Lie Groups and Lie Algebras

Homework 1

Due on September 30

1. Show that the Lie group $(\mathbb{R}, +)$ is isomorphic to the following matrix groups:
   (a) $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$
   (b) $B = \{\left(\begin{smallmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$
2. Show that the group of affine mappings

$$H = \{t \mapsto at + b \mid a > 0, b \in \mathbb{R}\}$$

   is isomorphic to a matrix group.
3. Let $u \in \mathbb{R}^3$ be a unit vector. Show that:
   (a) $g = \cos\theta + u \sin\theta$ is a unit quaternion.
   (b) The map $T_g : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T_g(v) = gvg^{-1}$$

   is a rotation about u by an angle 2θ.
   (c) $T : SU(2) \to SO(3)$ is a group homomorphism with kernel $\{\pm 1\}$.
4. For unit quaternions $g, h \in SU(2)$ define $T(g, h) : \mathbb{R}^4 \to \mathbb{R}^4$ through

$$T(g, h)(q) = gqh^{-1}.$$

   Show that:
   (a) $T(g, h) \in SO(4)$.
   (b) $T : SU(2) \times SU(2) \to SO(4)$ is a group homomorphism with kernel $\{\pm(1, 1)\}$.
   (c) T is surjective.
   (d) There is a surjective homomorphism $F : SO(4) \to SO(3) \times SO(3)$ with kernel $\{\pm 1\}$.