Lie Groups and Lie Algebras

Homework 5

Due on October 29

1. Consider the Lie groups

$$N = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \} \qquad \text{and} \qquad S = \{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \}.$$

   Show that:

   (a) S is a subgroup of the center of N.

   (b) Each element in S is a commutator.

2. Let G be a Lie group and $\rho : G \to GL(n, \mathbb{R})$ a finite dimensional representation. Show that $\widetilde{\rho} : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ defined by

$$\widetilde{\rho}(A)(v) = \frac{d}{dt}_{|t=0} \rho(\exp(tA))(v)$$

   is a Lie algebra homomorphism (called a **representation** of $\mathfrak{g}$).

3. Consider the **defining representation** $\rho : SU(2) \to GL(2, \mathbb{C})$ of $SU(2)$ on $\mathbb{C}^2$ (given by the usual multiplication of a matrix by a vector) and the representation $\rho \otimes \rho$ induced on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Show that:

   (a) ρ is irreducible.

   (b) $\rho \otimes \rho$ decomposes $\mathbb{C}^2 \otimes \mathbb{C}^2$ into two irreducible representations A and S, formed by the anti-symmetric and symmetric tensors.

   (c) $(\widetilde{\rho}(i))^2 + (\widetilde{\rho}(j))^2 + (\widetilde{\rho}(k))^2 = -3$.

   (d) $(\widetilde{\rho \otimes \rho}(i))^2 + (\widetilde{\rho \otimes \rho}(j))^2 + (\widetilde{\rho \otimes \rho}(k))^2 = 0$ on A.

   (e) $(\widetilde{\rho \otimes \rho}(i))^2 + (\widetilde{\rho \otimes \rho}(j))^2 + (\widetilde{\rho \otimes \rho}(k))^2 = -8$ on S.

   (In Quantum Mechanics $\mathbb{C}^2$ represents a particle of spin $\frac{1}{2}$, and A and S the total spin eigenstates of systems of 2 particles of spin $\frac{1}{2}$; the total spin is in each case the number s such that the sum of the squares of the linear operators associated to i, j and k is $-2s(2s+2)$).