Lie Groups and Lie Algebras

Homework 4

Due on October 21

1. Show that:

   (a) $\exp(\mathfrak{sl}(2,\mathbb{R})) = \{g \in SL(2,\mathbb{R}) \mid \operatorname{tr} g > -2\} \cup \{-1\}$.

   (b) $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ is not in $\exp(\mathfrak{sl}(2,\mathbb{R}))$ but is in $\exp(\mathfrak{sl}(2,\mathbb{C}))$.

   (c) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in $\exp(\mathfrak{sl}(2,\mathbb{C}))$ but is in $\exp(\mathfrak{gl}(2,\mathbb{C}))$.

   (d) $\exp : \mathfrak{gl}(2,\mathbb{C}) \to GL(2,\mathbb{C})$ is surjective.

2. Let G be a Lie group and $H \subset G$ a **Lie subgroup** (i.e. an embedded submanifold which is a subgroup). Show that:

   (a) $\mathfrak{h} \subset \mathfrak{g}$ is a **Lie subalgebra** (i.e. a subspace closed for $[\cdot,\cdot]$).

   (b) If H is a normal subgroup then $\mathfrak{h}$ is an **ideal** of $\mathfrak{g}$ (i.e. $[A,B] \in \mathfrak{h}$ for all $A \in \mathfrak{h}$ and $B \in \mathfrak{g}$).

   (c) If H is abelian then $(\mathfrak{h}, [\cdot,\cdot])$ is abelian (i.e. $[A,B] = 0$ for all $A, B \in \mathfrak{h}$).

   (d) If H is **central** (i.e. $gh = hg$ for all $g \in G, h \in H$) then $\mathfrak{h}$ is **central** (i.e. $[A,B] = 0$ for all $A \in \mathfrak{h}$ and $B \in \mathfrak{g}$).

3. (a) Let G be a connected Lie group and let $f : G \to H$ be a Lie group homomorphism which is a covering map. Show that the kernel of f is central.

   (b) Show that the fundamental group of a connected Lie group is abelian.

   (c) List all connected Lie groups with Lie algebra $\mathfrak{so}(2)$, $\mathfrak{so}(3)$ and $\mathfrak{so}(4)$.