# Differential Geometry of Curves and Surfaces

## Abbreviated lecture notes

### 1. Curves

1. If $U \subset \mathbb{R}^n$ is an open set then a **smooth map** (or a **differentiable map**) $\mathbf{F} : U \to \mathbb{R}^m$ is a $C^\infty$ map. If $D \subset \mathbb{R}^n$ is any set then $\mathbf{F} : D \to \mathbb{R}^m$ is **smooth** if there exist an open set $U \supset D$ and a smooth map $\mathbf{G} : U \to \mathbb{R}^m$ such that $\mathbf{G}|_D = \mathbf{F}$.
2. A **curve** in $\mathbb{R}^n$ is a smooth map $\mathbf{c} : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. The curve is called **regular** if $\dot{\mathbf{c}}(t) \neq \mathbf{0}$ for all $t \in I$.
3. If $\mathbf{c} : I \to \mathbb{R}^n$ is a curve and $t_0 \in I$ then the **arclength** measured from $t_0$ is
$$s(t) = \int_{t_0}^{t} \|\dot{\mathbf{c}}(u)\| du.$$
If $\mathbf{c}$ is regular then $s(t)$ is invertible, and we write $\mathbf{c}(s) = \mathbf{c}(t(s))$ (slightly abusing the notation). In this case we have $\|\mathbf{c}'(s)\| = 1$.
4. If $\mathbf{c} : I \to \mathbb{R}^2$ is a regular curve parameterized by arclength, we define the positive orthonormal frame $\{\mathbf{e}_1(s), \mathbf{e}_2(s)\}$ by taking $\mathbf{e}_1(s) = \mathbf{c}'(s)$ (tangent to the curve) and $\mathbf{e}_2(s) = R_{\frac{\pi}{2}}\mathbf{e}_1(s)$, where $R_{\frac{\pi}{2}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a rotation by $90^\circ$ in the positive direction. The **curvature** of $\mathbf{c}$ is the smooth function $\kappa : I \to \mathbb{R}$ such that $\mathbf{c}''(s) = \kappa(s)\mathbf{e}_2(s)$. We have
$$\begin{bmatrix} \mathbf{e}_1'(s) \\ \mathbf{e}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \end{bmatrix}.$$
5. If $\kappa(s_0) \neq 0$ then $r(s_0) = \frac{1}{|\kappa(s_0)|}$ is the radius of the circle that approximates $\mathbf{c}(s)$ to second order at $s_0$ (**radius of curvature**). We have
$$\ddot{\mathbf{c}}(t) = \ddot{s}(t)\mathbf{e}_1(s(t)) \pm \frac{\dot{s}^2(t)}{r(s(t))}\mathbf{e}_2(s(t)).$$
6. A **positive isometry** of $\mathbb{R}^2$ is a map $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A \in SO(2)$ is a **rotation matrix**, that is, $A = \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$ for some $\alpha \in \mathbb{R}$.
7. Two regular plane curves parameterized by arclength are related by a positive isometry if and only if their curvatures coincide.

8. If $\mathbf{c}: I \to \mathbb{R}^2$ is a regular curve (not necessarily parameterized by its arclength) then its curvature is given by
$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\left[(\dot{x}(t))^2 + (\dot{y}(t))^2\right]^{\frac{3}{2}}},$$
where $\mathbf{c}(t) = (x(t), y(t))$.

9. A regular plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is said to be **closed** if $\mathbf{c}(a) = \mathbf{c}(b)$ and moreover $\mathbf{c}^{(n)}(a) = \mathbf{c}^{(n)}(b)$ for any $n \in \mathbb{N}$ (so that it can be extended to a periodic curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$). A closed curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is said to be **simple** if its restriction to the interval $[a, b)$ is injective. A simple closed curve is said to be **convex** if it bounds a convex set. A **vertex** of a simple closed curve is a critical point (maximum, minimum or inflection point) of its curvature.

10. **Four Vertex Theorem:** Every simple closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ has at least four vertices on $[a, b)$ (in fact, at least two minima and two maxima).

11. If $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is a plane curve parameterized by arclength and we write its unit tangent vector as $\mathbf{c}'(s) = (\cos(\theta(s)), \sin(\theta(s)))$ then its curvature is $\kappa(s) = \theta'(s)$.

12. The **rotation index** of a closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$, parameterized by its arclength, with curvature $k : [a, b] \to \mathbb{R}$, is the integer
$$m = \frac{1}{2\pi} \int_a^b \kappa(s) ds.$$

13. A **(free) homotopy by closed regular curves** bewteen two closed regular plane curves $\mathbf{c}_0, \mathbf{c}_1 : [a, b] \to \mathbb{R}^2$ is a smooth map $\mathbf{H} : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that:
    (i) $\mathbf{H}(t, 0) = \mathbf{c}_0(t)$ for all $t \in [a, b]$;
    (ii) $\mathbf{H}(t, 1) = \mathbf{c}_1(t)$ for all $t \in [a, b]$;
    (iii) $\mathbf{c}_u(t) = \mathbf{H}(t, u)$ is a closed regular curve for all $u \in [0, 1]$.

14. If two closed regular plane curves are homotopic by closed regular curves then they have the same rotation index.

15. The **total curvature** of a closed plane curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$, parameterized by its arclength, with curvature $k : [a, b] \to \mathbb{R}$, is
$$\mu = \int_a^b |\kappa(s)| ds.$$

16. The total curvature $\mu$ of a closed regular curve satisfies $\mu \geq 2\pi$, and $\mu = 2\pi$ if and only if the curve is convex.

17. **Isoperimetric inequality:** If $\mathbf{c}$ is a simple closed curve with of minimal length enclosing a region of fixed area $A$ then $\mathbf{c}$ parameterizes a circle of radius $r = \sqrt{\frac{A}{\pi}}$. Conversely, if $\mathbf{c}$ is a simple closed curve of fixed length $l$ enclosing a region of maximal area then $\mathbf{c}$ parameterizes a circle of radius $r = \frac{l}{2\pi}$.

18. The **curvature** of a space curve $\mathbf{c} : I \to \mathbb{R}^3$ parameterized by arclength is
$$\kappa(s) = \|\mathbf{c}''(s)\| \geq 0.$$

If $\kappa(s) \neq 0$ we define the **normal vector** as

$$\mathbf{e}_2(s) = \frac{1}{\kappa(s)}\mathbf{c}''(s),$$

and the **binormal vector** as

$$\mathbf{e}_3(s) = \mathbf{e}_1(s) \times \mathbf{e}_2(s),$$

where

$$\mathbf{e}_1(s) = \mathbf{c}'(s)$$

is the unit tangent vector.

19. **Frenet-Serret formulas:**

$$\begin{bmatrix} \mathbf{e}_1'(s) \\ \mathbf{e}_2'(s) \\ \mathbf{e}_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1(s) \\ \mathbf{e}_2(s) \\ \mathbf{e}_3(s) \end{bmatrix},$$

where the function $\tau(s)$ is called the **torsion** of the curve.

20. A regular space curve $\mathbf{c} : I \to \mathbb{R}^3$ with nonvanishing curvature has zero torsion if and only if it lies on a plane.

21. A **positive isometry** of $\mathbb{R}^3$ is a map $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A \in SO(3)$ is a **rotation matrix**, that is, $A^tA = I$ and $\det A = 1$.

22. Two regular space curves with nonvanishing curvature are related by a positive isometry if and only if their curvatures and torsions coincide.

23. If $\mathbf{c} : I \to \mathbb{R}^3$ is a regular curve (not necessarily parameterized by its arclength) then its curvature and torsion are given by

$$\kappa(t) = \frac{\|\dot{\mathbf{c}}(t) \times \ddot{\mathbf{c}}(t)\|}{\|\dot{\mathbf{c}}(t)\|^3} \qquad \text{and} \qquad \tau(t) = \frac{\dot{\mathbf{c}}(t) \cdot (\ddot{\mathbf{c}}(t) \times \dddot{\mathbf{c}}(t))}{\|\dot{\mathbf{c}}(t) \times \ddot{\mathbf{c}}(t)\|^2}.$$

24. **Frenchel's Theorem:** Let $\mathbf{c} : [a,b] \to \mathbb{R}^3$ be a closed regular space curve parameterized by arclength, and let $\kappa(s) = \|\mathbf{c}''(s)\|$ be its curvature. Then

$$\int_a^b \kappa(s)ds \geq 2\pi,$$

and the equality holds if and only if $\mathbf{c}$ is a plane convex curve.

25. A simple closed regular curve in $\mathbb{R}^3$ is called a **knot**. Two knots are called **equivalent** if they are homotopic (up to reparameterization) by simple closed regular curves. A knot is called **trivial** if it is equivalent to the circle.

26. Let $\mathbf{c} : [a,b] \to \mathbb{R}^3$ be a nontrivial knot parameterized by arclength, and let $\kappa(s) = \|\mathbf{c}''(s)\|$ be its curvature. Then

$$\int_a^b \kappa(s)ds \geq 4\pi.$$

## 2. Differentiable manifolds

1. A set $M \subset \mathbb{R}^n$ is said to be a **differentiable manifold of dimension** $m \in \{1, \ldots, n-1\}$ if for any point $\mathbf{a} \in M$ there exists an open neighborhood $U \ni \mathbf{a}$, an open set $V \subset \mathbb{R}^m$ and a smooth function $\mathbf{f} : V \to \mathbb{R}^{n-m}$ such that

$$M \cap U = \operatorname{Graph}(\mathbf{f}) \cap U$$

   for some ordering of the Cartesian coordinates of $\mathbb{R}^n$. We also define a manifold of dimension $0$ as a set of isolated points, and a manifold of dimension $n$ as an open set.

2. $M \subset \mathbb{R}^n$ is a differentiable manifold of dimension $m$ if and only if for each point $\mathbf{a} \in M$ there exists an open set $U \ni \mathbf{a}$ and a smooth function $\mathbf{F} : U \to \mathbb{R}^{n-m}$ such that:
   (i) $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$;
   (ii) $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$.

3. A vector $\mathbf{v} \in \mathbb{R}^n$ is said to be **tangent** to a set $M \subset \mathbb{R}^n$ at the point $\mathbf{a} \in M$ if there exists a smooth curve $\mathbf{c} : \mathbb{R} \to M$ such that $\mathbf{c}(0) = \mathbf{a}$ and $\dot{\mathbf{c}}(0) = \mathbf{v}$. A vector $\mathbf{v} \in \mathbb{R}^n$ is said to be **orthogonal** to $M$ at the point $\mathbf{a}$ if it is orthogonal to all vectors tangent to $M$ at $\mathbf{a}$.

4. If $M \subset \mathbb{R}^n$ is a manifold of dimension $m$ then the set $T_{\mathbf{a}}M$ of all vectors tangent to $M$ at the point $\mathbf{a} \in M$ is a vector space of dimension $m$, called the **tangent space** to $M$ at $\mathbf{a}$. Its orthogonal complement $T_{\mathbf{a}}^{\perp}M$ is a vector space of dimension $(n - m)$, called the **normal space** to $M$ at $\mathbf{a}$.

5. Let $M \subset \mathbb{R}^n$ be an $m$-manifold, $\mathbf{a} \in M$, $U \ni \mathbf{a}$ an open set and $\mathbf{F} : U \to \mathbb{R}^{n-m}$ such that $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$ with $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$. Then $T_{\mathbf{a}}M = \ker D\mathbf{F}(\mathbf{a})$.

6. A **parameterization** of a given $m$-manifold $M \subset \mathbb{R}^n$ is a smooth injective map $\mathbf{g} : U \to M$, with $U \subset \mathbb{R}^m$ open, such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$. We have

$$T_{\mathbf{g}(\mathbf{t})}M = \operatorname{span}\left\{\frac{\partial \mathbf{g}}{\partial t^1}(\mathbf{t}), \ldots, \frac{\partial \mathbf{g}}{\partial t^m}(\mathbf{t})\right\}.$$

7. Given a smooth map $\mathbf{g} : U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ open, such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$, and given any point $\mathbf{t}_0 \in U$, there exists an open set $U_0 \subset U$ with $\mathbf{t}_0 \in U_0$ such that $\mathbf{g}(U_0)$ is an $m$-manifold.

## 3. Differential forms

1. The **dual vector space** to $\mathbb{R}^n$ is
$$(\mathbb{R}^n)^* = \{\alpha : \mathbb{R}^n \to \mathbb{R} : \alpha \text{ is linear}\}.$$
The elements of $(\mathbb{R}^n)^*$ are called **covectors**.
2. The covectors $dx^1, \ldots, dx^n \in (\mathbb{R}^n)^*$ defined through
$$dx^i(\mathbf{e}_j) = \begin{cases} 1 & \text{if } \; i = j \\ 0 & \text{if } \; i \neq j \end{cases}$$
form a basis for $(\mathbb{R}^n)^*$, whose dimension is then $n$.
3. A (covariant) $k$-**tensor** $T$ is a multilinear map $T : (\mathbb{R}^n)^k \to \mathbb{R}$, i.e.
   (i) $T(\mathbf{v}_1, \ldots, \mathbf{v}_i + \mathbf{w}_i, \ldots, \mathbf{v}_k) = T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k) + T(\mathbf{v}_1, \ldots, \mathbf{w}_i, \ldots, \mathbf{v}_k)$;
   (ii) $T(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_k) = \lambda T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k)$.
4. A $k$-tensor $\alpha$ is said to be **alternanting**, or a $k$-**covector**, if
$$\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j \ldots, \mathbf{v}_k) = -\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i \ldots, \mathbf{v}_k).$$
We denote by $\Lambda^k(\mathbb{R}^n)$ the vector space of all $k$-covectors.
5. Given $i_1, \ldots, i_k \in \{1, \ldots, n\}$, we define $dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Lambda^k(\mathbb{R}^n)$ as
$$dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det \begin{bmatrix} dx^{i_1}(\mathbf{v}_1) & \ldots & dx^{i_1}(\mathbf{v}_k) \\ \ldots & \ldots & \ldots \\ dx^{i_k}(\mathbf{v}_1) & \ldots & dx^{i_k}(\mathbf{v}_k) \end{bmatrix}.$$
The set $\left\{dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right\}_{1 \le i_1 < \ldots < i_k \le n}$ is a basis for $\Lambda^k(\mathbb{R}^n)$, whose dimension is then $\binom{n}{k}$. Since $\binom{n}{0} = 1$, we define $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.
6. If $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta \in \Lambda^l(\mathbb{R}^n)$,
$$\alpha \; = \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \qquad \beta \; = \sum_{j_1 < \ldots < j_l} \beta_{j_1 \ldots j_l} \, dx^{j_1} \wedge \ldots \wedge dx^{j_l},$$
we define their **wedge product** $\alpha \wedge \beta \in \Lambda^{k+l}(\mathbb{R}^n)$ as
$$\alpha \wedge \beta \; = \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_l}} \alpha_{i_1 \ldots i_k} \, \beta_{j_1 \ldots j_l} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$
If $\alpha$ is a $0$-covetor (real number), its wedge product by $\alpha$ is simply the product by a scalar.
7. **Properties of the wedge product:**
   (i) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$;
   (ii) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \text{if} \quad \alpha \in \Lambda^k(\mathbb{R}^n), \beta \in \Lambda^l(\mathbb{R}^n)$;
   (iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

8. A **differential form of degree** $k$ on $U \subset \mathbb{R}^n$ is a smooth function $\omega : U \to \Lambda^k(\mathbb{R}^n)$. We denote by $\Omega^k(U)$ the set of $k$-forms on $U$.
9. If $\mathbf{f} : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ is smooth and $\omega \in \Omega^k(V)$ then the **pull-back** of $\omega$ by $\mathbf{f}$ is the $k$-form $\mathbf{f}^*\omega \in \Omega^k(U)$ defined by

$$(\mathbf{f}^*\omega)(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega(\mathbf{f}(\mathbf{x}))(D\mathbf{f}(\mathbf{x})\mathbf{v}_1, \ldots, D\mathbf{f}(\mathbf{x})\mathbf{v}_k).$$

10. **Properties of the pull-back:**
    (i) $\mathbf{f}^*(\omega + \eta) = \mathbf{f}^*\omega + \mathbf{f}^*\eta$;
    (ii) $\mathbf{f}^*(\omega \wedge \eta) = \mathbf{f}^*\omega \wedge \mathbf{f}^*\eta$;
    (iii) $(\mathbf{g} \circ \mathbf{f})^*(\omega) = \mathbf{f}^*(\mathbf{g}^*\omega)$.
11. If $\omega \in \Omega^k(U)$ with $U \subset \mathbb{R}^n$,

$$\omega \;=\; \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k}(\mathbf{x})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

then its **exterior derivative** is the $(k+1)$-form $d\omega \in \Omega^{k+1}(U)$ defined by

$$d\omega \;=\; \sum_{i_1 < \ldots < i_k} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \ldots i_k}}{\partial x^i}\, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

12. **Properties of the exterior derivative:**
    (i) $d(\omega + \eta) = d\omega + d\eta$;
    (ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ if $\omega$ has degree $k$;
    (iii) $d(d\omega) = 0$;
    (iv) $\mathbf{f}^*(d\omega) = d(\mathbf{f}^*\omega)$.
13. We say that $\omega \in \Omega^k(U)$ is:
    (i) **closed** if $d\omega = 0$;
    (ii) **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$ (called a **potential** for $\omega$).
14. If $\omega \in \Omega^k(U)$ is exact then $\omega$ is closed.
15. **Poincaré Lemma:** If $\omega \in \Omega^k(U)$ is closed and the open set $U$ is star-shaped then $\omega$ is exact.
16. If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ are parameterizations of the $m$-manifold $M \subset \mathbb{R}^n$ then $\mathbf{h}^{-1} \circ \mathbf{g}$ is a **diffeomorphism** (smooth bijection with smooth inverse).
17. We say that two parameterizations $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ of the $m$-manifold $M \subset \mathbb{R}^n$ induce the **same orientation** if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) > 0$, and **opposite orientations** if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) < 0$. The manifold $M$ is called **orientable** if it is possible to choose parameterizations whose images cover $M$ and induce the same orientation. An **orientation** on an orientable manifold is a choice of a maximal family of parameterizations under these conditions, which are said to be **positive**. An orientable manifold with a choice of orientation is said to be **oriented**.

18. If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ is a positive parameterization of the oriented $m$-manifold $M \subset \mathbb{R}^n$ and $\omega \in \Omega^m(\mathbb{R}^n)$, we define the **integral** of $\omega$ along $\mathbf{g}(U)$ as

$$\int_{\mathbf{g}(U)} \omega = \int_U \omega(\mathbf{g}(\mathbf{t})) \left( \frac{\partial \mathbf{g}}{\partial t^1}, \ldots, \frac{\partial \mathbf{g}}{\partial t^m} \right) dt^1 \ldots dt^m$$
$$= \int_U \mathbf{g}^* \omega \left( \mathbf{e}_1, \ldots, \mathbf{e}_m \right) dt^1 \ldots dt^m.$$

19. If we think of an open set $U \subset \mathbb{R}^n$ as an $n$-manifold parameterized by the identity map (which we take to be positive), then

$$\int_U f(\mathbf{x})\, dx^1 \wedge \ldots \wedge dx^n = \int_U f(\mathbf{x})\, dx^1 \ldots dx^n,$$

and so

$$\int_{\mathbf{g}(U)} \omega = \int_U \mathbf{g}^* \omega.$$

20. The integral of a $m$-form on the image of a positive parameterization of an $m$-manifold is well defined, that is, it is independent of the choice of parameterization.
21. If $M \subset \mathbb{R}^n$ is an oriented $m$-manifold and $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \sum_{i=1}^{N} \int_{\mathbf{g}_i(U_i)} \omega,$$

where $\mathbf{g}_i : U_i \to M$ are positive parameterizations whose images are disjoint and cover $M$ except for a finite number of manifolds of dimension smaller than $m$. It can be shown that it is always possible to obtain a finite number of parameterizations of this kind, and that the definition above does not depend on the choice of these parameterizations.
22. Informally, an $m$-**manifold with boundary** is a subset $M \subset N$ of an $m$-manifold $N \subset \mathbb{R}^n$ delimited by an $(m-1)$-manifold $\partial M \subset M$, called the **boundary** of $M$, such that $M \setminus \partial M$ is again an $m$-manifold. We say that $M$ is **orientable** if $N$ is orientable. If $M$ is oriented, the **induced orientation** on $\partial M$ is defined as follows: if $\mathbf{g} : U \cap \{t^1 \leq 0\} \to M$ is a positive parameterization of $M$ such that $\mathbf{h}(t^2, \ldots, t^m) = \mathbf{g}(0, t^2, \ldots, t^m)$ is a parameterization of $\partial M$, then $\mathbf{h}$ is positive. Moreover, if $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \int_{M \setminus \partial M} \omega.$$

23. **Stokes Theorem:** If $M \subset \mathbb{R}^n$ is a compact, oriented $m$-manifold with boundary and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then

$$\int_M d\omega = \int_{\partial M} \omega,$$

where $\partial M$ has the induced orientation.
24. If $M$ is an oriented compact $m$-manifold (without boundary) and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then

$$\oint_M d\omega = 0.$$

## 4. <u>Surfaces</u>

1. A **surface** is a 2-dimensional differentiable manifold $S \subset \mathbb{R}^3$.
2. The **first fundamental form** of a surface $S$ parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is the quadratic form
$$\mathbf{I} = d\mathbf{g} \cdot d\mathbf{g} = E du^2 + 2F du\, dv + G dv^2,$$
where
$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{g}}{\partial u} & \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{g}}{\partial v} \\ \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{g}}{\partial u} & \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{g}}{\partial v} \end{bmatrix}$$
is a positive definite matrix of functions, called the **matrix of the metric**.
3. The squared length of a vector tangent to a surface $S$ parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is
$$\left\| v^1 \frac{\partial \mathbf{g}}{\partial u} + v^2 \frac{\partial \mathbf{g}}{\partial v} \right\|^2 = \mathbf{I}(v^1, v^2) = E(v^1)^2 + 2F v^1 v^2 + G(v^2)^2.$$
In particular, the length of a curve $\mathbf{c} : [a,b] \to S$ given by $\mathbf{c}(t) = \mathbf{g}(u(t), v(t))$ is
$$\int_a^b \sqrt{\mathbf{I}(\dot{u}(t), \dot{v}(t))}\, dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$
4. The **second fundamental form** of a surface $S$ parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is the quadratic form
$$\mathbf{II} = -d\mathbf{g} \cdot d\mathbf{n} = L du^2 + 2M du\, dv + N dv^2,$$
where
$$\mathbf{n} = \frac{\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}}{\left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\|}$$
is a unit normal vector to $S$ and
$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = - \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial u} & \frac{\partial \mathbf{g}}{\partial u} \cdot \frac{\partial \mathbf{n}}{\partial v} \\ \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial u} & \frac{\partial \mathbf{g}}{\partial v} \cdot \frac{\partial \mathbf{n}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathbf{g}}{\partial u^2} \cdot \mathbf{n} & \frac{\partial^2 \mathbf{g}}{\partial u \partial v} \cdot \mathbf{n} \\ \frac{\partial^2 \mathbf{g}}{\partial v \partial u} \cdot \mathbf{n} & \frac{\partial^2 \mathbf{g}}{\partial v^2} \cdot \mathbf{n} \end{bmatrix}.$$
5. At a point where the second fundamental form is definite ($LN - M^2 > 0$) the surface is convex (i.e. it lies on the same side of the tangent plane); at a point where the second fundamental form is indefinite ($LN - M^2 < 0$) the surface is not convex (i.e. it lies on both sides of the tangent plane).
6. **Gauss's equations:**
$$\frac{\partial^2 \mathbf{g}}{\partial u^2} = \Gamma^u_{uu} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{uu} \frac{\partial \mathbf{g}}{\partial v} + L\mathbf{n};$$
$$\frac{\partial^2 \mathbf{g}}{\partial u \partial v} = \Gamma^u_{uv} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{uv} \frac{\partial \mathbf{g}}{\partial v} + M\mathbf{n};$$
$$\frac{\partial^2 \mathbf{g}}{\partial v \partial u} = \Gamma^u_{vu} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{vu} \frac{\partial \mathbf{g}}{\partial v} + M\mathbf{n};$$
$$\frac{\partial^2 \mathbf{g}}{\partial v^2} = \Gamma^u_{vv} \frac{\partial \mathbf{g}}{\partial u} + \Gamma^v_{vv} \frac{\partial \mathbf{g}}{\partial v} + N\mathbf{n},$$
where the functions $\Gamma^u_{uu}, \Gamma^u_{uv} = \Gamma^u_{vu}, \Gamma^u_{vv}, \Gamma^v_{uu}, \Gamma^v_{uv} = \Gamma^v_{vu}, \Gamma^v_{vv}$ are called the **Christoffel symbols**.

7. **Weingarten's equations:**

$$\frac{\partial \mathbf{n}}{\partial u} = \frac{FM - GL}{EG - F^2}\frac{\partial \mathbf{g}}{\partial u} + \frac{FL - EM}{EG - F^2}\frac{\partial \mathbf{g}}{\partial v};$$
$$\frac{\partial \mathbf{n}}{\partial v} = \frac{FN - GM}{EG - F^2}\frac{\partial \mathbf{g}}{\partial u} + \frac{FM - EN}{EG - F^2}\frac{\partial \mathbf{g}}{\partial v}.$$

8. The **normal curvature** of a curve $\mathbf{c} : I \to S$ on a surface $S$, parameterized by arclength, is $\kappa_n(s) = \mathbf{c}''(s) \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit normal vector to $S$ at $\mathbf{c}(s)$. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization and $\mathbf{c}(s) = \mathbf{g}(u(s), v(s))$ then $\kappa_n(s) = \mathbf{II}(u'(s), v'(s))$.
9. The maximum and the minimum of $\mathbf{II}(v^1, v^2)$ subject to the constraint $\mathbf{I}(v^1, v^2) = 1$ are called the **principal curvatures** of $S$ at the point under consideration. The directions of the corresponding unit tangent vectors are called the **principal directions** of $S$ at that point. If the principal curvatures are different then the principal directions are orthogonal.
10. The **mean curvature** of a surface $S$ at a given point is

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

where $\kappa_1$ and $\kappa_2$ are the principal curvatures at that point. The **Gauss curvature** of $S$ at the same point is

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}.$$

$S$ is said to be **minimal** if $H \equiv 0$, and **flat** if $K \equiv 0$.
11. If $\kappa_1 = \kappa_2$ at some point then that point is called **umbillic**. Moreover, we call the point **elliptic** if $K > 0$, **hyperbolic** if $K < 0$, and **parabolic** if $K = 0$. The surface is convex at elliptic points, and is not convex at hyperbolic points.
12. The principal direction corresponding to the principal curvature $\kappa_1$ is given by tangent vectors of the form

$$v^1 \frac{\partial \mathbf{g}}{\partial u} + v^2 \frac{\partial \mathbf{g}}{\partial v}$$

such that

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \kappa_1 \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}.$$

13. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization then the **area** of $\mathbf{g}(U) \subset S$ is

$$A = \iint_U \left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| du\, dv = \iint_U \sqrt{EG - F^2}\, du\, dv.$$

14. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization then

$$\frac{\partial \mathbf{n}}{\partial u} \times \frac{\partial \mathbf{n}}{\partial v} = K \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}.$$

In particular, if $K(u_0, v_0) \neq 0$ then

$$|K(u_0, v_0)| = \lim_{\varepsilon \to 0} \frac{A'(\varepsilon)}{A(\varepsilon)},$$

where $A(\varepsilon)$ is the area of $\mathbf{g}(B_\varepsilon(u_0, v_0)) \subset S$ and $A'(\varepsilon)$ is the area of $\mathbf{n}(B_\varepsilon(u_0, v_0)) \subset S^2$.

15. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization,

$$\mathbf{g}_\varepsilon(u, v) = \mathbf{g}(u, v) + \varepsilon f(u, v)\mathbf{n}(u, v)$$

is a small deformation of $\mathbf{g}$ and $A(\varepsilon)$ is the area of $\mathbf{g}_\varepsilon(U)$ then

$$\frac{dA}{d\varepsilon}(0) = -2 \iint_U fH\sqrt{EG - F^2}\, du\, dv.$$

In particular, if $S$ has minimal area (for a fixed boundary) then $H \equiv 0$. If $V(\varepsilon)$ is the volume bounded by $\mathbf{g}_\varepsilon(U)$ (possibly together with another, fixed surface) then

$$\frac{dV}{d\varepsilon}(0) = \iint_U f\sqrt{EG - F^2}\, du\, dv.$$

In particular, if $S$ has minimal area while bounding a fixed volume then $H$ is constant.

16. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}\}$ is an orthonormal frame and $\theta^1, \theta^2 \in \Omega^1(U)$ are such that

$$d\mathbf{g} = \theta^1\mathbf{e}_1 + \theta^2\mathbf{e}_2$$

then the first fundamental form is

$$\mathbf{I} = (\theta^1)^2 + (\theta^2)^2.$$

Moreover, if ${\omega_i}^j \in \Omega^1(U)$ are such that

$$d\mathbf{e}_i = \sum_{j=1}^{3} {\omega_i}^j \mathbf{e}_j,$$

we have

$${\omega_i}^j = -{\omega_j}^i.$$

Defining the symmetric $2 \times 2$ matrix $B$ through

$$\begin{cases} {\omega_1}^3 = b_{11}\theta^1 + b_{12}\theta^2 \\ {\omega_2}^3 = b_{21}\theta^1 + b_{22}\theta^2 \end{cases},$$

we have

$$\mathbf{II} = \sum_{i,j=1}^{2} b_{ij}\theta^i\theta^j.$$

In particular,

$$H = \frac{1}{2}\operatorname{tr} B \qquad \text{and} \qquad K = \det B$$

(that is, the eigenvalues of $B$ are $\kappa_1$ and $\kappa_2$).

17. **First structure equations:** $d\theta^i = \sum_{j=1}^{2} \theta^j \wedge {\omega_j}^i \Leftrightarrow \begin{cases} d\theta^1 = \theta^2 \wedge {\omega_2}^1 \\ d\theta^2 = \theta^1 \wedge {\omega_1}^2 \end{cases}$.

18. **Second structure equation:** $d{\omega_2}^1 = K\theta^1 \wedge \theta^2$.

## 5. Intrinsic geometry of Riemannian surfaces

1. A **Riemannian surface** is a choice of a first fundamental form (also called a **Riemannian metric**) $\mathbf{I} = ds^2$ on some open set $U \subset \mathbb{R}^2$ (possibly arising from a parameterization of a surface in $\mathbb{R}^3$ or of a general 2-manifold in $\mathbb{R}^n$). Given 1-forms $\{\theta^1, \theta^2\}$ such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2,$$

we define the **connection form** associated to $\{\theta^1, \theta^2\}$ as the unique 1-form ${\omega_2}^1$ such that

$$\begin{cases} d\theta^1 = \theta^2 \wedge {\omega_2}^1 \\ d\theta^2 = -\theta^1 \wedge {\omega_2}^1 \end{cases},$$

and the **Gauss curvature** as the function $K$ such that

$$d{\omega_2}^1 = K\theta^1 \wedge \theta^2.$$

It turns out that the Gauss curvature is well defined, that is, it does not depend on the choice of $\{\theta^1, \theta^2\}$.

2. **Gauss's Theorema Egregium:** The Gauss curvature of a surface $S \subset \mathbb{R}^3$ depends only on its first fundamental form.

3. If the first fundamental form is of the type

$$ds^2 = E\left(du^2 + dv^2\right)$$

then

$$K = -\frac{1}{E}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\sqrt{E}.$$

4. On a general Riemannian surface parameterized by the coordinates $(t^1, t^2)$ we **define** the tangent vector with components $(v^1, v^2) \in \mathbb{R}^2$ to be the derivative operator

$$v^1\frac{\partial}{\partial t^1} + v^2\frac{\partial}{\partial t^2}.$$

If we change coordinates to $(s^1, s^2)$ then this operator changes to

$$w^1\frac{\partial}{\partial s^1} + w^2\frac{\partial}{\partial s^2},$$

where

$$w^i = \sum_{j=1}^{2}\frac{\partial s^i}{\partial t^j}v^j.$$

If we correspondingly change the first fundamental form as

$$\mathbf{I} = \sum_{i,j=1}^{2} g_{ij}dt^i dt^j = \sum_{i,j,k,l=1}^{2} g_{ij}\frac{\partial t^i}{\partial s^k}\frac{\partial t^j}{\partial s^l}ds^k ds^l$$

then $\mathbf{I}(\mathbf{v}) = \mathbf{I}(\mathbf{w})$.

5. The inner product of two tangent vectors on a Riemannian surface is defined by the formula $\mathbf{I}(\mathbf{v}+\mathbf{w}) = \mathbf{I}(\mathbf{v}) + \mathbf{I}(\mathbf{w}) + 2\langle \mathbf{v}, \mathbf{w}\rangle$. If $\mathbf{I} = Edu^2 + 2F du\, dv + G dv^2$ then this inner product is given by

$$\left\langle v^1 \frac{\partial}{\partial u} + v^2 \frac{\partial}{\partial v}, w^1 \frac{\partial}{\partial u} + w^2 \frac{\partial}{\partial v} \right\rangle = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}.$$

If $ds^2 = (\theta^1)^2 + (\theta^2)^2$ then the dual frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\theta^1, \theta^2\}$ is an orthonormal frame.

6. If $S \subset \mathbb{R}^3$ is a surface and $\mathbf{c} : I \subset \mathbb{R} \to S$ is a curve then a **vector field along c** is a function $\mathbf{V} : I \to \mathbb{R}^3$ such that $\mathbf{V}(t) \in T_{\mathbf{c}(t)}S$ for all $t \in I$, and the **covariant derivative** of $\mathbf{V}$ along $\mathbf{c}$ is the vector field defined by

$$\frac{D\mathbf{V}}{dt}(t) = \frac{d\mathbf{V}}{dt}(t) - \left( \frac{d\mathbf{V}}{dt}(t) \cdot \mathbf{n}(t) \right) \mathbf{n}(t),$$

where $\mathbf{n}(t)$ is the unit normal vector to $S$ at $\mathbf{c}(t)$.

7. Given a Riemannian metric $ds^2 = (\theta^1)^2 + (\theta^2)^2$ on a open set $U \subset \mathbb{R}^2$, and a curve $(u, v) : I \to U$, we define the **covariant derivative** of the **vector field along the curve** $\mathbf{V} : I \to \mathbb{R}^2$, defined as

$$\mathbf{V} = V^1(t)\, \mathbf{e}_1(u(t), v(t)) + V^2(t)\, \mathbf{e}_2(u(t), v(t)),$$

as the vector field along the curve

$$\frac{D\mathbf{V}}{dt} = \left( \frac{dV^1}{dt} + V^2 {\omega_2}^1(\dot{u}, \dot{v}) \right) \mathbf{e}_1 + \left( \frac{dV^2}{dt} - V^1 {\omega_2}^1(\dot{u}, \dot{v}) \right) \mathbf{e}_2,$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the orthonormal frame dual to $\{\theta^1, \theta^2\}$ and ${\omega_2}^1$ is the connection form associated to $\{\theta^1, \theta^2\}$. It turns out that the covariant derivative is well defined, that is, it does not depend on the choice of $\{\theta^1, \theta^2\}$.

8. If $\mathbf{V} : I \to \mathbb{R}^2$ and $\mathbf{W} : I \to \mathbb{R}^2$ are vector fields along a curve then

$$\frac{d}{dt} \langle \mathbf{V}(t), \mathbf{W}(t) \rangle = \left\langle \frac{D\mathbf{V}}{dt}(t), \mathbf{W}(t) \right\rangle + \left\langle \mathbf{V}(t), \frac{D\mathbf{W}}{dt}(t) \right\rangle.$$

9. A vector field $\mathbf{V}$ is said to be **parallel** (or **parallel transported**) along a given curve if $\frac{D\mathbf{V}}{dt} = 0$ along that curve. If $\mathbf{V}$ and $\mathbf{W}$ are both parallel along a curve then $\langle \mathbf{V}, \mathbf{W} \rangle$ is constant along that curve; in particular, $\mathbf{I}(\mathbf{V})$, $\mathbf{I}(\mathbf{W})$ and $\sphericalangle(\mathbf{V}, \mathbf{W})$ are constant along the curve.

10. If $\mathbf{c} : I \to S$ is a curve on a surface $S \subset \mathbb{R}^3$, parameterized by arclength, then we have decomposition $\mathbf{c}''(s) = \boldsymbol{\kappa}_g(s) + \boldsymbol{\kappa}_n(s)$, where $\boldsymbol{\kappa}_g(s) \in T_{\mathbf{c}(s)}S$ is the **geodesic curvature vector** and $\boldsymbol{\kappa}_n(s) \in T^{\perp}_{\mathbf{c}(s)}S$ is the **normal curvature vector**. We have

$$\boldsymbol{\kappa}_g(s) = \frac{D\mathbf{c}'}{ds}(s) \qquad \text{and} \qquad \boldsymbol{\kappa}_n(s) = \mathbf{II}(u'(s), v'(s))\mathbf{n}.$$

11. A **geodesic** on a Riemannian surface is a curve whose velocity vector is parallel along the curve, that is, a solution of the equation

$$\frac{D\dot{\mathbf{c}}}{dt}(t) = 0.$$

In particular, the length of the velocity vector is constant, and so the parameter is an affine function of the arclength (**affine parameter**).

12. Curves with minimal length (among all curves connecting two given points) are necessarily geodesics (up to reparameterization).

13. The geodesic equations for a surface $S \subset \mathbb{R}^3$ can be written as

$$\begin{cases} \ddot{u} + \Gamma^u_{uu}\dot{u}^2 + 2\Gamma^u_{uv}\dot{u}\dot{v} + \Gamma^u_{vv}\dot{v}^2 = 0 \\ \ddot{v} + \Gamma^v_{uu}\dot{u}^2 + 2\Gamma^v_{uv}\dot{u}\dot{v} + \Gamma^v_{vv}\dot{v}^2 = 0 \end{cases}.$$

In particular, the Christoffel symbols can only depend on the first fundamental form, and are indeed given by

$$\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial u} & \frac{1}{2}\frac{\partial E}{\partial v} & \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} & \frac{1}{2}\frac{\partial G}{\partial u} & \frac{1}{2}\frac{\partial G}{\partial v} \end{bmatrix}$$

(so that these equations also hold for an abstract Riemannian surface).

14. If $\mathbf{c}(s)$ is a geodesic parameterized by arclength then its length between $\mathbf{c}(s_0)$ and $\mathbf{c}(s_1)$ is minimal (among all curves connecting $\mathbf{c}(s_0)$ and $\mathbf{c}(s_1)$) provided that $s_1$ is sufficiently close to $s_0$.

## 6. Gauss-Bonnet Theorem

1. If $\mathbf{c}(s)$ is a curve parameterized by arclength on an oriented Riemannian surface then its (scalar) **geodesic curvature** is the function

$$\kappa_g(s) = \left\langle \frac{D\mathbf{c}'}{ds}(s), \mathbf{n}(s) \right\rangle,$$

where

$$\mathbf{n}(s) = -\left\langle \mathbf{c}'(s), \mathbf{e}_2 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{c}'(s), \mathbf{e}_1 \right\rangle \mathbf{e}_2$$

is the unit normal to the curve obtained by rotating $\mathbf{c}'(s)$ by $90^\circ$ in the positive direction. Here $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a positive orthonormal frame, that is, $du \wedge dv(\mathbf{e}_1, \mathbf{e}_2) > 0$, where the coordinate system $(u, v)$ is assumed to be positive.

2. A **domain** on $\mathbb{R}^2$ is a compact 2-dimensional manifold with boundary, that is, a compact set $A \subset \mathbb{R}^2$ whose boundary $\partial A$ is a 1-dimensional manifold. Informally, a **domain with corners** is a generalization where we allow $\partial A$ to have a finite number of vertices (that is, it can be parameterized by a sectionally smooth regular curve).

3. If $ds^2 = Edu^2 + 2F du\, dv + Gdv^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2$ is the line element of an oriented Riemannian surface, where $\{\theta^1, \theta^2\}$ is dual to a positive orthonormal frame, then

$$\theta^1 \wedge \theta^2 = \sqrt{EG - F^2}\, du \wedge dv.$$

The **area** of a domain $A$ is

$$\operatorname{area}(A) = \int_A \theta^1 \wedge \theta^2.$$

4. **Gauss-Bonnet Theorem for domains:** If $A$ is a simply connected domain on an oriented Riemannian surface with metric $ds^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2$, with $\{\theta^1, \theta^2\}$ dual to a positive orthonormal frame, then

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} \kappa_g(s) ds = 2\pi,$$

where $\partial A$ has the induced orientation.

5. **Gauss-Bonnet Theorem for domains with corners:** If $A$ is a simply connected domain with corners on an oriented Riemannian surface with metric $ds^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2$, with $\{\theta^1, \theta^2\}$ dual to a positive orthonormal frame, then

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} \kappa_g(s) ds + \sum_{i=1}^{n} \varepsilon_i = 2\pi,$$

where $\partial A$ has the induced orientation and $\varepsilon_1, \ldots, \varepsilon_n$ are the angles by which the velocity vector rotates at each corner.

6. If $A$ is a **geodesic triangle**, that is, a simply connected domain with $3$ corners whose boundary is the union of the images of $3$ geodesics, and if $\alpha$, $\beta$ and $\gamma$ are its internal angles, then

$$\alpha + \beta + \gamma = \pi + \int_A K\theta^1 \wedge \theta^2.$$

7. If $\mathbf{V} : [t_0, t_1] \to \mathbb{R}^2$ is parallel along the boundary of a simply connected domain parameterized by a closed curve $\mathbf{c} : [t_0, t_1] \to \partial A$ then the angle $\alpha$ between $\mathbf{V}(t_1)$ and $\mathbf{V}(t_0)$ at the point $\mathbf{c}(t_0) = \mathbf{c}(t_1)$ is
$$\alpha = \int_A K\theta^1 \wedge \theta^2.$$
8. A **triangle** on a compact 2-manifold (surface) $S \subset \mathbb{R}^n$ is the image of an Euclidean triangle by a parameterization $\mathbf{g} : U \subset \mathbb{R}^2 \to S$. A **triangulation** of $S$ is a decomposition of $S$ into a finite number of triangles such that the intersection of any two triangles is precisely a common edge, a common vertex or empty. The **Euler characteristic** of $S$ is the integer $\chi(S) = V - E + F$, where $V$, $E$ and $F$ are the total numbers of vertices, edges and triangles on any triangulation.
9. **Gauss-Bonnet Theorem for compact surfaces:** If $S \subset \mathbb{R}^n$ is a compact (orientable) surface then
$$\int_S K = 2\pi\chi(S)$$
(also true for non-orientable surfaces).
10. We consider compact surfaces up to **homeomorphism** (i.e. continuous deformation), which preserves the Euler characteristic. The **connected sum** $S_1 \# S_2$ of two surfaces $S_1$ and $S_2$ is the surface obtained by removing a small disk on both surfaces and gluing them along the disk's boundary. We have $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.
11. Any orientable surface is homeomorphic to either the sphere $S^2$ or a connected sum of $g$ tori $T^2$, and so its Euler characteristic is $2 - 2g$ (with $g = 0$ for the sphere). The integer $g$ is known as the **genus** of the surface.
12. Examples of non-orientable surfaces are the **Klein bottle** $K^2$ and the **projective plane** $P^2$. We have $\chi(K^2) = 0$ and $\chi(P^2) = 1$. In fact, $K^2 = P^2 \# P^2$, and any non-orientable surface is homeomorphic to a connected sum of projective planes.

## 7. Minimal surfaces

1. The graph of a smooth function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a minimal surface (i.e. has vanishing mean curvature) if and only if

$$\left[1+\left(\frac{\partial f}{\partial y}\right)^2\right]\frac{\partial^2 f}{\partial x^2}+\left[1+\left(\frac{\partial f}{\partial x}\right)^2\right]\frac{\partial^2 f}{\partial y^2}-2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\frac{\partial^2 f}{\partial x \partial y}=0,$$

   or, equivalently,

$$\frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{W}\frac{\partial f}{\partial y}\right)=0,$$

   where

$$W=\left[1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]^{\frac{1}{2}}.$$

2. A coordinate system $(u, v)$ on a Riemannian surface is called **isothermal** if the metric in these coordinates has the form $ds^2 = E(du^2 + dv^2)$ (so that the angle between two vectors coincides with the Euclidean angle). Any (minimal) surface can be parameterized by isothermal coordinates.
3. If $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ is a parameterization of the surface $S \subset \mathbb{R}^3$ by isothermal coordinates then

$$\frac{\partial^2 \mathbf{g}}{\partial u^2}+\frac{\partial^2 \mathbf{g}}{\partial v^2}\equiv \Delta \mathbf{g}=2EH\mathbf{n}.$$

   In particular, $S$ is minimal if and only if the components $x(u, v)$, $y(u, v)$ and $z(u, v)$ of the parameterization are harmonic functions,

$$\Delta x=\Delta y=\Delta z=0,$$

   implying that there are no compact minimal surfaces (without boundary).
4. **Weierstrass-Enneper Theorem:** Any simply connected minimal surface can be parameterized by $\mathbf{g} : U \to \mathbb{R}^3$, with $U \subset \mathbb{C}$ simply connected, given by

$$\mathbf{g}(w)=\left(\operatorname{Re}\int \frac{1}{2}f(w)(1-g^2(w))dw, \operatorname{Re}\int \frac{i}{2}f(w)(1+g^2(w))dw, \operatorname{Re}\int f(w)g(w)dw\right),$$

   where $f$ is a holomorphic function in $U$ and $g$ is a meromorphic function in $U$. The zeros of $f$ coincide with the poles of $g$, and the order of the zeros of $f$ is twice the order of the poles of $g$. Moreover, the first fundamental form is given by

$$\mathbf{I}=\frac{1}{4}|f(w)|^2\left(1+|g(w)|^2\right)^2 dw d\bar{w}.$$

5. Minimal surfaces corresponding to the Weierstrass-Enneper data $f_\theta(w) = e^{i\theta} f(w)$ are called **associated minimal surfaces**, and in particular are **isometric** (that is, have the same first fundamental form). The minimal surfaces corresponding to $f(w)$ and to $-if(w)$ are called **conjugate minimal surfaces**, as the corresponding coordinate functions are conjugate harmonic functions.

6. The Gauss curvature of a minimal surface with Weierstrass-Enneper data $f(w)$ and $g(w)$ is

$$K(w) = -\left(\frac{4|g'(w)|}{|f(w)|\,(1+|g(w)|^2)^2}\right)^2,$$

   and the principal curvatures are

$$\kappa_1(w) = \frac{4|g'(w)|}{|f(w)|\,(1+|g(w)|^2)^2} \qquad \text{and} \qquad \kappa_2(w) = -\frac{4|g'(w)|}{|f(w)|\,(1+|g(w)|^2)^2}.$$

7. **Ricci Theorem:** Let $ds^2$ be the metric of a simply connected Riemannian surface with Gauss curvature $K < 0$. Then there exists a minimal surface parameterized by $\mathbf{g} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $ds^2 = d\mathbf{g} \cdot d\mathbf{g}$ if and only if the Gauss curvature of $d\tilde{s}^2 = \sqrt{-K}ds^2$ is zero.