Lie Groups and Lie Algebras

Homework 3

Due on October 14

1. Factorize the matrix

$$g = \begin{pmatrix} 6 & 1 & 3 & -2 \\ 4 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

   as a product $g = n\pi b$, where n is an upper triangular matrix with 1's on the diagonal, π is a permutation matrix and b is upper triangular.

2. (a) Show that

$$T = \{\cos\theta + i\sin\theta \mid \theta \in \mathbb{R}\}$$

   is a maximal torus of

$$SU(2) = \{a + bi + cj + dk \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

   (b) Show that the set of matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   is a maximal torus of $SO(3)$.

   (c) Show that the set of matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

   is a maximal torus of $SO(4)$.

3. Let G be a Lie group. Show that the following are Lie groups:

   (a) The connected component of the identity G_0.

   (b) The **center** of G,

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

   (c) The kernel of any Lie group homomorphism $f : G \to H$.