Lie Groups and Lie Algebras

Homework 6

Due on November 5

1. Let V be a finite-dimensional unitary representation of a compact Lie group G.

   (a) The **dual representation** is the representation defined on V^* by

   $$(g\alpha)(\xi) = \alpha(g^{-1}\xi)$$

   for all $\xi \in V$. Show that the dual representation is indeed a representation.

   (b) If $\{\xi_1, \dots, \xi_n\}$ is a basis for V then the **dual basis** $\{\alpha_1, \dots, \alpha_n\}$ for V^* is defined by $\alpha_i(\xi_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker delta). Show that if the action of $g \in G$ on V is represented with respect to the basis $\{\xi_1, \dots, \xi_n\}$ by the matrix (g_{ij}) then its action on V^* is represented with respect to the basis $\{\alpha_1, \dots, \alpha_n\}$ by the matrix $(g_{ij})^{-t}$.

   (c) Define the **conjugate vector space** to V as a complex vector space $\overline{V}$ which coincides with V as a set, and denote by $\xi \mapsto \overline{\xi}$ the identity map. The complex vector space structure on $\overline{V}$ is defined by $\overline{\xi} + \overline{\eta} = \overline{\xi + \eta}$ and $\lambda\overline{\xi} = \overline{\overline{\lambda}\xi}$. The **conjugate representation** is the representation on $\overline{V}$ defined by $g\overline{\xi} = \overline{g\xi}$. Show that the conjugate representation is indeed a representation.

   (d) Show that if the action of $g \in G$ on V is represented with respect to the basis $\{\xi_1, \dots, \xi_n\}$ by the matrix (g_{ij}) then its action on $\overline{V}$ is represented with respect to the basis $\{\overline{\xi}_1, \dots, \overline{\xi}_n\}$ by the matrix $(\overline{g}_{ij})$.

   (e) Show that the map $f : \overline{V} \to V^*$ given by

   $$f(\overline{\xi})(\eta) = \langle \xi, \eta \rangle$$

   is an isomorphism.

   (f) Show that the dual and the conjugate representations are equivalent. Would this still be true for a non-unitary representation?

2. Let G be a Lie group. Show that

   $$gA = \frac{d}{dt}_{|t=0} g \exp(tA) g^{-1}$$

   defines an action of G on $\mathfrak{g}$ (called the **adjoint action**).