3 \supset \dots$$

    defined by

$$\mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n]$$

    is such that $\mathfrak{g}^i = 0$ for some $i \in \mathbb{N}$.

17. A Lie algebra $\mathfrak{g}$ is called **soluble** if the descending series of ideals

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \dots$$

defined by

$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$$

is such that $\mathfrak{g}^{(i)} = 0$ for some $i \in \mathbb{N}$.

18. Every abelian Lie algebra is nilpotent. Every nilpotent algebra is soluble.

19. A Lie algebra is called **simple** if it has no ideals other than 0 and $\mathfrak{g}$.

20. $\mathfrak{sl}(n, \mathbb{C})$ is simple.

2.2 Simple Lie algebras over $\mathbb{C}$

1. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. The **idealizer** of $\mathfrak{h}$ is

$$I(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [y, x] \in \mathfrak{h} \text{ for all } y \in \mathfrak{h}\},$$

i.e. it is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.

2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h}$ is nilpotent and $I(\mathfrak{h}) = \mathfrak{h}$.

3. Every finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a Cartan subalgebra. Moreover, given any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ there exists an automorphism θ of $\mathfrak{g}$ such that $\theta(\mathfrak{h}_1) = \mathfrak{h}_2$.

4. If $\mathfrak{g}$ is simple then its Cartan subalgebra is abelian.

5. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha} \mathbb{C} e_\alpha$$

as $\mathfrak{h}$-modules (**Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$**).

6. The set $\Phi \subset \mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$ of linear functionals such that

$$[x, e_\alpha] = \alpha(x) e_\alpha$$

is called the set of **roots** of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

7. The set Φ of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ satisfies:

   (a) $\Phi = -\Phi$;

   (b) $\operatorname{span} \Phi = \mathfrak{h}^*$.

8. There exists a subset $\Pi \subset \Phi$ (**fundamental roots**) such that:

   (a) Π is linearly independent;

   (b) Each $\alpha \in \Phi$ is a linear combination of elements in Π with coefficitents in either $\mathbb{Z}_0^+$ or $\mathbb{Z}_0^-$.

9. We define $\mathfrak{h}_{\mathbb{R}}^* = \operatorname{span}_{\mathbb{R}} \Pi = \operatorname{span}_{\mathbb{R}} \Phi$. The **rank** of $\mathfrak{g}$ is $\dim_{\mathbb{C}} \mathfrak{h} = \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$.

10. The **Killing form** on $\mathfrak{g}$ is the symmetric bilinear map $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by

$$\langle x, y \rangle = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y),$$

where $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $\operatorname{ad}_x(y) = [x, y]$.

11. If $\mathfrak{g}$ is a finite-dimensional nontrivial simple Lie algebra over $\mathbb{C}$ then the Killing form is nondegenerate.

12. Let $\mathfrak{g}$ be a finite-dimensional nontrivial simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ its Cartan subalgebra. Then:

    (a) The restriction of the Killing form to $\mathfrak{h}$ is nondegenerate;

    (b) The restriction of the corresponding form on $\mathfrak{h}^*$ to $\mathfrak{h}_{\mathbb{R}}^*$ is an inner product.

13. Let $\mathfrak{g}$ be a finite-dimensional nontrivial simple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ its Cartan subalgebra and $\Phi \subset \mathfrak{h}^*$ its root system. The group

$$W = \langle s_\alpha \rangle_{\alpha \in \Phi}$$

is called the **Weyl group** of $\mathfrak{g}$, where $s_\alpha : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$ is the linear isometry

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

(reflection on the hyperplane orthogonal to α).

14. The Weyl group W satisfies:

    (a) $W(\Phi) = \Phi$;

    (b) $\Phi = W(\Pi)$;

    (c) $W = \langle s_\alpha \rangle_{\alpha \in \Phi}$

    (where $\Pi \subset \Phi$ is a set of fundamental roots).

15. The **Cartan matrix** associated to a set of fundamental roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is the matrix $A = (A_{ij})_{i,j=1}^l$ defined by

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

We have $A_{ii} = 2$ and $A_{ij} \in \mathbb{Z}_0^-$ for $i \neq j$.

16. The quantities $n_{ij} = A_{ij}A_{ji}$ satisfy $n_{ij} \in \mathbb{Z}_0^+$ and $n_{ij} \in \{0, 1, 2, 3\}$ for $i \neq j$.

17. The **Dynkin diagram** of a finite-dimensional nontrivial simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is a graph with vertices labelled $1, \ldots, l$ in bijective correspondence with a set of fundamental roots such that the vertices i, j with $i \neq j$ are joined by n_{ij} edges. The Dynkin diagram is uniquely determined by $\mathfrak{g}$.

18. Let Δ be the Dynkin diagram of a finite-dimensional nontrivial simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Then

(a) Δ is connected;

(b) Any two vertices are joined by at most 3 edges;

(c) The quadratic form

$$Q(x_1, \dots, x_l) = 2\sum_{i=1}^{l} {x_i}^2 - \sum_{i \neq j} \sqrt{n_{ij}} x_i x_j$$

is positive definite.

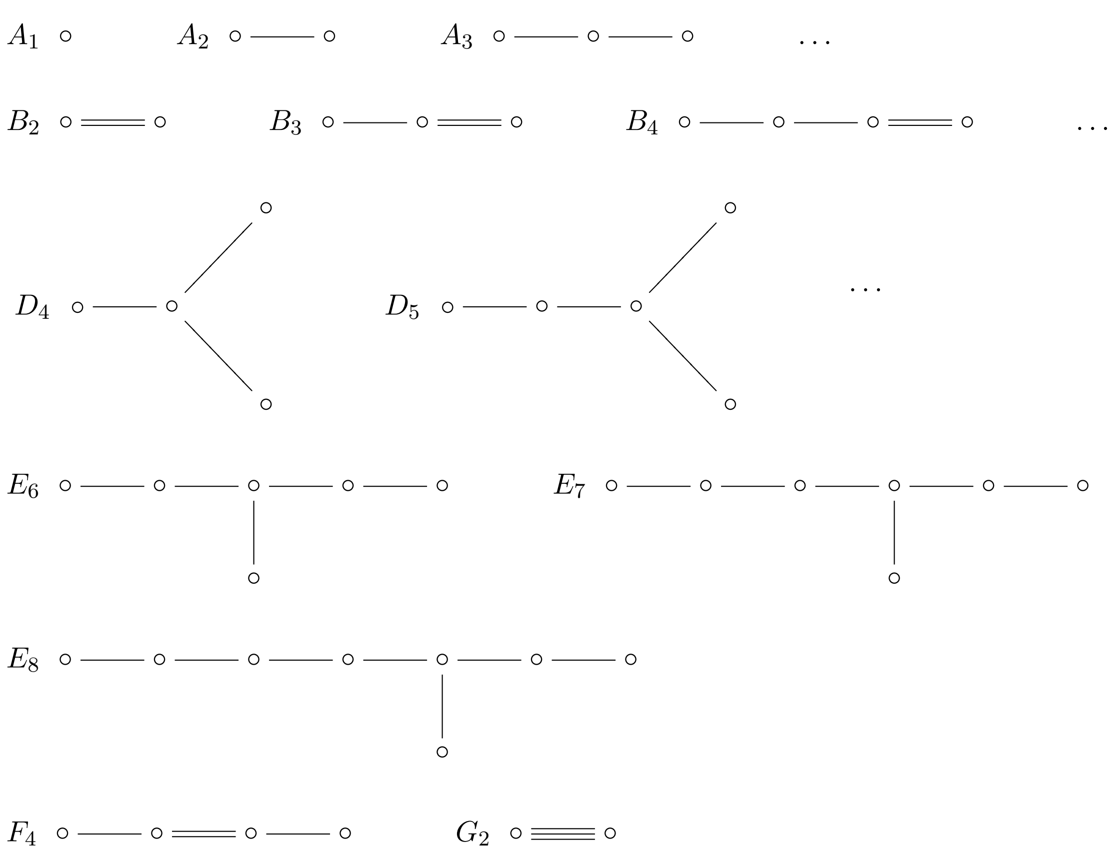
19. Consider graphs Δ with the following properties:

    (a) Δ is connected;

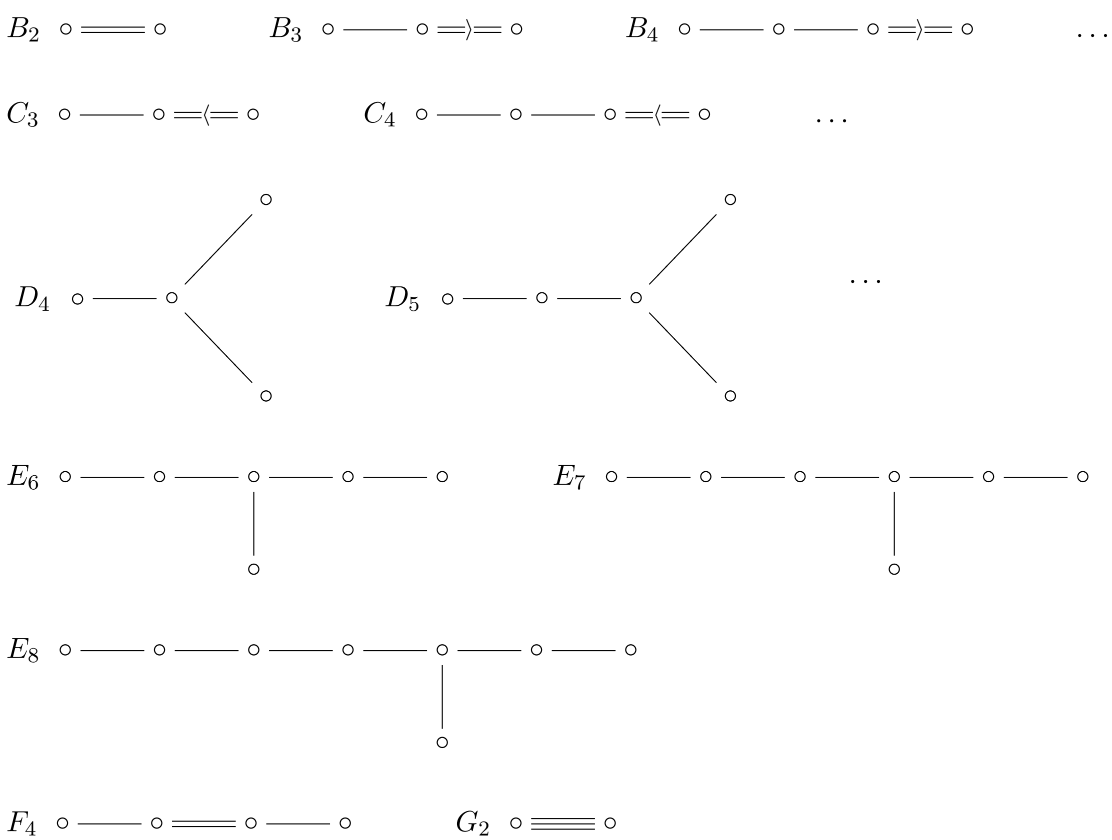
    (b) The number of edges joining any two vertices is $0, 1, 2$ or 3;

    (c) The quadratic form Q determined by Δ is positive definite.

    The Δ must be one of the graphs in the following list:



20. Let $\mathfrak{g}$ be a finite-dimensional simple nontrivial Lie algebra over $\mathbb{C}$. Then the Cartan matrix of $\mathfrak{g}$ is given by one of the following Dynkin diagrams (where the arrow points towards the shorter root):



Moreover, each of these Cartan matrices corresponds to a unique Lie algebra (up to isomorphism).

21. $A_l = \mathfrak{sl}(l+1, \mathbb{C})$ and $\dim A_l = l(l+2)$ $(l \geq 1)$;
    $B_l = \mathfrak{so}(2l+1, \mathbb{C})$ and $\dim B_l = l(2l+1)$ $(l \geq 2)$;
    $C_l = \mathfrak{sp}(2l, \mathbb{C})$ and $\dim C_l = l(2l+1)$ $(l \geq 3)$;
    $D_l = \mathfrak{so}(2l, \mathbb{C})$ and $\dim D_l = l(2l-1)$ $(l \geq 4)$;
    $\dim G_2 = 14$;
    $\dim F_4 = 52$;
    $\dim E_6 = 78$;
    $\dim E_7 = 133$;
    $\dim E_8 = 248$.

2.3 Representations of simple Lie algebras

1. Let
$$T(\mathfrak{g}) = \mathbb{C}1 \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$$
be the tensor algebra of $\mathfrak{g}$ and let $I \subset T(\mathfrak{g})$ be the two-sided ideal generated by
$$\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}.$$
The **universal enveloping algebra** of $\mathfrak{g}$ is $U(\mathfrak{g}) = T(\mathfrak{g})/I$.

2. **Poincaré-Birkhoff-Witt Theorem:** If $\{x_1, \ldots, x_n\}$ is a basis for $\mathfrak{g}$ then

$$\{{x_1}^{i_1} \ldots {x_n}^{i_n} \mid i_1, \ldots, i_n \in \mathbb{Z}_0^+\}$$

is a basis for $U(\mathfrak{g})$.

3. $U(\mathfrak{g})$-modules coincide with $\mathfrak{g}$-modules.

4. Let $\mathfrak{g}$ be a finite-dimensional nontrivial simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$, its Cartan subalgebra and $\Phi = \{\alpha_1, \ldots, \alpha_l\}$ a set of fundamental roots. The images h_i of $\frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ under the isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ given by the Killing form are called the **fundamental coroots**.

5. The **Verma module with highest weight** $\lambda \in \mathfrak{h}^*$ is the $\mathfrak{g}$-module obtained by taking the quotient

$$M(\lambda) = U(\mathfrak{g})/J(\lambda)$$

where

$$J(\lambda) = \sum_{\alpha \in \Phi^+} U(\mathfrak{g}) e_\alpha + \sum_{i=1}^{l} (h_i - \lambda(h_i)1) \subset U(\mathfrak{g})$$

and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C} e_\alpha$$

as $\mathfrak{h}$-modules.

6. Regarded as an $\mathfrak{h}$-module, $M(\lambda)$ decomposes as a sum of 1-dimensional $\mathfrak{h}$-modules. The corresponding 1-dimensional representations $\mu \in \mathfrak{h}^*$ are called the **weights** of $M(\lambda)$.

7. All the weights of $M(\lambda)$ are of the form $\lambda - n_1\alpha_1 - \ldots - n_l\alpha_l$, where $n_1, \ldots, n_l \in \mathbb{Z}_0^+$.

8. $M(\lambda)$ has a unique maximal $\mathfrak{g}$-submodule $K(\lambda)$, so that $L(\lambda) = M(\lambda)/K(\lambda)$ is an irreducible $\mathfrak{g}$-module.

9. $\dim L(\lambda)$ is finite if and only if $\lambda(h_i) \in \mathbb{Z}_0^+$ for $i = 1, \ldots, l$.

10. $\lambda \in \mathfrak{h}^*$ is called **integral** if $\lambda(h_i) \in \mathbb{Z}$ for $i = 1, \ldots, l$, and **dominant integral** if $\lambda(h_i) \in \mathbb{Z}_0^+$ for $i = 1, \ldots, l$. In terms of the dual basis $\{\omega_1, \ldots, \omega_l\}$ of $\{h_1, \ldots, h_l\}$ (called the **fundamental weights**), integral means a linear combination with integer coefficients, and dominant integral means a linear combination with nonnegative integer coefficients.

11. Every finite-dimensional irreducible $\mathfrak{g}$-module is of the form $L(\lambda)$ for some dominant integral $\lambda \in \mathfrak{h}^*$.

12. For $\lambda \in \mathfrak{h}^*$ dominant integral and $\mu \in \mathfrak{h}^*$ define

$$L(\lambda)_\mu = \{v \in L(\lambda) \mid xv = \mu(x)v \text{ for all } x \in \mathfrak{h}\}.$$

The values of $\mu \in \mathfrak{h}^*$ for which $L(\lambda)_\mu \neq 0$ are called the **weights** of $L(\lambda)$. $L(\lambda)_\mu \neq 0$ is called the μ-**weight eigenspace**, and $\dim L(\lambda)_\mu$ is called the **multiplicity** of the weight μ in $L(\lambda)$.

13. The weights μ of $L(\lambda)$ are integral.

14. Let $X \cong \mathbb{Z}^l$ be the set of integral weights, and $e : X \to e(X)$ an isomorphism such that $e(\lambda_1 + \lambda_2) = e(\lambda_1)e(\lambda_2)$. Let $\mathbb{Z}e(X)$ be the free abelian group on $e(X)$. Then $\mathbb{Z}e(X)$ is an integral domain where the sum is the group operation on $\mathbb{Z}e(X)$ and the product is the group operation on X extended to $\mathbb{Z}e(X)$. The **character** of $L(\lambda)$ is

$$\operatorname{char} L(\lambda) = \sum_{\mu \in X} \dim L(\lambda)_\mu e(\mu) \in \mathbb{Z}e(X).$$

15. **Weyl's character formula:**

$$\operatorname{char} L(\lambda) = \frac{\sum_{w \in W} (\det w) e(w(\lambda + \rho))}{\sum_{w \in W} (\det w) e(w(\rho))}$$

where W is the Weyl group, $\rho = \omega_1 + \ldots + \omega_l = \frac{1}{2} \sum_{\alpha \in \Phi^+ \alpha}$ and the equality holds in the field of fractions of $\mathbb{Z}e(X)$.

16. **Weyl's dimension formula:**

$$\dim L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in \Phi^+} \langle \alpha, \lambda + \rho \rangle} = \prod_{\alpha \in \Phi^+} \frac{\sum_{i=1}^{l} k_i w_i (m_i + 1)}{\sum_{i=1}^{l} k_i w_i},$$

where

$$\alpha = \sum_{i=1}^{l} k_i \alpha_i, \qquad \lambda = \sum_{i=1}^{l} m_i \omega_i$$

and $w_i = 1, 2, 3$ is such that

$$\langle \alpha_i, \alpha_i \rangle = w_i \langle \alpha_0, \alpha_0 \rangle$$

for $\alpha_0 \in \Pi$ a fundamental root of minimal length.