Lie Groups and Lie Algebras

Homework 2

Due on October 7

1. (a) Show that $SL(2,\mathbb{R})$ acts on $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

   (b) Show that H is a homogeneous space under this action.

   (c) By considering the isotropy subgroup of $z = i$ show that $H \cong SL(2,\mathbb{R})/SO(2)$.

   (d) Consider the automorphism $\alpha : SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$ given by $\alpha(g) = hgh^{-1}$, where

$$h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

   What is the set of points of $SL(2,\mathbb{R})$ fixed by α? What is the geometric interpretation of the action of h on H?

   (e) Show that H with the Riemannian metric

$$ds^2 = \frac{1}{(\mathrm{Im}(z))^2} dz d\overline{z}$$

   is a symmetric space.

2. (a) Show that the Riemann tensor R of a symmetric space M satisfies $\nabla_X R = 0$ for all $X \in \mathfrak{X}(M)$.

   (b) List all simply connected symmetric spaces of dimension 2.

   (c) Give an example of a non-simply connected symmetric space of dimension 2.

3. Show that the square root of a symmetric positive-definite matrix is unique.

4. If $\alpha = \rho e^{i\theta}$ is a complex number then the map $z \mapsto \alpha z$ is a linear transformation of $\mathbb{C} \cong \mathbb{R}^2$. Determine the polar decomposition of the matrix which represents this linear transformation with respect to the canonical basis.