Lie Groups and Lie Algebras

Homework 7

Due on November 19

1. The map

$$(x, y, z, w) \mapsto \begin{pmatrix} x+iy & -z+iw \\ z+iw & x-iy \end{pmatrix}$$

maps the unit sphere $S^3 \subset \mathbb{R}^4$ diffeomorphically to $SU(2)$. Show that:

   (a) The usual volume element of S^3 (normalized so that the total volume is 1) is the bi-invariant volume element of $SU(2)$.

   (b) The character $\chi_2 : SU(2) \to \mathbb{C}$ of the fundamental representation of $SU(2)$ on $\mathbb{C}^2$ is $\chi_2(x, y, z, w) = 2x$.

   (c) The characters $\chi_1, \chi_3 : SU(2) \to \mathbb{C}$ of the representations of $SU(2)$ on $\Lambda^2\mathbb{C}^2$ and $S^2\mathbb{C}^2$ are $\chi_1(x, y, z, w) = 1$ and $\chi_3(x, y, z, w) = 4x^2 - 1$.

   (d) $\{\chi_1, \chi_2, \chi_3\}$ is an orthonormal set in $L^2(SU(2))$.

2. (a) Show that $\rho_S, \rho_A : S_k \to GL(1, \mathbb{C})$ given by $\rho_S(\pi) = 1$ and $\rho_A(\pi) = \mathrm{sgn}(\pi)$ determine irreducible representations Q_S, Q_A of the symmetric group S_k.

   (b) Show that the S_k-equivariant homomorphisms of $V_{Q_S} = \mathrm{Hom}_{S_k}(Q_S; V^{\otimes k})$ and $V_{Q_A} = \mathrm{Hom}_{S_k}(Q_A; V^{\otimes k})$ can be identified with the symmetric and anti-symmetric tensors in $V^{\otimes k}$.