Differential Geometry of Curves and Surfaces

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Author: Tiago Mourão

Based on lectures from Professor José Natário

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1 Curves

Definition 1.1. (Smooth) If $U \subset \mathbb{R}^n$ is an open set then a **smooth** map (or a differentiable map) $F : U \to \mathbb{R}^m$ is a C^∞ map. If $D \subset \mathbb{R}^n$ is any set then $F : D \to \mathbb{R}^m$ is a smooth map if there exists an open set U with $D \subset U$ and a smooth map $G : D \to \mathbb{R}^m$ such that $G|_D \equiv F$.

Definition 1.2. (Curve) A **curve** in $\mathbb{R}^n$ is a smooth map $c : I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$ is an interval.

Definition 1.3. (Regular curve) A curve $c : I \to \mathbb{R}^n$ is said to be **regular** if $c'(t) \neq 0$ for all $t \in I$.

Proposition 1.4. (Arclength) If $c : I \to \mathbb{R}^n$ is a curve and $t_0 \in I$ then the arclength measured from t_0 is

$$s(t) := \int_{t_0}^{t} \|c'(u)\| \, du.$$

Notation 1.5. If c is regular then $s(t)$ is invertible, and we write $c(s) = c(t(s))$ (slightly abusing the notation). Then we can use s as a new parameter for the curve, and this new parameter satisfies $\|c'(s)\| = 1$.

1.1 Curves in the plane

Definition 1.6. If $c : I \to \mathbb{R}^2$ is a regular curve parameterized by arclength, we define the positive orthonormal frame $\{\vec{e_1}(s), \vec{e_2}(s)\}$ by taking $\vec{e_1}(s) = c'(s)$ (tangent to the curve) and $\vec{e_2}(s) = R_{\frac{\pi}{2}}\vec{e_1}(s)$, where $R_{\frac{\pi}{2}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a rotation by 90° in the positive direction.

Proposition 1.7. The vectors $\vec{e_i}$ and $\vec{e_i}'$ are orthogonal.

Proof. We have

$$\vec{e_i} \cdot \vec{e_i} = 1 \Rightarrow \vec{e_i} \cdot \vec{e_i}' = \frac{1}{2}(\vec{e_i} \cdot \vec{e_i})' = 0,$$

as required. □

This implies that $\vec{e_1}$ and $\vec{e_2}'$ are parallel and that $\vec{e_1}'$ and $\vec{e_2}$ are parallel.

Definition 1.8. (Curvature) The **curvature** of the regular curve $c : I \to \mathbb{R}^2$ parameterized by arclength is the smooth function $k : I \to \mathbb{R}$ such that $c''(s) = k(s)\vec{e_2}(s)$.

Proposition 1.9. The curvature of $c(s)$ is zero if and only if $c(s)$ is a straight line.

Proof. If the curvature is zero then

$$c''(s) = 0 \Leftrightarrow c(s) = as + b,$$

as required. □

Remark 1.10. From Taylor's formula we have

$$\begin{aligned} c(s) &= c(s_0) + c'(s_0)(s - s_0) + c''(s_0)(s - s_0)^2 + \mathcal{O}((s - s_0)^3) \\ &= c(s_0) + \vec{e_1}(s_0)(s - s_0) + \frac{1}{2}k(s_0)\vec{e_2}(s_0)(s - s_0) + \mathcal{O}((s - s_0)^3). \end{aligned}$$

Hence, if $k(s_0) \neq 0$ then $r(s_0) = \frac{1}{|k(s_0)|}$ is the radius of the circle that approximates $c(s)$ to the second order around $c(s_0)$ (radius of curvature).

Proposition 1.11. We have

$$\begin{bmatrix} \vec{e_1}'(s) \\ \vec{e_2}'(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) \\ -k(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{e_1}(s) \\ \vec{e_2}(s) \end{bmatrix}$$

Proof. By definition we already have

$$k(s)\vec{e_2}(s) = \vec{e_1}'(s).$$

Furthermore,

$$\begin{aligned} \vec{e_1}(s) \cdot \vec{e_2}(s) = 0 \Rightarrow \vec{e_1}'(s) \cdot \vec{e_2}(s) + \vec{e_1}(s) \cdot \vec{e_2}'(s) = 0 \\ \Leftrightarrow k(s)(\vec{e_2}(s) \cdot \vec{e_2}(s)) + \vec{e_1}(s) \cdot \vec{e_2}'(s) = 0 \\ \Leftrightarrow k(s) + \vec{e_1}(s) \cdot \vec{e_2}'(s) = 0. \end{aligned}$$

Because $\vec{e_1}(s)$ and $\vec{e_2}'(s)$ are parallel, there exists some function α such that

$$\vec{e_2}'(s) = \alpha(s)\vec{e_1}(s),$$

and then we have

$$k(s) + \alpha(s) = 0 \Rightarrow \vec{e_2}'(s) = -k(s)\vec{e_1}(s),$$

as required. □

Definition 1.12. (Positive isometry) A **positive isometry** of $\mathbb{R}^2$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $F(x) = Ax + b$, where $A \in SO(2)$ is a rotation matrix, that is,

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad \alpha \in \mathbb{R}.$$

Theorem 1.13. *Two curves c and d, parameterized by arclength, are related by a positive isometry (i.e., $d(s) = Ac(s) + b$) if and only if their curvatures coincide, i.e., $k_c(s) = k_d(s)$.*

Proof. (⇒) If $d(s) = Ac(s) + b$ then we have, using the fact that rotation matrices commute,

$$\begin{aligned} d'(s) = Ac'(s) \Rightarrow d''(s) = Ac''(s) = A\Big(k_c(s)R_{\frac{\pi}{2}}c'(s)\Big) &= k_c(s)AR_{\frac{\pi}{2}}c'(s) \\ &= k_c(s)R_{\frac{\pi}{2}}\Big(Ac'(s)\Big) \\ &= k_c(s)R_{\frac{\pi}{2}}d'(s). \end{aligned}$$

On the other hand,

$$d''(s) = k_d(s)R_{\frac{\pi}{2}}d'(s),$$

and therefore $k_c(s) = k_d(s)$.

(⇐) Assume that $k_c(s) = k_d(s)$. Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ and $\{\overrightarrow{f_1}, \overrightarrow{f_2}\}$ be the positive orthonormal frames associated to c and d, respectively. Let A be the rotation matrix such that

$$\overrightarrow{f_1}(s_0) = A\overrightarrow{e_1}(s_0).$$

We then have

$$\overrightarrow{f_2}(s_0) = R_{\frac{\pi}{2}}\overrightarrow{f_1}(s_0) = R_{\frac{\pi}{2}}A\overrightarrow{e_1}(s_0) = AR_{\frac{\pi}{2}}\overrightarrow{e_1}(s_0) = A\overrightarrow{e_2}(s_0).$$

Now define

$$\begin{aligned} \overrightarrow{g_1}(s) &= A\overrightarrow{e_1}(s), \\ \overrightarrow{g_2}(s) &= A\overrightarrow{e_2}(s). \end{aligned}$$

We have

$$\begin{aligned} \overrightarrow{g_1}(s_0) &= \overrightarrow{f_1}(s_0), \\ \overrightarrow{g_2}(s_0) &= \overrightarrow{f_2}(s_0), \end{aligned}$$

and furthermore

$$\begin{bmatrix} \overrightarrow{g_1}'(s) \\ \overrightarrow{g_2}'(s) \end{bmatrix} = \begin{bmatrix} A\overrightarrow{e_1}'(s) \\ A\overrightarrow{e_2}'(s) \end{bmatrix} = \begin{bmatrix} Ak_c(s)\overrightarrow{e_2}(s) \\ -Ak_c(s)\overrightarrow{e_1}(s) \end{bmatrix} = \begin{bmatrix} 0 & k_c(s) \\ -k_c(s) & 0 \end{bmatrix} \begin{bmatrix} A\overrightarrow{e_1}'(s) \\ A\overrightarrow{e_2}'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_d(s) \\ -k_d(s) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{g_1}(s) \\ \overrightarrow{g_2}(s) \end{bmatrix}.$$

Since $(\overrightarrow{g_1}, \overrightarrow{g_2})$ and $(\overrightarrow{f_1}, \overrightarrow{f_2})$ satisfy the same system of linear differential equations with the same initial conditions, by Picard-Lindelof we must have

$$\begin{aligned}\overrightarrow{f_1}(s) &= \overrightarrow{g_1}(s) = A\overrightarrow{e_1}(s),\\ \overrightarrow{f_2}(s) &= \overrightarrow{g_2}(s),\end{aligned}$$

and therefore

$$d'(s) = \overrightarrow{f_1}(s) = A\overrightarrow{e_1}(s) = Ac'(s) \Rightarrow d(s) = Ac(s) + b,$$

as required. □

Definition 1.14. Let $c : I \to \mathbb{R}^2$ be a curve and let k be its curvature, obtained by reparameterizing c with arclength. We define the curvature k_t, the curvature function with respect to the parameter t, as

$$k_t(t) = k(s(t)).$$

Proposition 1.15. If $c : I \to \mathbb{R}^2$ is a curve (not necessarily parameterized by its arclength) written as $c(t) = (x(t), y(t))$, then its curvature is given by

$$k_t(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

Proof. Let c_s denote the curve c parameterized by arclength. We have

$$c(t) = c_s(s(t)) \Rightarrow c'(t) = s'(t)c_s'(s(t))$$

and therefore

$$\begin{aligned}c''(t) = s''(t)c_s'(s(t)) + s'(t)^2c_s''(s(t)) &= s''(t)\overrightarrow{e_1}(s(t)) + s'(t)^2k(s(t))\overrightarrow{e_2}(s(t))\\ &= s''(t)\overrightarrow{e_1}(s(t)) + s'(t)^2k_t(t)\overrightarrow{e_2}(s(t)).\end{aligned}$$

For simplicity, let us drop the arguments. We now obtain

$$
\begin{aligned}
x'y'' - x''y' = \det \begin{bmatrix} x' & y' \\ x'' & y'' \end{bmatrix} = \det(c', c'') &= \det(s'\overrightarrow{e_1}, s'^2 k_t \overrightarrow{e_2} + s''\overrightarrow{e_1}) \\
&= \det(s'\overrightarrow{e_1}, s'^2 k_t \overrightarrow{e_2}) + \underbrace{\det(s'\overrightarrow{e_1}, s''\overrightarrow{e_1})}_{0} \\
&= s'^3 k_t \underbrace{\det(\overrightarrow{e_1}, \overrightarrow{e_2})}_{1} \\
&= s'^3 k_t.
\end{aligned}
$$

Furthermore, we know that

$$
s(t) = \int_{t_0}^{t} \|c'(u)\| \, du \Rightarrow s'(t) = \|c'(t)\|,
$$

and hence we have

$$
k_t = \frac{x'y'' - x''y'}{s'^3} = \frac{x'y'' - x''y'}{(x' + y')^{\frac{3}{2}}},
$$

as required. □

Definition 1.16. (Closed curve) A regular curve $c : I \to \mathbb{R}^2$ is said to be **closed** if $c(a) = c(b)$ and moreover $c^{(n)}(a) = c^{(n)}(b)$ for all $n \in \mathbb{N}$.

Definition 1.17. (Simple closed curve) A regular curve $c : I \to \mathbb{R}^2$ is said to be a **simple closed curve** if it is closed and its restriction to $[a, b)$ is injective.

Definition 1.18. (Convex curve) A regular curve $c : I \to \mathbb{R}^2$ is said to be **convex** if it is simple closed and it bounds a convex region.

Definition 1.19. (Vertex) A **vertex** of a closed curve is a critical point of its curvature (i.e. a maxima, minima and inflection points).

Remark 1.20. Any closed curve can be extended to a periodic curve $c : \mathbb{R} \to \mathbb{R}^2$.

Theorem 1.21. *(4 Vertex Theorem) The curvature function of any simple closed convex curve $c : [a,b] \to \mathbb{R}^2$ has at least two maxima and two minima on $[a,b)$.*

Proof. Since k is continuous and $k(a) = k(b)$ we know that k has at least a minimum s_m and a maximum s_M. If $s_m = s_M$ then k is constant and the proof is done; else, assume $s_m \neq s_M$ and thus $s_m, s_M \in [a,b)$. Furthermore, assume WLOG that $s_m = a$.

By using a rigid motion we may assume that $c(a)$ and $c(s_M)$ are on the x–axis. Since c is convex, we may then assume that $y(s) > 0$ for $s \in (a, s_M)$ and $y(s) < 0$ for $s \in (s_M, b)$.

Now, if there are no more maxima nor minima, we know that $k'(s) \geq 0$ for $s \in (a, s_M)$ and $k'(s) \leq 0$ for $s \in (s_M, b)$. Furthermore, since $k(a) < k(s_M)$, we know that $k'(s)$ must be nonzero in some interval in (a, s_M) and in some interval in (s_M, b). Thus we have

$$\int_a^b k'(s)y(s)\,ds > 0 \Rightarrow \underbrace{\Big[k(s)y(s)\Big]_a^b}_{0} - \int_a^b k(s)y'(s)\,ds > 0 \Leftrightarrow \int_a^b -k(s)y'(s)\,ds > 0.$$

Now let $\vec{e_2}(s) = (u(s), v(s))$. We know that

$$\vec{e_2}'(s) = -k(s)\vec{e_1}(s) \Rightarrow v'(s) = -k(s)y'(s)$$

and therefore we obtain

$$\int_a^b v'(s)\,ds > 0 \Leftrightarrow v(b) - v(a) > 0$$

which is a contradiction since c is a closed curve.

This implies that there must be another maximum (or minimum), and since we already had a maximum and a minimum this in turn implies that there must also be another minimum (or maximum). □

Remark 1.22. The theorem also holds true for general simple closed curves, but the proof is much harder.

Given a curve $c : [a,b] \to \mathbb{R}^2$ parameterized by arclength, as the vector $\vec{e_1}$ has norm equal to 1 it is possible to write this function as

$$\vec{e_1}(s) = \Big(\cos(\theta(s)), \sin(\theta(s))\Big)$$

for some function $\theta : [a,b] \to \mathbb{R}$. We then have

$$\vec{e_1}'(s) = \Big(-\sin(\theta(s)) \cdot \theta'(s), \cos(\theta(s)) \cdot \theta'(s) \Big) = \theta'(s) \cdot \Big(-\sin(\theta(s)), \cos(\theta(s)) \Big).$$

On the other hand,

$$\vec{e_1}'(s) = k(s)\vec{e_2}(s) = k(s)\Big(-\sin(\theta(s)), \cos(\theta(s)) \Big)$$

and therefore

$$k(s) = \theta'(s) \Rightarrow \int_a^b k(s)\, ds = \int_a^b \theta'(s)\, ds = \theta(b) - \theta(a).$$

Because the curve is closed we know that $\frac{1}{2\pi}(\theta(b) - \theta(a))$ is an integer. This motivates us to make the following definition.

Definition 1.23. (Rotation index) The **rotation index** of a closed curve $c : [a,b] \to \mathbb{R}^2$, parameterized by its arclength, is the integer

$$m = \frac{1}{2\pi} \int_a^b k(s)\, ds.$$

Definition 1.24. (Homotopy) A (free) **homotopy by closed regular curves** between two closed regular curves $c_0, c_1 : [a,b] \to \mathbb{R}^2$ is a smooth map $H : [a,b] \times [0,1] \to \mathbb{R}^n$ such that

i) $H(t,0) = c_0(t)$, $\forall t \in [a,b]$;

ii) $H(t,1) = c_1(t)$, $\forall t \in [a,b]$;

iii) $c_u(t) := H(t,u)$ is a closed regular curve for each value of $u \in [0,1]$.

Theorem 1.25. *If two closed regular plane curves are homotopic by closed regular curves then they have the same rotation index.*

Proof. For each $u \in [0,1]$, let $m(u)$ denote the rotation index of the curve $c_u(t) := H(t,u)$. We have

$$m(u) = \frac{1}{2\pi} \int_a^b k_u(t) \|c_u'(t)\| dt = \frac{1}{2\pi} \int_a^b \frac{x_u'(t) y_u''(t) - x_u''(t) y_u'(t)}{(x_u'(t)^2 + y_u'(t)^2)}\, dt$$

$$= \frac{1}{2\pi} \int_a^b \frac{\frac{\partial H^1}{\partial t}(t,u) \frac{\partial^2 H^2}{\partial t^2}(t,u) - \frac{\partial H^2}{\partial t}(t,u) \frac{\partial^2 H^1}{\partial t^2}(t,u)}{\big(\frac{\partial H^1}{\partial t}(t,u)\big)^2 + \big(\frac{\partial H^2}{\partial t}(t,u)\big)^2}\, dt.$$

Since

$$\left(\frac{\partial H^1}{\partial t}(t,u)\right)^2 + \left(\frac{\partial H^2}{\partial t}(t,u)\right)^2 = \left\|\frac{\partial H}{\partial t}(t,u)\right\|^2 \neq 0,$$

the expression we obtained for $m(u)$ is a smooth function of u, and therefore since $m(u) \in \mathbb{Z}$ for all $u \in [0,1]$ this implies that m must be constant. □

Remark 1.26. In particular, the rotation index of a convex planar curve $c : [a,b] \to \mathbb{R}^2$ is either $m = 1$ or $m = -1$: in fact, the map $H : [a,b] \times [0,1] \to \mathbb{R}^n$ defined as

$$H(t,u) = \frac{c(t)}{1 - u + u\|c(t)\|}$$

is a homotopy by regular curves between c and a parameterization of the unit circle (where we assumed, for simplicity, that the convex region bounded by c contains the origin).

Definition 1.27. (Total curvature) The **total curvature** of a regular plane curve $c : [a,b] \to \mathbb{R}^2$, parameterized by arclength, is defined as

$$\mu := \int_a^b |k(s)|\, ds = \int_a^b \|c''(s)\|\, ds = \int_a^b \|\vec{e_1}'(s)\| ds.$$

Remark 1.28. The total curvature is the total distance length traveled by the vector $\vec{e}_i$ along the unit circle

$$S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}.$$

Remark 1.29. If m and k are the rotation index and the curvature of a regular plane curve $c : [a,b] \to \mathbb{R}^2$, respectively, it is always true that

$$|2\pi m| = \left|\int_a^b k(s)\, ds\right| \leq \int_a^b |k(s)|\, ds = \mu.$$

Theorem 1.30. *The total curvature μ of a closed regular plane curve satisfies $\mu \geq 2\pi$, and $\mu = 2\pi$ if and only if the curve is convex.*

Proof. First, note that the image of the continuous function $\vec{e_1} : [a,b] \to \mathbb{R}^2$ must cover at least half a circle: if that was not the case, we could assume that $e_1^y(s) > 0$ for all $s \in [a,b]$ but then

$$\begin{aligned} y'(s) = e_1^y(s) \Rightarrow y'(s) > 0 \quad \forall s \in [a,b] \\ \Rightarrow \int_a^b y'(s)\,ds > 0 \\ \Rightarrow y(b) - y(a) > 0, \end{aligned}$$

which contradicts the curve being closed.

Suppose then that $\vec{e_1}(s)$ covers exactly half a circle for $s \in [a, s_0]$. Then it must cover at least half a circle for $s \in [s_0, b]$. Hence,

$$\mu = \int_a^{s_0} \|\vec{e_1}'(s)\| ds + \int_{s_0}^b \|\vec{e_1}'(s)\| ds \geq \pi + \pi = 2\pi,$$

which proves the first assertion.

To prove the second assertion, suppose that $\mu = 2\pi$. Since the image of $\vec{e_1}(s)$ must in fact be more than half a circle (otherwise the same argument as above leads to a contradiction), $\vec{e_1}(s)$ cannot turn back, and so we can assume WLOG that $k(s) \geq 0$ (that is, $\vec{e_1}(s)$ always rotates anticlockwise). For each point $s_0 \in [a,b]$ rotate the curve such that $\vec{e_1}(s_0)$ is horizontal. If the image of c is on both sides of the tangent to c at $c(s_0)$ then $y(s)$ has at least a maximum and a minimum different from $y(s_0)$, and so $\vec{e_1}(s)$ is horizontal at 3 points in $[a,b)$ with $k(s) > 0$ somewhere between any two of these points. But then the image of $\vec{e_1}(s)$ must be more than a full circle, and so $\mu > 2\pi$, which is a contradiction. Therefore the image of c is always on one side of any of its tangents, implying that c is convex. Finally, if c is convex then $k(s)$ cannot change sign (or else the image of c would be on both sides of the tangent to c at the point where $k(s)$ changes sign), and since $m = \pm 1$ we must have $\mu = 2\pi$. □

Theorem 1.31. *(Isoperimetric inequality) If c is a closed simple curve of minimal length enclosing a fixed area A then c is a circle of radius $r = \sqrt{\frac{A}{\pi}}$.*

Proof. Assume that c is parameterized by arclength. As is well known from Green's Theorem (which is a particular case of the Stokes Theorem – see next section), the area of the region R enclosed by the curve c is

$$A = \iint_R 1 dx dy = \frac{1}{2} \oint_{\partial R} x dy - y dx = \frac{1}{2} \int_0^l \left[x(s)y'(s) - y(s)x'(s)\right] ds,$$

where we have assumed that $c(s) = (x(s), y(s))$ is oriented in the positive direction (that is, has rotation index $m = 1$) and has length l. This formula can also be written as

$$A = \frac{1}{2}\int_0^l \det(c(s), c'(s))ds.$$

Now let $H : [0, l] \times [0, 1]$ be a homotopy by closed regular curves such that $H(s, 0) = c(s)$ (also called a **variation** of c). Then the curves $c_u(s) = H(s, u)$ have length

$$l(u) = \int_0^l \left\| \frac{\partial H}{\partial s}(s, u) \right\| ds$$

and enclose an area

$$A(u) = \frac{1}{2}\int_0^l \det\left(H(s, u), \frac{\partial H}{\partial s}(s, u)\right) ds,$$

where $l(0) = l$ and $A(0) = A$. We then have

$$\begin{aligned}
\frac{dl}{du}(0) &= \int_0^l \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial s} \cdot \frac{\partial H}{\partial s}\right)^{\frac{1}{2}} ds = \int_0^l \frac{\partial H}{\partial s} \cdot \frac{\partial^2 H}{\partial u \partial s} \underbrace{\left\| \frac{\partial H}{\partial s} \right\|^{-1}}_{1} ds \\
&= \int_0^l \left[\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial s} \cdot \frac{\partial H}{\partial u}\right) - \frac{\partial^2 H}{\partial s^2} \cdot \frac{\partial H}{\partial u}\right] ds = -\int_0^l c''(s) \cdot \frac{\partial H}{\partial u}(s, 0)ds
\end{aligned}$$

and

$$\begin{aligned}
\frac{dA}{du}(0) &= \frac{1}{2}\int_0^l \frac{\partial}{\partial u} \det\left(H, \frac{\partial H}{\partial s}\right) ds = \frac{1}{2}\int_0^l \left[\det\left(\frac{\partial H}{\partial u}, \frac{\partial H}{\partial s}\right) + \det\left(H, \frac{\partial^2 H}{\partial u \partial s}\right)\right] ds \\
&= \frac{1}{2}\int_0^l \left[\det\left(\frac{\partial H}{\partial u}, \frac{\partial H}{\partial s}\right) + \frac{\partial}{\partial s}\det\left(H, \frac{\partial H}{\partial u}\right) - \det\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial u}\right)\right] ds \\
&= \int_0^l \det\left(\frac{\partial H}{\partial u}(s, 0), c'(s)\right) ds = -\int_0^l \vec{e_2}(s) \cdot \frac{\partial H}{\partial u}(s, 0)ds,
\end{aligned}$$

where we used

$$\det(\vec{v}, c'(s)) = \det\begin{bmatrix} v^1 & v^2 \\ x'(s) & y'(s) \end{bmatrix} = (y'(s), -x'(s)) \cdot \vec{v} = -\vec{e_2}(s) \cdot \vec{v}.$$

It can be shown that the method of Lagrange multipliers still applies to this infinite-dimensional situation: if c minimizes length among all closed curves enclosing a fixed area A then there exists a constant λ such that

$$\frac{d}{du}\Big[l(u) + \lambda A(u)\Big] = 0$$

for any variation H. Therefore we must have

$$\int_0^l \Big[c''(s) + \lambda \vec{e_2}(s)\Big] \cdot \frac{\partial H}{\partial u}(s,0) ds = 0 \Leftrightarrow \int_0^l \Big[k(s) + \lambda\Big] \vec{e_2}(s) \cdot \frac{\partial H}{\partial u}(s,0) ds = 0.$$

In particular, if we choose the variation H such that

$$\frac{\partial H}{\partial u}(s,0) = \Big[k(s) + \lambda\Big] \vec{e_2}(s)$$

(which is easily seen to be possible), we obtain

$$k(s) + \lambda = 0,$$

and so $k(s)$ must be constant. □

Remark 1.32. Using similar techniques one can prove that if $c : [a,b] \to \mathbb{R}^2$ connects $c(a) = x_0$ e $c(b) = x_1$ and has minimal length then $k(s) = 0$, and so c parameterizes a line segment.

Remark 1.33. It is also true that the curve with a given fixed length that encloses the largest possible area is a circle.

Remark 1.34. The theorem is also true if part of the curve is given: the remaining part must be an arc of a circle.

1.2 Curves in space

Definition 1.35. If $c : I \to \mathbb{R}^3$ is a curve, parametrized by arclength, we define $\vec{e_1} : I \to \mathbb{R}^3$ by $\vec{e_1}(s) = c'(s)$.

As with plane curves, we have $\|\vec{e_1}(s)\| = 1$ for all $s \in I$.

Definition 1.36. (Curvature) The curvature of a curve $c : I \to \mathbb{R}^3$, parametrized by arclength, is defined as

$$k(s) := \|c''(s)\| = \|\vec{e_1}'(s)\| \geq 0.$$

Definition 1.37. Given a curve $c : I \to \mathbb{R}^3$, parametrized by arclength, for each $s \in I$ such that $k(s) \neq 0$ we define

$$\vec{e_2}(s) := \frac{\vec{e_1}'(s)}{\|\vec{e_1}'(s)\|} = \frac{\vec{e_1}'(s)}{k(s)} \Rightarrow \vec{e_1}'(s) = k(s)\vec{e_2}(s).$$

If we now define $\vec{e_3} : I \to \mathbb{R}^3$ by

$$\vec{e_3}(s) = \vec{e_1}(s) \times \vec{e_2}(s),$$

sometimes called the **binormal vector**, we obtain, at each point $c(s)$ of the the curve with $k(s) \neq 0$, an orthonormal frame $\{\vec{e_1}(s), \vec{e_2}(s), \vec{e_3}(s)\}$ for $\mathbb{R}^3$. This is sometimes called a **moving orthonormal frame**, something we will also define for surfaces later on.

By the exact same argument as in Proposition 1.7. we have the following.

Proposition 1.38. The vectors $\vec{e_i}$ and $\vec{e_i}'$ are orthogonal.

Now, note that we have

$$\vec{e_1}(s) \cdot \vec{e_2}(s) = 0 \Rightarrow \underbrace{\vec{e_1}'(s) \cdot \vec{e_2}(s)}_{k(s)} + \vec{e_1}(s) \cdot \vec{e_2}'(s) = 0 \Leftrightarrow \vec{e_1}(s) \cdot \vec{e_2}'(s) = -k(s).$$

Since $\vec{e_2}'(s)$ is orthogonal to $\vec{e_2}(s)$, we know that $\vec{e_2}'(s)$ must be some linear combination of $\vec{e_1}(s)$ and $\vec{e_3}(s)$. Therefore, let us define the **torsion** τ as the function such that

$$\vec{e_2}'(s) = -k(s)\vec{e_1}(s) + \tau(s)\vec{e_3}(s).$$

We can also obtain

$$\vec{e_3}'(s) = \underbrace{\vec{e_1}'(s) \times \vec{e_2}(s)}_{0} + \vec{e_1}(s) \times \vec{e_2}'(s) = \vec{e_1}(s) \times \Big(-k(s)\vec{e_1}(s) + \tau(s)\vec{e_3}(s) \Big)$$
$$= -\tau(s)\vec{e_2}(s).$$

These computations lead to the following formulas, known as the Frenet-Serret formulas.

Proposition 1.39. (Frenet-Serret Formulas) The unit vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$, the curvature and the torsion of a curve $c : I \to \mathbb{R}^3$ satisfy the following equations:

$$\begin{bmatrix} \vec{e_1}'(s) \\ \vec{e_2}'(s) \\ \vec{e_3}'(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} = \begin{bmatrix} \vec{e_1}(s) \\ \vec{e_2}(s) \\ \vec{e_3}(s) \end{bmatrix}.$$

Our goal now will be to study the functions k and τ and what they say about the curve.

Proposition 1.40. If $c : I \to \mathbb{R}^3$ has zero curvature then c parametrizes a straight line.

Proof. We have
$$k(s) = 0 \Leftrightarrow c''(s) = 0 \Leftrightarrow c(s) = sa + b$$
for some vectors $a, b \in \mathbb{R}^3$. □

Proposition 1.41. If $c : I \to \mathbb{R}^3$ has nonzero curvature then it has zero torsion if and only if it is a plane curve.

Proof. If the torsion vanishes then

$$\vec{e_3}'(s) = 0 \Leftrightarrow \vec{e_3}(s) = a \Rightarrow \vec{e_1}(s) \cdot a = 0 \Leftrightarrow c'(s) \cdot a = 0 \Leftrightarrow c(s) \cdot a = b,$$

where $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$ are constants, and so $c(s)$ lies in the plane with equation $a \cdot x = b$.

Conversely, if $c(s)$ lies in the plane with equation $a \cdot x = b$ then

$$\vec{e_1}(s) \cdot a = 0 \Rightarrow \vec{e_1}'(s) \cdot a = 0 \Rightarrow \vec{e_2}(s) \cdot a = 0 \Rightarrow \vec{e_2}'(s) \cdot a = 0$$
$$\Rightarrow \Big(-k(s)\vec{e_1}(s) + \tau(s)\vec{e_3}(s) \Big) \cdot a = 0 \Rightarrow \tau(s) = 0,$$

since a is nonzero and proportional to $\vec{e_3}(s)$. □

Definition 1.42. (Positive isometry) A **positive isometry** of $\mathbb{R}^3$ is a map $F : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $F(x) = Ax + b$, where $A \in SO(3)$ is a rotation matrix, that is, $A^t A = I$ and $\det A = 1$.

Theorem 1.43. *Two curves c and d, parametrized by arclength, are related by a positive isometry (i.e., $d(s) = Ac(s) + b$) if and only if their curvatures and torsions coincide, i.e., $k_c(s) = k_d(s)$ and $\tau_c(s) = \tau_d(s)$.*

Proof. ($\Rightarrow$) If $d(s) = Ac(s) + b$ then we have

$$d'(s) = Ac'(s) \Rightarrow d''(s) = Ac''(s) \Rightarrow \|d''(s)\| = \|Ac''(s)\| = \|c''(s)\| \Leftrightarrow = k_c(s) = k_d(s).$$

On the other hand, if $\{\vec{e_1}(s), \vec{e_2}(s), \vec{e_3}(s)\}$ and $\{\vec{f_1}(s), \vec{f_2}(s), \vec{f_3}(s)\}$ are the Frenet-Serret frames for $c(s)$ and $d(s)$ then we have

$$\vec{f_2}(s) = \frac{1}{k_d(s)}\vec{f_1}'(s) = \frac{1}{k_c(s)}A\vec{e_1}'(s) = A\vec{e_2}(s)$$

and

$$\vec{f_3}(s) = \vec{f_1}(s) \times \vec{f_2}(s) = \left(A\vec{e_1}(s)\right) \times \left(A\vec{e_2}(s)\right) = A\left(\vec{e_1}(s) \times \vec{e_2}(s)\right) = A\vec{e_3}(s)$$

(since rotations preserve the cross product), and so

$$vv{f_3}'(s) = A\vec{e_3}'(s) = -\tau_c(s)A\vec{e_2}(s) = -\tau_c(s)\vec{f_2}(s),$$

implying that $\tau_c(s) = \tau_d(s)$.

($\Leftarrow$) Let A be the rotation matrix such that

$$A\vec{e_i}(s_0) = \vec{f_i}(s_0) \qquad (i = 1, 2, 3)$$

for some s_0. Then

$$\vec{g_i}(s) = A\vec{e_i}(s)$$

satisfy the same linear ODE system as $\vec{f_i}(s)$ (since $k_c(s) = k_d(s)$ and $\tau_c(s) = \tau_d(s)$) and have the same initial values at $s = s_0$. By the Picard-Lindelöf Theorem, they must coincide. In particular,

$$d'(s) = \vec{f_1}(s) = A\vec{e_1}(s) = Ac'(s) \Rightarrow d(s) = Ac(s) + b,$$

as required. □

The following proposition will be needed for the proof of Theorem 2.9..

Proposition 1.44. Let $c : I \to \mathbb{R}^3$ be a regular curve and let s be the arclength function. Then we have

$$k(s(t)) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

and

$$\tau(s(t)) = \frac{c'(t) \cdot (c''(t) \times c'''(t))}{\|c'(t) \times c''(t)\|^2}.$$

Proof. These formulas can be checked by direct substitution, using

$$\begin{aligned}
c'(t) &= s'(t)\vec{e_1}(s), \\
c''(t) &= s''(t)\vec{e_1}(s) + (s'(t))^2 k(s)\vec{e_2}(s), \\
c'''(t) &= \left(s'''(t) - (s'(t))^3 (k(s))^2\right) \vec{e_1}(s) + \left(3s'(t)s''(t)k(s) + (s'(t))^3 k'(s)\right) \vec{e_2}(s) \\
&\qquad + (s'(t))^3 k(s)\tau(s)\vec{e_3}(s).
\end{aligned}$$

□

Theorem 1.45. *(Frenchel) Let $c : [a, b] \to \mathbb{R}^3$ be a closed regular curve, parametrized by arclength, with curvature $k(s) = \|c''(s)\|$. Then*

$$\int_a^b k(s)\, ds \geq 2\pi,$$

and the equality holds if and only if c is a convex plane curve.

Proof. For each unit vector

$$\vec{n} \in S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$$

let us define the curve d_n as the curve obtained from projecting the curve c on the plane through the origin perpendicular to $\vec{n}$; that is, define $d_n(s)$ as

$$d_n(s) := c(s) - (c(s) \cdot \vec{n})\vec{n}, \qquad \forall s \in [a, b].$$

Furthermore, let μ_n denote the total curvature of the curve d_n. We will prove the following lemma.

Lemma 1.46. The total curvature of c equals the average of the total curvatures of the curves

d_n; that is,

$$\int_a^b k(s)\,ds = \frac{1}{4\pi}\int_{S^2} \mu_n \,\mathrm{dVol}_2(\vec{n}).$$

Proof. Let $\vec{n} \in S^2$ be a unit vector. The total curvature of d_n is

$$\mu_n = \int_a^b \frac{\|d_n' \times d_n''\|}{\|d_n'\|^3}\|d_n'\|\,ds.$$

Using the vector identity

$$\|a \times b\|^2 = \|a\|^2\|b\|^2 - (a \cdot b)^2$$

we get

$$\mu_n = \int_a^b \frac{\left[\|d_n'\|^2\|d_n''\|^2 - (d_n' \cdot d_n'')^2\right]^{\frac{1}{2}}}{\|d_n'\|^2}\,ds.$$

Now,

$$\|d_n'\|^2 = \|c' - (c' \cdot \vec{n})\vec{n}\| = \|c'\|^2 - 2(c' \cdot \vec{n})^2 + (c' \cdot \vec{n})^2 = 1 - (c' \cdot \vec{n})^2$$

and

$$\|d_n''\|^2 = \|c'' - (c'' \cdot \vec{n})\vec{n}\| = \|c''\|^2 - 2(c'' \cdot \vec{n})^2 + (c'' \cdot \vec{n})^2 = k(s)^2 - (c'' \cdot \vec{n})^2.$$

Furthermore,

$$\begin{aligned} d_n' \cdot d_n'' &= \Big(c' - (c' \cdot \vec{n})\vec{n}\Big) \cdot \Big(c'' - (c'' \cdot \vec{n})\vec{n}\Big) \\ &= c' \cdot c'' - (c' \cdot \vec{n})(c'' \cdot \vec{n}) - (c' \cdot \vec{n})(c'' \cdot \vec{n}) + (c' \cdot \vec{n})(c'' \cdot \vec{n}) \\ &= -(c' \cdot \vec{n})(c'' \cdot \vec{n}) \end{aligned}$$

and therefore

$$\begin{aligned} \|d_n'\|^2\|d_n''\|^2 - (d_n' \cdot d_n'')^2 &= (1 - (c' \cdot \vec{n})^2)(k(s)^2 - (c'' \cdot \vec{n})^2) - (c' \cdot \vec{n})^2(c'' \cdot \vec{n})^2 \\ &= k(s)^2 - k(s)^2(c' \cdot \vec{n})^2 - (c'' \cdot \vec{n})^2. \end{aligned}$$

Hence

$$\mu_n = \int_a^b \frac{\left[k(s)^2 - k(s)^2(c' \cdot \vec{n})^2 - (c'' \cdot \vec{n})^2\right]^{\frac{1}{2}}}{1 - (c' \cdot \vec{n})^2}\,ds.$$

Let us choose $\vec{e_2}(s)$ to be an arbitrary unit vector whenever it is undefined, that it, whenever

$k(s) = 0$. This choice does not change the value of the integral below, since it is clear that the integrand is zero if $k(s) = 0$, and so we can write

$$\mu_n = \int_a^b \frac{k(s)\left[1 - (\vec{e_1} \cdot \vec{n})^2 - (\vec{e_2} \cdot \vec{n})^2\right]^{\frac{1}{2}}}{1 - (\vec{e_1} \cdot \vec{n})^2} ds.$$

Note that this integral is undefined when

$$1 - (\vec{e_1}(s) \cdot \vec{n}) = 0 \Leftrightarrow \vec{e_1}(s) = \pm\vec{n}.$$

However, the subset of vectors $\vec{n}$ for which this happens has measure zero on S^2, and therefore these values of $\vec{n}$ may be ignored when computing the value of the integral.

Finally, since the functions $\vec{e_1}, \vec{e_2} : [a, b] \to \mathbb{R}^3$ do not depend on $\vec{n}$, we assume that $\vec{e_1}$ and $\vec{e_2}$ are constant when integrating over all possible unit vectors $\vec{n}$; in this case, we will assume that $\vec{e_1}(s) = (0, 0, 1)$ and that $\vec{e_2}(s) = (0, 1, 0)$. We then obtain

$$\begin{aligned}
\frac{1}{4\pi} \int_{S^2} \mu_n \, \mathrm{dVol}_2(\vec{n}) &= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \int_a^b \frac{k(s)(1 - \cos^2\theta - \sin^2\theta \sin^2\varphi)^{\frac{1}{2}}}{1 - \cos^2\theta} \sin\theta \, ds \, d\varphi \, d\theta \\
&= \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \int_a^b \frac{k(s) \sin\theta |\cos\varphi|}{\sin^2\theta} \sin\theta \, ds \, d\varphi \, d\theta \\
&= \frac{1}{4} \int_a^b k(s) \left(\int_0^{2\pi} |\cos\varphi| \, d\varphi \right) ds \\
&= \int_a^b k(s) \, ds
\end{aligned}$$

as required. $\square$

Now, since every curve d_n is a plane curve, we may use Theorem 1.30 to get

$$\int_a^b k(s) \, ds = \frac{1}{4\pi} \int_{S^2} \mu_n \, \mathrm{dVol}_2(\vec{n}) \geq \frac{1}{4\pi} \int_{S^2} 2\pi \, \mathrm{dVol}_2(\vec{n}) = 2\pi.$$

Finally, suppose that the total curvature of c is 2π. Since $\mu_n \geq 2\pi$ and this function is continuous on the dense open subset of S^2 where it is defined, we must have $\mu_n = 2\pi$ on this set, and so all projected curves are convex. To show that c is a plane curve, we proceed by contradiction: suppose that $c([s_0, s_1])$ is not contained in any plane. Choose $s' \in (s_0, s_1)$ such that $c(s_0)$, $c(s')$ and $c(s_1)$ are not collinear, and therefore define a plane π_0. Let π be a plane through the origin orthogonal to π_0 (so that vectors orthogonal to π_0 are perpendicular to vectors orthogonal to π). Moving π slightly if necessary, the orthogonal projection d of c on π (which takes place along π_0) can be chosen such that $d(s_0)$, $d(s')$ and $d(s_1)$ are distinct points on $\ell = \pi_0 \cap \pi$. Since d is convex (again it may be necessary to move π slightlaay), this implies that $d(s) \in \ell$ for all $s \in [s_0, s_1]$. But then $c(s) \in \pi_0$ for all $s \in [s_0, s_1]$, which is a contradiction. This shows that c must be a plane curve, and since its total curvature is 2π it must be convex. □

Definition 1.47. (Knot) A simple closed regular curve in $\mathbb{R}^3$ is called a **knot**. Two knots are called **equivalent** if they are homotopic (up to reparameterization) by simple closed regular curves. A knot is called **trivial** if it is equivalent to the circle.

Theorem 1.48. *Let $c : [a, b] \to \mathbb{R}^3$ be a nontrivial knot, parametrized by arclength, with curvature $k(s) = \|c''(s)\|$. Then*

$$\int_a^b k(s)\, ds \geq 4\pi.$$

Proof. We offer only a sketch of the proof. Again it suffices to show that $\mu_n \geq 4\pi$ for almost all $\vec{n} \in S^2$. Assume that $\mu_n < 4\pi$ for some $\vec{n} \in S^2$. Then

$$\left\{ \vec{e} \in S^1 : d_n'(s)/\|d_n'(s)\| = \pm\vec{e} \text{ for at least 4 values of } s \in [a, b) \right\} \neq S^1$$

(as otherwise $d_n'(s)/\|d_n'(s)$ would cover S^1 twice, moving through a total distance $\mu_n \geq 4\pi$). Let us then take $\vec{e} \in S^1$ such that $d_n'(s)/\|d_n'(s) = \pm\vec{e}$ at most 3 times. Assume for simplicity that $\vec{n} = (1, 0, 0)$ and $\vec{e} = (1, 0)$. Then $d_n'(s)$ can be horizontal at most 3 times. Since $d_n(s) = (y(s), z(s))$ and $d_n'(s) = (y'(s), z'(s))$, we see that $z(s)$ has at most one local minimum z_m and one local maximum z_M (which must in fact be both global). So for each $z \in (z_m, z_M)$ there are exactly 2 points in the curve such that $z(s) = z$. Connecting them by a line allows us to define a homotopy between c and the circle as z decreases from z_M to z_m. Therefore c must be the trivial knot. □

2 Differentiable manifolds and differential forms

2.1 Differentiable manifolds

Definition 2.1. (Differentiable manifold) A set $M \subset \mathbb{R}^n$ is said to be a **differentiable manifold** of dimension $m \in \{1, 2, \dots, n-1\}$ if for all $a \in M$ there exists an open set U with $a \in U$ and a smooth function $f : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that

$$M \cap U = \mathscr{G}(f) \cap U$$

for some ordering of the cartesian coordinates of $\mathbb{R}^n$.

Remark 2.2. We define a differentiable manifold $M \subset \mathbb{R}^n$ of dimension 0 to be a set of isolated points, and of dimension n to be an open set of $\mathbb{R}^n$.

Theorem 2.3. *A set $M \subset \mathbb{R}^n$ is a differentiable manifold of dimension m if and only if for all $a \in M$ there exists an open set $U \subset \mathbb{R}^n$ with $a \in U$ and a smooth function $F : U \to \mathbb{R}^{n-m}$ such that*

i) $M \cap U = \{x \in U : F(x) = 0\}$;

ii) rank $DF(a) = n - m$.

Proof. ($\Rightarrow$) Assume M is a manifold of dimension m and $a \in M$. Then there exists an open set $U \subset \mathbb{R}^n$ and a smooth function $f : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that

$$M \cap U = \mathscr{G}(f) \cap U = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : y = f(x)\} \cap U.$$

If we define $F : U \to \mathbb{R}^{n-m}$ where for each $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$ we have

$$F(x, y) := f(x) - y$$

then

$$M \cap U = \{x \in U : F(x) = 0\}.$$

Furthermore,

$$DF(a) = [Df(a) \quad -I_{n-m}] \Rightarrow \operatorname{rank} DF(a) = n - m$$

as required.

($\Leftarrow$) Since rank $DF(a) = n - m$ we can assume that

$$\det \frac{\partial F}{\partial y}(a) \neq 0$$

where y corresponds to the last $n - m$ variables of F.

By the Implicit Function Theorem, there exists some open $U \in \mathbb{R}^n$ with $a \in U$ and some smooth function $f : U \to \mathbb{R}^{n-m}$ such that

$$F(x, y) = 0 \Leftrightarrow y = f(x).$$

Hence

$$M \cap U = \{x \in U : F(x) = 0\} = \{(x, y) \in U : y = f(x)\} = \mathscr{G}(f) \cap U$$

as required. □

Definition 2.4. (Tangent vector) A vector $v \in \mathbb{R}^n$ is said to be **tangent** to a set $M \subset \mathbb{R}^n$ at the point $a \in M$ if there exists a smooth curve $c : \mathbb{R} \to M$ such that $c(0) = a$ and $\dot{c}(0) = v$. A vector $v \in \mathbb{R}^n$ is said to be **orthogonal** to M at the point a if it is orthogonal to all vectors tangent to M at a.

Proposition 2.5. If $M \subset \mathbb{R}^n$ is a differentiable manifold of dimension m then the set $T_a M$ of all vectors tangent to M at the point $a \in M$ is a vector space of dimension m (called the **tangent space** to M at the point a).

Proof. Assume without loss of generality that M is given by $y = f(x)$ in a neighborhood of the point a, where we use the notation in the proof of the theorem above. Any curve $c : \mathbb{R} \to M$ with $c(0) = a$ is given in this neighborhood by $c(t) = (d(t), f(d(t)))$, where $d : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$. Therefore, $\dot{c}(0) = (\dot{d}(0), Df(a)\dot{d}(0))$, and so any vector tangent to M at the point a is contained in the image of $\mathbb{R}^m$ by the injective linear map $u \mapsto (u, Df(a)u)$. On the other hand, given $u \in \mathbb{R}^m$, its image by this map is the vector tangent to the curve $c(t) = (b + tu, f(b + tu))$, where we write $a = (b, c)$, and so it is tangent to M at the point a. We conclude that $T_a M$ is an m-dimensional vector subspace of $\mathbb{R}^n$. □

Definition 2.6. (Normal space) The **normal space** to an m-manifold $M \subset \mathbb{R}^n$ at the point $a \in M$ is the $(n - m)$-dimensional vector space $T_a^\perp M$ obtained by taking the orthogonal complement of $T_a M$.

Proposition 2.7. Let $M \subset \mathbb{R}^n$ be an m-manifold, $a \in M$ a point in M, $U \ni a$ an open set and $F : U \to \mathbb{R}^{n-m}$ such that $M \cap U = \{x \in U : F(x) = 0\}$ with $\operatorname{rank} DF(a) = n - m$. Then $T_a M = \ker DF(a)$.

Proof. Since $\dim(\ker DF(a)) = m$, it suffices to show that $T_a M \subset \ker DF(a)$. For any smooth curve $c : \mathbb{R} \to M$ satisfying $c(0) = a$, we have $F(c(t)) = 0$ whenever $c(t) \in U$; differentiating at $t = 0$ we obtain $DF(a)\dot{c}(0) = 0$. □

Definition 2.8. (Parameterization) A **parameterization** of an m-manifold $M \subset \mathbb{R}^n$ is a smooth injective map $g : U \to M$ (with $U \subset \mathbb{R}^m$ open) such that $\operatorname{rank} Dg(t) = m$ for all $t \in U$.

Proposition 2.9. If $g : U \to M$ is a parameterization of the m-manifold $M \subset \mathbb{R}^n$ then

$$T_{g(t)} M = \operatorname{span}\left\{\frac{\partial g}{\partial t^1}(t), \dots, \frac{\partial g}{\partial t^m}(t)\right\}.$$

Proof. Obvious from the fact that $\frac{\partial g}{\partial t^1}(t), \dots, \frac{\partial g}{\partial t^m}(t)$ are linearly independent velocities of curves on M. □

Theorem 2.10. *If $M \subset \mathbb{R}^n$ is an m-manifold and $a \in M$ then there exists an open set $V \ni a$ such that $M \cap V$ is the image of a parameterization $g : U \subset \mathbb{R}^m \to M$. Conversely, given a smooth map $g : U \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $\operatorname{rank} Dg(t) = m$ for all $t \in U$, and given any point $t_0 \in U$, there exists an open set $U_0 \subset U$ with $t_0 \in U_0$ such that $g(U_0)$ is an m-manifold.*

Proof. Since there exists an open set $V \ni a$ such that $M \cap V$ is the graph of a smooth function $f : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$, we can choose the parameterization to be $g(t) = (t, f(t))$. To prove the converse, assume that the first m lines of $Dg(t_0)$ are linearly independent. Then, writing $(x, y) \in \mathbb{R}^n$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$, we have, from the Inverse Function Theorem, that the equations $(x, y) = g(t)$ can be solved to yield $t = h(x)$ in some open neighborhood U_0 of t_0, with h a smooth function. Therefore, $g(h(x)) = (x, k(x))$, and so $g(U_0)$ is the graph of the smooth function k. □

2.2 Differential forms

Definition 2.11. The dual vector space to $\mathbb{R}^n$ is

$$(\mathbb{R}^n)^* = \{\alpha : \mathbb{R}^n \to \mathbb{R} : \alpha \text{ is linear}\}.$$

The elements of $(\mathbb{R}^n)^*$ are called **covectors**.

We denote the basis of $\mathbb{R}^n$ by $\{e_1, \dots, e_n\}$.

Definition 2.12. For each $k = 1, \dots, n$ we define $dx^k \in (\mathbb{R}^n)^*$ such that

$$dx^k(e_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

Proposition 2.13. The set of covectors $\{dx^1, \dots, dx^n\}$ is a basis for $(\mathbb{R}^n)^*$.

Proof. Let α be a covector and v be a vector in $\mathbb{R}^n$. Setting $\alpha_i := \alpha(e_i)$ for $i = 1, \dots, n$ it is clear that

$$\alpha(v) = \alpha\left(\sum_{i=1}^n v^i e_i\right) = \sum_{i=1}^n v^i \alpha(e_i) = \sum_{i=1}^n \alpha_i v^i.$$

In particular, $dx^i(v) = v^i$, and so

$$\alpha(v) = \sum_{i=1}^n \alpha_i dx^i(v),$$

implying that α can be written as a linear combination of the covectors in $\{dx^1, \dots, dx^n\}$.

To prove that these covectors are linearly independent, let $c_1, \dots, c_n$ be such that

$$\left(\sum_{i=1}^n c_i dx^i\right)(v) = 0 \qquad \forall v \in \mathbb{R}^n.$$

Then we have, for all $j = 1, \dots, n$,

$$\left(\sum_{i=1}^n c_i dx^i\right)(e_j) = 0 \Leftrightarrow \sum_{i=1}^n c^i dx^i(e_j) = 0 \Leftrightarrow c_j = 0$$

as required. ☐

Definition 2.14. A **k-tensor** is a multilinear map $T : (\mathbb{R}^n)^k \to \mathbb{R}$.

Definition 2.15. A k-tensor T is called **alternating** if for all $v_1, \dots, v_k \in \mathbb{R}^n$ and $i \neq j$,

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Definition 2.16. For all sets $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ we define

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}(v_1, \dots, v_k) = \det \begin{bmatrix} dx^{i_1}(v_1) & \dots & dx^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ dx^{i_k}(v_1) & \dots & dx^{i_k}(v_k) \end{bmatrix}$$

Remark 2.17. These k-tensors are all alternating. Furthermore, if $i_1, \dots, i_k$ are not all different then

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}(v_1, \dots, v_k) = 0 \qquad \forall v_1, \dots, v_k \in \mathbb{R}^n.$$

Definition 2.18. We define $\Lambda^k(\mathbb{R}^n)$ as the set of alternating tensors in $\mathbb{R}^n$.

Remark 2.19. $\Lambda^k(\mathbb{R}^n)$ is a vector space.

Remark 2.20. $\Lambda^0(\mathbb{R}^n)$ is defined to be the set of real numbers $\mathbb{R}$.

Proposition 2.21. The set of k-tensors

$$\{dx^{i_1} \wedge \dots \wedge dx^{i_k}\}_{1 \leq i_1 < \dots < i_k \leq n}$$

is a basis for the vector space $\Lambda^k(\mathbb{R}^n)$.

Proof. To prove that these k-tensors are linearly independent, let $a_{i_1 \dots i_k}$ for each $\{i_1, \dots, i_k\} \subset$

$\{1, \ldots, n\}$ be such that

$$\sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}(v_1, \ldots, v_k) = 0, \quad \forall v_1, \ldots, v_k \in \mathbb{R}^n.$$

We have, for all $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$,

$$\begin{aligned}
&\sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx_{i_k}(e_{j_1}, \ldots, e_{j_k}) = 0 \\
&\Rightarrow \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} \det \begin{bmatrix} dx^{i_1}(e_{j_1}) & \ldots & dx^{i_1}(e_{j_k}) \\ \vdots & \ddots & \vdots \\ dx^{i_k}(e_{j_1}) & \ldots & dx^{i_k}(e_{j_k}) \end{bmatrix} = 0 \\
&\Rightarrow a_{j_1 \ldots j_k} = 0
\end{aligned}$$

as required.

We now prove that these k-tensors generate $\Lambda^k(\mathbb{R}^n)$. For a tensor $T \in \Lambda^k(\mathbb{R}^n)$ set for each $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$$a_{i_1 \ldots i_k} := T(e_{i_1}, \ldots, e_{i_k}).$$

Now define the k-tensor $S \in \Lambda^k(\mathbb{R}^n)$ by

$$\sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Because tensors are multilinear it is enough to check that for any $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ we have

$$S(e_{j_1}, \ldots, e_{j_k}) = a_{j_1 \ldots j_k} = T(e_{j_1}, \ldots, e_{j_k}).$$

□

Remark 2.22. The vector space $\Lambda^k(\mathbb{R}^n)$ has dimension $\binom{n}{k}$.

Definition 2.23. (Wedge product) If

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

is an alternating k-tensor and

$$\beta = \sum_{1 \leq i_1 < \ldots < i_l \leq n} \beta_{i_1, \ldots, i_l} dx^{i_1} \wedge \ldots \wedge dx^{i_l}$$

is an alternating l-tensor then their **wedge product** is the alternating $(k+l)$-tensor

$$\alpha \wedge \beta := \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ 1 \leq j_1 < \ldots < j_l \leq n}} \alpha_{i_1 \ldots i_k} \beta_{j_1, \ldots, j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$

Proposition 2.24. For tensors $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta, \gamma \in \Lambda^l(\mathbb{R}^n)$ the wedge product satisfies the following identities:

i) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$;

ii) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$;

iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

Proof. Left as an exercise. □

Definition 2.25. (Differential forms) A **differential form of degree** k in $\mathbb{R}^n$ is a smooth function $\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$. The set of k-forms in $\mathbb{R}^n$ is represented by $\Omega^k(\mathbb{R}^n)$.

Remark 2.26. The set $\Omega^0(\mathbb{R}^n)$ is the set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 2.27. Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\eta \in \Omega^l(\mathbb{R}^n)$ be two differential forms. Then their wedge product at the point $x \in \mathbb{R}^n$ is defined as

$$(\omega \wedge \eta)(x) := (\omega(x)) \wedge (\eta(x)).$$

Definition 2.28. (Pullback) If $\omega \in \Omega^k(\mathbb{R}^m)$ is a form and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function then the **pullback** of ω by f is $f^*\omega \in \Omega^k(\mathbb{R}^n)$ defined by

$$f^*\omega(x)(v_1, \ldots, v_k) = \omega(f(x))(Df(x)v_1, \ldots, Df(x)v_k).$$

Remark 2.29. If $\omega \in \Omega^0(\mathbb{R}^m)$ then $f^*\omega = \omega \circ f$.

Proposition 2.30. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Then the pullback satisfies the following identities:

i) $f^*(\omega + \eta) = f^*\omega + f^*\eta$, for $\omega, \eta \in \Omega^k(\mathbb{R}^m)$;

ii) $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$, for $\omega \in \Omega^k(\mathbb{R}^m)$ and $\omega \in \Omega^l(\mathbb{R}^m)$;

iii) $f^*(g^*\omega) = (g \circ f)^*\omega$, for any smooth function $g : \mathbb{R}^m \to \mathbb{R}^p$ and $\omega \in \Omega^k(\mathbb{R}^p)$.

Proof. Left as an exercise. □

Definition 2.31. (Exterior derivative) If $\omega \in \Omega^k(\mathbb{R}^n)$ is the k-form given by

$$\omega(x) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1 \ldots i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad \forall x \in \mathbb{R}^n$$

then $d\omega$ is the $(k+1)$-form given by

$$d\omega(x) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \ldots i_k}}{\partial x_i}(x) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad \forall x \in \mathbb{R}^n.$$

Remark 2.32. If $\omega \in \Omega^0(\mathbb{R}^n)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ then

$$d\omega(x) = \sum_{i=1}^{n} \frac{\partial \omega}{\partial x_i}(x) dx^i \in \Omega^1(\mathbb{R}^n).$$

Proposition 2.33. The exterior derivative satisfies the following properties:

i) $d(\omega + \eta) = d\omega + d\eta$, for $\omega, \eta \in \Omega^k(\mathbb{R}^n)$;

ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, for $\omega \in \Omega^k(\mathbb{R}^n)$ and $\eta \in \Omega^l(\mathbb{R}^n)$;

iii) $d(d\omega) = 0$, for $\omega \in \Omega^k(\mathbb{R}^n)$;

iv) $d(f^*\omega) = f^*(d\omega)$, for $\omega \in \Omega^k(\mathbb{R}^n)$ and a smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$;

Proof. Left as an exercise. □

Definition 2.34. (Closed and exact) Let $U \subset \mathbb{R}^n$ be an open set. We say that a differential form $\omega \in \Omega^k(U)$ is

- **Closed** if $d\omega = 0$;
- **Exact** if $\omega = d\eta$ for some $\Omega^{k-1}(U)$.

Remark 2.35. By property *iii)* of Proposition 2.33, if a differential form is exact then it is also closed.

Definition 2.36. (Diffeomorphism) A **diffeomorphism** between two open sets $U, V \subset \mathbb{R}^n$ is a smooth bijection $f : U \to V$ such that its inverse is also smooth. If such a function exists we say that U and V are **diffeomorphic**.

Proposition 2.37. Let $U \subset \mathbb{R}^n$ be an open set such that all closed forms in U are exact and let $V \subset \mathbb{R}^n$ be some other open set such that there exists in V a differential form which is closed but not exact. Then U and V are not diffeomorphic.

Proof. Let $\omega \in \Omega^k(V)$ such that ω is closed but not exact and assume, by contradiction, that there exists a diffeomorphism $f : U \to V$. Then

$$d(f^*\omega) = f^*(d\omega) = 0$$

which implies that $f^*\omega \in \Omega^k(U)$ is closed. Then we know that $f^*\omega$ is exact, and so let $\eta \in \Omega^k(U)$ be such that $f^*\omega = d\eta$. We have

$$(f^{-1})^* f^* \omega = (f^{-1})^* d\eta \Leftrightarrow (f \circ f^{-1})^* \omega = d((f^{-1})^* \eta) \Leftrightarrow \omega = d((f^{-1})^* \eta),$$

implying that ω is exact, which is a contradiction. □

Theorem 2.38. *(Poincaré Lemma) If $U \subset \mathbb{R}^n$ is a **star-shaped** open set in $\mathbb{R}^n$ then all closed differential forms in U are also exact.*

Proof. Assume without loss of generality that 0 is a center for U. If

$$\omega(x) \;=\; \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

we define $I\Omega \in \Omega^{k-1}(U)$ as

$$I\omega(x) \;=\; \sum_{i_1<\ldots<i_k} \left(\int_0^1 t^{k-1} \omega_{i_1\ldots i_k}(tx)\, dt \right) \sum_{l=1}^{n} (-1)^{l-1} x^{i_l}\, dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_l}} \wedge \ldots \wedge dx^{i_k}$$

(where $\widehat{\ }$ means omission). We have

$$\omega = d(I\omega) + I(d\omega),$$

and so if $d\omega = 0$ then $\omega = d(I\omega)$. □

2.3 Integration of differential forms

Theorem 2.39. *Let M be an m-manifold in $\mathbb{R}^n$. If $g : U \subset \mathbb{R}^m \to M$ and $h : V \subset \mathbb{R}^m \to M$ are parametrizations of M then*

$$h^{-1} \circ g : g^{-1}(g(U) \cap h(V)) \to h^{-1}(g(U) \cap h(V))$$

is a diffeomorphism.

Proof. It is clear that $h^{-1} \circ g$ is a bijection because both g and h are injective. Now, since

$$\operatorname{rank} Dg = m = \operatorname{rank} Dh$$

we may assume WLOG that

$$\det \begin{bmatrix} \frac{\partial g^1}{\partial t^1} & \cdots & \frac{\partial g^1}{\partial t^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g^m}{\partial t^1} & \cdots & \frac{\partial g^m}{\partial t^m} \end{bmatrix} \neq 0 \qquad \text{and} \qquad \det \begin{bmatrix} \frac{\partial h^1}{\partial s^1} & \cdots & \frac{\partial h^1}{\partial s^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^m}{\partial s^1} & \cdots & \frac{\partial h^m}{\partial s^m} \end{bmatrix} \neq 0.$$

Let us define $G : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by

$$G(t, u) := g(t) + u^1 e_{m+1} + \ldots + u^{n-m} e_n$$

and $H : V \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by

$$H(s, v) := h(s) + v^1 e_{m+1} + \ldots + v^{n-m} e_n.$$

Consider any $x_0 \in g(U) \cap h(V)$ and let t_0 and s_0 be such that

$$x_0 = g(t_0) = h(s_0).$$

It is clear that

$$\det DG(t_0, 0) = \det \begin{bmatrix} | & 0 \\ Dg(t_0) & \\ | & I_{n-m} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial g^1}{\partial t^1} & \cdots & \frac{\partial g^1}{\partial t^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g^m}{\partial t^1} & \cdots & \frac{\partial g^m}{\partial t^m} \end{bmatrix} \neq 0,$$

and so G is locally invertible around $(t_0, 0)$ with a smooth inverse. Similarly, H is locally invertible around $(s_0, 0)$ with a smooth inverse.

This implies that the two functions $H^{-1} \circ G$ and $G^{-1} \circ H$ are smooth where defined, and since

$$H^{-1} \circ G(t, 0) = \big(h^{-1} \circ g(t), 0\big),$$
$$G^{-1} \circ H(s, 0) = \big(g^{-1} \circ h(s), 0\big),$$

we obtain that $h^{-1} \circ g$ and $g^{-1} \circ h$ are both smooth, and because $g^{-1} \circ h = (h^{-1} \circ g)^{-1}$ we conclude that $h^{-1} \circ g$ is a diffeomorphism, as desired. □

Remark 2.40. Because the inverse of a diffeomorphism is smooth we know that the Jacobian matrix of a diffeomorphism can never vanish.

Definition 2.41. Let M be an m-manifold in $\mathbb{R}^n$. Two parametrizations $g : U \subset \mathbb{R}^m \to M$ and $h : V \subset \mathbb{R}^m \to M$ induce the **same orientation** in M if $\det D(h^{-1} \circ g) > 0$, and induce **opposite orientations** if $\det D(h^{-1} \circ g) < 0$. We say M is **orientable** if we can find parametrizations whose images cover M which induce the same orientation on M. Furthermore, an **orientation** in M is a choice of a maximal family of such parametrizations (which are called **positive**). When such a choice is made, M is said to be **oriented**.

Proposition 2.42. A surface $M \subset \mathbb{R}^3$ is orientable if and only if it admits a continuous normal $n : M \to S^2$.

Proof. ($\Leftarrow$) We cover M by parametrizations $g : U \subset \mathbb{R}^2 \to M$ with U connected. Since the normal n is continuous, the function $f : U \to \mathbb{R}$ defined as

$$f(t) = n(g(t)) \cdot \left(\frac{\partial g}{\partial t^1}(t) \times \frac{\partial g}{\partial t^2}(t) \right)$$

cannot change sign. If f is positive we keep g, and if f is negative we replace it with $\tilde{g}(t^1, t^2) = g(t^2, t^1)$. With these choices, we obtain parametrizations g covering M for which f is always positive; but then the vectors $\frac{\partial g}{\partial t^1}(t) \times \frac{\partial g}{\partial t^2}(t)$ are all positive multiples of each other, meaning that the nonzero determinants occurring in these cross products have the same sign. From this it is easy to see that the diffeomorphisms obtained from these parametrizations have positive determinant.

($\Rightarrow$) Fix an orientation for M. Given $x_0 \in M$, choose a positive parametrization $g : U \subset \mathbb{R}^2 \to$

M such that $g(t_0) = x_0$ for some $t_0 \in U$, and define

$$n(x_0) = \frac{\frac{\partial g}{\partial t^1}(t_0) \times \frac{\partial g}{\partial t^2}(t_0)}{\left\| \frac{\partial g}{\partial t^1}(t_0) \times \frac{\partial g}{\partial t^2}(t_0) \right\|}.$$

Since $\dim T_{x_0}^{\perp} M = 1$ and g is positive, $n(x_0)$ does not depend on the choice of g, and since $g : U \to M$ is continuous at t_0 then $n : M \to S^2$ is also continuous at x_0. □

Definition 2.43. Let M be an oriented m-manifold in $\mathbb{R}^n$. If $g : U \subset \mathbb{R}^m \to M$ is a positive parametrization then the **integral** of a differential form $\omega \in \Omega^m(\mathbb{R}^n)$ on $g(U)$ is defined by

$$\int_{g(U)} \omega := \int_U \omega(g(t)) \left(\frac{\partial g}{\partial t^1}(t), \ldots, \frac{\partial g}{\partial t^m}(t) \right) dt^1 \ldots dt^m.$$

There are a few things we can prove right away about the integral of forms.

Firstly, note that if we see $U \subset \mathbb{R}^k$ as an k-manifold in $\mathbb{R}^k$ and parametrize it using the identity function $\mathrm{id} : U \to U$ defined by $\mathrm{id}(t) = t$ for all $t \in U$ (which we take to be positive), then for any k-form $\eta \in \Omega^k(U)$ we have

$$\int_U \eta = \int_{\mathrm{id}(U)} \eta = \int_U \eta(t)(e_1, \ldots, e_k) dt^1 \ldots dt^k.$$

If we let $f : \mathbb{R}^k \to \mathbb{R}$ be the smooth function such that $\eta(t) = f(t) dt^1 \wedge \ldots \wedge dt^k$ then

$$\int_U f(t) dt^1 \wedge \ldots \wedge dt^k = \int_U \eta = \int_U f(t) dt^1 \ldots dt^k.$$

Hence, the integral of forms of maximal degree just becomes the usual integral in $\mathbb{R}^k$.

Secondly, using the same notation as in the definition above, we have

$$\begin{aligned} \int_{g(U)} \omega &= \int_U \omega(g(t)) \left(\frac{\partial g}{\partial t^1}(t), \ldots, \frac{\partial g}{\partial t^m}(t) \right) dt^1 \ldots dt^m \\ &= \int_U (g^* \omega)(t)(e_1, \ldots, e_m) dt^1 \ldots dt^m \\ &= \int_U g^* \omega \end{aligned}$$

where the last equality comes from $g^* \omega$ being an m-form in U, i.e., a form of maximal degree.

Proposition 2.44. Let M be an oriented m-manifold in $\mathbb{R}^n$ and let $\omega \in \Omega^m(\mathbb{R}^n)$ be a differential form. The integral $\int_{g(U)} \omega$ is well defined, i.e., it does not depend on the positive parametrization g of M.

Proof. We start by proving the proposition for the case $m = n$. Consider an open set M in $\mathbb{R}^n$. Furthermore, fix an open set $W \subset M$ and let $g : U \subset \mathbb{R}^n \to M$ be a positive parametrization (that is, a diffeomorphism with positive Jacobian) such that $g(U) = W$. We have

$$\int_W \omega = \int_U \omega(g(t)) \left(\frac{\partial g}{\partial t^1}, \ldots, \frac{\partial g}{\partial t^n} \right) dt^1 \ldots dt^n.$$

If we let $\omega = f dx^1 \wedge \ldots \wedge dx^n$, where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, then

$$\begin{aligned} \int_W \omega &= \int_U f(g(t)) dx^1 \wedge \ldots \wedge dx^n \left(\frac{\partial g}{\partial t^1}, \ldots, \frac{\partial g}{\partial t^n} \right) dt^1 \ldots dt^n \\ &= \int_U f(g(t)) \det Dg(t) dt^1 \ldots dt^n. \end{aligned}$$

Since g is a diffeomorphism with positive Jacobian, we have

$$\int_W \omega = \int_U f(g(t)) \det |Dg(t)| dt^1 \ldots dt^n,$$

and by the change of variables theorem we get

$$\int_W \omega = \int_{g(U)} f(x) dx^1 \ldots dx^n = \int_V f(x) dx^1 \ldots dx^n,$$

which does not depend on g, as desired.

Now let M be an m-manifold in $\mathbb{R}^n$, and let $g : U \subset \mathbb{R}^m \to M$ and $h : V \subset \mathbb{R}^m \to M$ be positive parametrizations such that $g(U) = h(V) = W \subset M$. Then there exists a positive diffeomorfism $k : U \to V$ such that $g = h \circ k$. Hence

$$\int_W \omega = \int_{g(U)} \omega = \int_U g^* \omega = \int_U (h \circ k)^* \omega = \int_U k^*(h^* \omega) = \int_V h^* \omega,$$

as required. ∎

Definition 2.45. If $M \subset \mathbb{R}^n$ is an oriented m-manifold and $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \sum_{i=1}^{N} \int_{g_i(U_i)} \omega,$$

where $g_i : U_i \to M$ are positive parametrizations whose images are disjoint and cover M except for the union of a finite number of manifolds of dimension smaller than m.

Remark 2.46. It can be shown that it is always possible to obtain a finite number of parametrizations of this kind, and that the definition above does not depend on the choice of these parametrizations.

Definition 2.47. Informally, an m**-manifold with boundary** is a subset $M \subset N$ of an m-manifold $N \subset \mathbb{R}^n$ delimited by an $(m-1)$-manifold $\partial M \subset M$, called the **boundary** of M, such that $M \setminus \partial M$ is again an m-manifold. We say that M is **orientable** if N is orientable. If M is oriented, the **induced orientation** on ∂M is defined as follows: if $g : U \cap \{t^1 \leq 0\} \to M$ is a positive parametrization of M such that $h(t^2, \ldots, t^m) = g(0, t^2, \ldots, t^m)$ is a parametrization of ∂M, then h is positive. Moreover, if $\omega \in \Omega^m(\mathbb{R}^n)$, we define

$$\int_M \omega = \int_{M \setminus \partial M} \omega.$$

Remark 2.48. A manifold is a particular case of a manifold with boundary, but a manifold with boundary is not a manifold in general.

Theorem 2.49. *(Stokes) If $M \subset \mathbb{R}^n$ is a compact, oriented m-manifold with boundary and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the induced orientation.

Proof. We assume that M can be decomposed into images of cubes by positive parametrizations. Since the integrals along adjacent faces correspond to opposite orientations, and consequently cancel out, we can assume without loss of generality that M is the image of a cube.

Let $g : [0,1]^m \to M$ be a parametrization. Since

$$\int_M d\omega = \int_{[0,1]^m} g^*(d\omega) = \int_{[0,1]^m} d(g^*\omega)$$

and

$$\int_{\partial M} \omega = \int_{\partial[0,1]^m} g^*\omega,$$

it suffices to prove the Stokes Theorem in the case when $M = [0,1]^m$. If $\omega \in \Omega^{m-1}(\mathbb{R}^m)$ then

$$\omega = \sum_{i=1}^{m} \omega_i(x)\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m,$$

and so

$$d\omega = \sum_{i=1}^{m} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \ldots \wedge dx^m.$$

Consequently,

$$\begin{aligned}
\int_{[0,1]^m} d\omega &= \sum_{i=1}^{m} (-1)^{i-1} \int_{[0,1]^m} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \ldots \wedge dx^m \\
&= \sum_{i=1}^{m} (-1)^{i-1} \int_{[0,1]^m} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \ldots dx^m \\
&= \sum_{i=1}^{m} (-1)^{i-1} \left(\int_{\{x^i=1\}} \omega_i\, dx^1 \ldots \widehat{dx^i} \ldots dx^m - \int_{\{x^i=0\}} \omega_i\, dx^1 \ldots \widehat{dx^i} \ldots dx^m \right) \\
&= \int_{\partial[0,1]^m} \omega,
\end{aligned}$$

where we used the definition of induced orientation (note that the orientation is reversed each time the coordinate functions are switched). □

Corollary 2.50. If $\partial M = \varnothing$ then $\int_M d\omega = 0$.

3 Surfaces

Definition 3.1. A surface is a 2-dimensional manifold $S \subset \mathbb{R}^3$.

Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of S and fix $(u_0, v_0) \in U$. For any vectors $\vec{v}, \vec{w} \in T_{g(u_0,v_0)}S$ we may write

$$\vec{v} = v^1 \frac{\partial g}{\partial u}(u_0, v_0) + v^2 \frac{\partial g}{\partial v}(u_0, v_0) = Dg(u_0, v_0) \begin{bmatrix} v^1 \\ v^2 \end{bmatrix},$$
$$\vec{w} = w^1 \frac{\partial g}{\partial u}(u_0, v_0) + w^2 \frac{\partial g}{\partial v}(u_0, v_0) = Dg(u_0, v_0) \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}.$$

For simplicity, let us drop the argument (u_0, v_0). Hence we have

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= \left\langle v^1 \frac{\partial g}{\partial u} + v^2 \frac{\partial g}{\partial v}, w^1 \frac{\partial g}{\partial u} + w^2 \frac{\partial g}{\partial v} \right\rangle \\ &= \left\| \frac{\partial g}{\partial u} \right\|^2 v^1 w^1 + \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} (v^1 w^2 + v^2 w^1) + \left\| \frac{\partial g}{\partial v} \right\|^2 v^2 w^2. \end{aligned}$$

Let $v = (v^1, v^2)$ and $w = (w^1, w^2)$ and recall that we may write

$$v^1 = du(v), \qquad v^2 = dv(v), \qquad w^1 = dv(w), \qquad w^2 = dv(w).$$

We define

$$E(u_0, v_0) = \left\| \frac{\partial g}{\partial u}(u_0, v_0) \right\|^2, \qquad F(u_0, v_0) = \frac{\partial g}{\partial u}(u_0, v_0) \cdot \frac{\partial g}{\partial v}(u_0, v_0), \qquad G(u_0, v_0) = \left\| \frac{\partial g}{\partial v}(u_0, v_0) \right\|^2$$

and therefore we have

$$\langle \vec{v}, \vec{w} \rangle = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}.$$

In particular, we have

$$\| \vec{v} \|^2 = E(u_0, v_0) du(v) du(v) + 2F(u_0, v_0) du(v) dv(v) + G(u_0, v_0) dv(v) dv(v).$$

Based on this, we make the following definition, where $\text{Pol}_2(\mathbb{R}^2)$ is the set of homogeneous polynomials of degree 2.

Definition 3.2. (First Fundamental Form) Let $S \subset \mathbb{R}^3$ be a surface parametrized by $g : U \to \mathbb{R}^3$ and define the functions $E, F, G : U \to \mathbb{R}$ by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} \\ \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial v} \end{bmatrix}.$$

We define the **first fundamental form** of S to be the function $\mathbb{I} : U \to \mathrm{Pol}_2(\mathbb{R}^2)$ given by

$$\mathbb{I} = Edu^2 + 2Fdudv + Gdv^2.$$

We call

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

the **matrix of the metric**.

Remark 3.3. When the context is clear, we will drop the (u_0, v_0) in $\mathbb{I}(u_0, v_0)(v)$ and simply write $\mathbb{I}(v)$ for

$$\mathbb{I}(v) = E(u_0, v_0)du^2(v) + 2F(u_0, v_0)dudv(v) + G(u_0, v_0)dv^2(v).$$

The same will be done for the second fundamental form, defined later in the text.

Remark 3.4. We have, for all $v = (v^1, v^2)$,

$$\mathbb{I}(v) = Edu^2(v) + 2Fdudv(v) + Gdv^2(v) = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}.$$

Note that, again, the above relation should be interpreted as a equality between functions, for any (u_0, v_0). If we wanted to write it in the most correct way, we could say that for any (u_0, v_0) and any $v = (v^1, v^2)$ we have

$$\mathbb{I}(u_0, v_0)(v) = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} E(u_0, v_0) & F(u_0, v_0) \\ F(u_0, v_0) & G(u_0, v_0) \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}.$$

What we saw before Definition 3.2 was the following proposition.

Proposition 3.5. Let $S \subset \mathbb{R}^3$ be a surface and $g : U \to \mathbb{R}^3$ be a parametrization of S. If $\vec{v} \in T_{g(u_0,v_0)}S$ is a vector written in the basis $\left\{\frac{\partial g}{\partial u}(u_0, v_0), \frac{\partial g}{\partial v}(u_0, v_0)\right\}$ as

$$\vec{v} = v^1 \frac{\partial g}{\partial u}(u_0, v_0) + v^2 \frac{\partial g}{\partial v}(u_0, v_0)$$

then $\|\vec{v}\|^2 = \mathbb{I}(v^1, v^2)$.

Remark 3.6. Now note that if we set $g(u, v) = (g^1(u, v), g^2(u, v), g^3(u, v))$ we may think of g^1, g^2, g^3 as 0-forms. Hence it makes sense to write dg as the triple of 1-forms

$$dg := \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv = \left(\frac{\partial g^1}{\partial u}du + \frac{\partial g^1}{\partial v}dv, \frac{\partial g^2}{\partial u}du + \frac{\partial g^2}{\partial v}dv, \frac{\partial g^3}{\partial u}du + \frac{\partial g^3}{\partial v}dv\right).$$

Now, if we compute the dot product $dg \cdot dg$ just as we would a vector, we obtain

$$dg \cdot dg = Edu^2 + 2Fdudv + Gdv^2$$

and so we frequently write $\mathbb{I} = dg \cdot dg$.

Remark 3.7. If $c(t) = g(u(t), v(t))$ is a curve on a surface S then

$$c'(t) = u'(t)\frac{\partial g}{\partial u} + v'(t)\frac{\partial g}{\partial v}$$

and so the length of c is

$$\int_a^b \|c'(t)\|\, dt = \int_a^b \sqrt{\mathbb{I}(u'(t), v'(t))}\, dt.$$

We will now introduce the second fundamental form. In order to do that, we need the unit normal of the surface with respect to a parametrization.

Definition 3.8. (Unit normal) Given a surface $S \subset \mathbb{R}^3$ and a parametrization $g : U \to \mathbb{R}^3$, we define the **unit normal** with respect to g by

$$\vec{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\|}.$$

Since $\vec{n}$ is normal to the surface we know that

$$\frac{\partial g}{\partial u} \cdot \vec{n} = 0 = \frac{\partial g}{\partial v} \cdot \vec{n}.$$

This implies that

$$\frac{\partial^2 g}{\partial u^2} \cdot \vec{n} + \frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial u} = 0, \qquad \frac{\partial^2 g}{\partial u \partial v} \cdot \vec{n} + \frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial v} = 0, \tag{1}$$
$$\frac{\partial^2 g}{\partial v \partial u} \cdot \vec{n} + \frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial u} = 0, \qquad \frac{\partial^2 g}{\partial v^2} \cdot \vec{n} + \frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial v} = 0, \tag{2}$$

and so let us define

$$L = \frac{\partial^2 g}{\partial u^2} \cdot \vec{n}, \qquad M = \frac{\partial^2 g}{\partial u \partial v} \cdot \vec{n} = \frac{\partial^2 g}{\partial v \partial u} \cdot \vec{n}, \qquad N = \frac{\partial^2 g}{\partial v^2} \cdot \vec{n}.$$

From the previous formulas we get

$$L = -\frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial u}, \qquad M = -\frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial v} = -\frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial u}, \qquad N = -\frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial v}, \tag{3}$$

and so we make the following definition.

Definition 3.9. (Second Fundamental Form) Let $S \subset \mathbb{R}^3$ be a surface parametrized by $g : U \to \mathbb{R}^3$ and let $\vec{n}$ be the unit normal with respect to g. Define the functions $L, M, N : U \to \mathbb{R}$ by

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} -\frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial u} & -\frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial v} \\ -\frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial u} & -\frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial v} \end{bmatrix}.$$

We define the **second fundamental form** of S to be the function $\mathbb{II} : U \to \mathrm{Pol}_2(\mathbb{R}^2)$ given by

$$\mathbb{II} = Ldu^2 + 2Mdudv + Ndv^2.$$

Definition 3.10. (Second Fundamental Form) Similarly to Remark 3.6, we have

$$\mathbb{II} = Ldu^2 + 2Mdudv + Ndv^2 = -\left(\frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv\right) \cdot \left(\frac{\partial \vec{n}}{\partial u}du + \frac{\partial \vec{n}}{\partial u}dv\right)$$

and thus sometimes we write

$$\mathbb{II} = -dg \cdot d\vec{n}.$$

We have

$$\mathrm{III}(v^1, v^2) = \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

Note now that we may write

$$\frac{\partial \vec{n}}{\partial u} = \alpha \frac{\partial g}{\partial u} + \beta \frac{\partial g}{\partial v}$$

and therefore the equations in (1) and (2)

$$\frac{\partial^2 g}{\partial u^2} \cdot \vec{n} + \frac{\partial g}{\partial u} \cdot \frac{\partial \vec{n}}{\partial u} = 0 \qquad \text{and} \qquad \frac{\partial^2 g}{\partial v \partial u} \cdot \vec{n} + \frac{\partial g}{\partial v} \cdot \frac{\partial \vec{n}}{\partial u} = 0$$

become

$$L + \alpha E + \beta F = 0 \qquad \text{and} \qquad M + \alpha F + \beta G = 0.$$

We can solve this system of equations to get α and β. The same can be done for $\frac{\partial \vec{n}}{\partial v}$, giving us Weingarten's equations.

Proposition 3.11. (Weingarten's equations) We have

$$\frac{\partial \vec{n}}{\partial u} = \frac{FM - GL}{EG - F^2} \frac{\partial g}{\partial u} + \frac{FL - EM}{EG - F^2} \frac{\partial g}{\partial v}$$

and

$$\frac{\partial \vec{n}}{\partial v} = \frac{FN - GM}{EG - F^2} \frac{\partial g}{\partial u} + \frac{FM - EN}{EG - F^2} \frac{\partial g}{\partial v}.$$

3.1 Curvatures

Definition 3.12. Let $c : I \subset \mathbb{R} \to S$ be a curve along S parametrized by arclength. We write

$$c''(s) = \overrightarrow{k_g}(s) + \overrightarrow{k_n}(s)$$

where $\overrightarrow{k_g}$, the **geodesic curvature vector**, is tangent to S and $\overrightarrow{k_n}$, the **normal curvature vector**, is orthogonal to S. Furthermore, define the **normal curvature** $k_n(s)$ as the function such that $\overrightarrow{k_n} = k_n \vec{n}$.

Proposition 3.13. The normal curvature of a curve $c : I \subset \mathbb{R} \to S$ given by $c(s) = g(u(s), v(s))$, where g is a parametrization of S, satisfies

$$k_n(s) = \mathbb{II}(u'(s), v'(s)).$$

Proof. We have $\overrightarrow{k_n} = k_n \vec{n}$, and therefore

$$\begin{aligned} k_n &= \overrightarrow{k_n} \cdot \vec{n} = (c'' - \overrightarrow{k_g}) \cdot \vec{n} = c'' \cdot \vec{n} = (\underbrace{c' \cdot \vec{n}}_{=0})' - c' \cdot \vec{n}' \\ &= -\left(u'\frac{\partial g}{\partial u} + v'\frac{\partial g}{\partial v}\right) \cdot \left(u'\frac{\partial \vec{n}}{\partial u} + v'\frac{\partial \vec{n}}{\partial v}\right) \\ &= L(u')^2 + 2Mu'v' + N(v')^2 \\ &= \mathbb{II}(u', v') \end{aligned}$$

as required. □

Note that, since we had the curve c parametrized by arclength,

$$\|c'(s)\| = 1 \Rightarrow u'(s)^2 + v'(s)^2 = 1 \Leftrightarrow \mathbb{I}(u', v') = 1$$

We now want to study the maximum and minimum values of k_n at some point $g(u_0, v_0) \in S$. It is clear that $u'(s)$ and $v'(s)$ can take on any values, and therefore we may simply study the extrema of $\mathbb{II}(v^1, v^2)$ with the restriction that $\mathbb{I}(v^1, v^2) = 1$.

We will use the method of Lagrange multipliers. Let

$$f(v^1, v^2) := L(v^1)^2 + 2Mv^1v^2 + N(v^2)^2 - \lambda\Big(E(v^1)^2 + 2Fv^1v^2 + G(v^2)^2\Big).$$

We want to solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial v^1} = 0 \\ \frac{\partial f}{\partial v^2} = 0 \end{cases} \Leftrightarrow \begin{cases} 2Lv^1 + 2Mv^2 - 2\lambda E v^1 - 2\lambda F v^2 & = 0 \\ 2Mv^1 + 2Nv^2 - 2\lambda F v^1 - 2\lambda G v^2 & = 0 \end{cases}$$

which we may write as

$$\begin{bmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We are then looking for the values of λ such that

$$\det \left(\underbrace{\begin{bmatrix} L & M \\ M & N \end{bmatrix}}_{B} - \lambda \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{A} \right) = 0 \Leftrightarrow \det(B - \lambda A) = 0.$$

This is equivalent to

$$\det \Big(A(A^{-1}B - \lambda I) \Big) = 0 \Leftrightarrow \underbrace{\det(A)}_{\neq 0} \det(A^{-1}B - \lambda I) = 0 \Leftrightarrow \det(A^{-1}B - \lambda I) = 0.$$

We have

$$A^{-1}B = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} GL - FM & GM - FN \\ EM - FL & EN - FM \end{bmatrix}.$$

The solutions k_1, k_2 for λ correspond to the eigenvalues of the matrix $A^{-1}B$, and so we obtain

$$k_1 + k_2 = \operatorname{tr}(A^{-1}B) = \frac{GL - 2FM + EN}{EG - F^2}$$

and

$$k_1 k_2 = \det(A^{-1}B) = \frac{\det(B)}{\det(A)} = \frac{LN - M^2}{EG - F^2}.$$

Furthermore, if (v^1, v^2) is a extremizer corresponding to $\lambda = k_1$ then

$$(B - k_1 A) \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} v^1 & v^2 \end{bmatrix} (B - k_1 A) \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0 \Leftrightarrow \mathbb{II}(v^1, v^2) - k_1 \mathbb{I}(v^1, v^2) = 0$$

and because $\mathbb{I}(v^1, v^2) = 1$ we get

$$\mathbb{II}(v^1, v^2) = k_1.$$

Definition 3.14. The maximum and the minimum of $\mathbb{II}(v^1, v^2)$ subject to the constraint $\mathbb{I}(v^1, v^2) = 1$ are called the **principal curvatures** of S at the point under consideration. The directions of the corresponding unit tangent vectors are called the **principal directions** of S at that point.

Definition 3.15. Let k_1, k_2 be the principal curvatures of S at a given point. We define the **mean curvature** at that point by

$$H = \frac{1}{2}(k_1 + k_2) = \frac{GL - 2FM + EN}{2(EG - F^2)}$$

and the **Gauss curvature** at that point by

$$K = k_1 k_2 = \frac{\det \mathbb{II}}{\det \mathbb{I}} = \frac{LN - M^2}{EG - F^2}.$$

If $K \equiv 0$ then S is said to be **flat**.
If $H \equiv 0$ then S is said to be **minimal**.

Proposition 3.16. If $k_1 \neq k_2$ then the principal directions are orthogonal.

Proof. Let (v^1, v^2) and (w^1, w^2) be such that

$$B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = k_1 A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \qquad \text{and} \qquad B \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = k_2 A \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}.$$

We have

$$k_1 \begin{bmatrix} w^1 & w^2 \end{bmatrix} A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} w^1 & w^2 \end{bmatrix} B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \left(\begin{bmatrix} w^1 & w^2 \end{bmatrix} B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \right)^T.$$

Using the fact that $B = B^T$, we get

$$k_1 \begin{bmatrix} w^1 & w^2 \end{bmatrix} A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 & v^2 \end{bmatrix} B \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = k_2 \begin{bmatrix} v^1 & v^2 \end{bmatrix} A \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = k_2 \left(\begin{bmatrix} v^1 & v^2 \end{bmatrix} A \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \right)^T,$$

and using the fact that $A = A^T$, we get

$$k_1 \begin{bmatrix} w^1 & w^2 \end{bmatrix} A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = k_2 \begin{bmatrix} w^1 & w^2 \end{bmatrix} A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \Leftrightarrow (k_1 - k_2) \begin{bmatrix} w^1 & w^2 \end{bmatrix} A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0.$$

Assuming that $k_1 \neq k_2$ we can divide by $k_1 - k_2$ and therefore we have

$$\begin{bmatrix} w^1 & w^2 \end{bmatrix} \begin{bmatrix} \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} \\ \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial v} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0 \Rightarrow \langle \vec{v}, \vec{w} \rangle = 0$$

as required. □

Definition 3.17. If $k_1 = k_2$ at some point in S then that point is called **umbillic**. Furthermore, we call the point

- **elliptic** if $K > 0$;
- **hyperbolic** if $K < 0$;
- **parabolic** if $K = 0$.

We will now deduce a different type of formula for K. Recall the formula for the area of the image of a parametrization.

Definition 3.18. (Area) Let $g : U \to S$ be a parametrization for some surface $S \subset \mathbb{R}^3$ and let $V \subset U$ be a open set. The area of $g(V)$ is given by

$$\iint_V \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dudv.$$

Consider some parametrization $g : U \to S$ of S and let $\vec{n}$ denote the unit normal. Recall that Weingarten's formulas give us

$$\frac{\partial \vec{n}}{\partial u} = \frac{FM - GL}{EG - F^2} \frac{\partial g}{\partial u} + \frac{FL - EM}{EG - F^2} \frac{\partial g}{\partial v}$$

and

$$\frac{\partial \vec{n}}{\partial v} = \frac{FN - GM}{EG - F^2} \frac{\partial g}{\partial u} + \frac{FM - EN}{EG - F^2} \frac{\partial g}{\partial v}.$$

and therefore

$$\frac{\partial \vec{n}}{\partial u} \times \frac{\partial \vec{n}}{\partial v} = \frac{(FM - GL)(FM - EN) - (FL - EM)(FN - GM)}{(EG - F^2)^2} \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$$

which simplifies to

$$\frac{\partial \vec{n}}{\partial u} \times \frac{\partial \vec{n}}{\partial v} = \frac{LN - M^2}{EG - F^2} \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = K \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}.$$

Hence note that if for some $(u_0, v_0) \in U$ we have $K(u_0, v_0) \neq 0$ then the function $n : U \to S^2$ is a parametrization of S^2in some neighborhood of (u_0, v_0).

Therefore, for $\varepsilon > 0$ let us define

$$A(\varepsilon) = \iint_{B_\varepsilon(u_0,v_0)} \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dudv$$

as the area of $g(B_\varepsilon(u_0, v_0))$ and

$$A'(\varepsilon) = \iint_{B_\varepsilon(u_0,v_0)} \left\| \frac{\partial \vec{n}}{\partial u} \times \frac{\partial \vec{n}}{\partial v} \right\| dudv$$

as the area of $\vec{n}(B_\varepsilon(u_0, v_0))$.

From our previous work we have the following result.

Proposition 3.19. Let $(u_0, v_0) \in U$ be a point such that $K(u_0, v_0) \neq 0$. Then we have

$$|K(u_0, v_0)| = \lim_{\varepsilon \to 0} \frac{A'(\varepsilon)}{A(\varepsilon)}.$$

Proof. We saw that

$$A'(\varepsilon) = \iint_{B_\varepsilon(u_0,v_0)} \left\| \frac{\partial \vec{n}}{\partial u} \times \frac{\partial \vec{n}}{\partial v} \right\| dudv = \iint_{B_\varepsilon(u_0,v_0)} |K| \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dudv,$$

which yields the result. □

3.2 Minimal surfaces

Proposition 3.20. Let g be a parametrization of S. The area of $g(U)$ is given by

$$A(U) = \iint_U \sqrt{EG - F^2}\, dudv.$$

Proof. The relation

$$\|v \times w\| = \sqrt{\|v\|^2\|w\|^2 - (v \cdot w)^2}$$

holds for any vectors $v, w \in \mathbb{R}^3$, and therefore

$$\begin{aligned} A(U) = \iint_U \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dudv &= \iint_U \left(\left\| \frac{\partial g}{\partial u} \right\|^2 \left\| \frac{\partial g}{\partial v} \right\|^2 - \left(\frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} \right)^2 \right)^{1/2} \\ &= \iint_U \sqrt{EG - F^2}\, dudv, \end{aligned}$$

as required. □

Now let $f : U \to \mathbb{R}$ be a smooth function. We define deformations of g along the normal $\vec{n}$ by

$$g_\varepsilon(u, v) := g(u, v) + \varepsilon f(u, v)\vec{n}(u, v).$$

Let $A(\varepsilon)$ denote the area of $g_\varepsilon(U)$. We will start by computing $A(\varepsilon)$.

We have

$$\begin{aligned} \frac{\partial g_\varepsilon}{\partial u} &= \frac{\partial g}{\partial u} + \varepsilon \frac{\partial f}{\partial u}\vec{n} + \varepsilon f \frac{\partial \vec{n}}{\partial u}, \\ \frac{\partial g_\varepsilon}{\partial v} &= \frac{\partial g}{\partial v} + \varepsilon \frac{\partial f}{\partial v}\vec{n} + \varepsilon f \frac{\partial \vec{n}}{\partial v}, \end{aligned}$$

and therefore

$$\begin{aligned} E_\varepsilon &= \frac{\partial g_\varepsilon}{\partial u} \cdot \frac{\partial g_\varepsilon}{\partial u} = E - 2\varepsilon f L + \mathcal{O}(\varepsilon^2), \\ F_\varepsilon &= \frac{\partial g_\varepsilon}{\partial u} \cdot \frac{\partial g_\varepsilon}{\partial v} = F - 2\varepsilon f M + \mathcal{O}(\varepsilon^2), \\ G_\varepsilon &= \frac{\partial g_\varepsilon}{\partial v} \cdot \frac{\partial g_\varepsilon}{\partial v} = G - 2\varepsilon f N + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Hence, we have

$$\begin{aligned} A(\varepsilon) &= \iint_U \sqrt{E_\varepsilon G_\varepsilon - F_\varepsilon^2}\, dudv \\ &= \iint_U \sqrt{EG - F^2 - 2\varepsilon f(EN + GL - 2FM) + \mathcal{O}(\varepsilon^2)}\, dudv. \end{aligned}$$

With this expression for $A(\varepsilon)$, we can compute $\frac{dA}{d\varepsilon}(0)$ to better understand what happens to the area of $g(U)$ after a small deformation:

$$\begin{aligned} \frac{dA}{d\varepsilon}(0) &= \iint_U \frac{1}{2}\Big(-2f(EN + GL - 2FM)\Big)(EG - F^2)^{-1/2}\, dudv \\ &= \iint_U -2f \frac{EN + GL - 2FM}{2(EG - F^2)} \sqrt{EG - F^2}\, dudv \\ &= \iint_U -2fH\sqrt{EG - F^2} dudv. \end{aligned}$$

From this we get the following proposition.

Proposition 3.21. If S has minimal area for some fixed boundary, then $H \equiv 0$, i.e., S is minimal (definition 6.13.).

Proof. Let φ be a smooth positive function with support away from the boundary (so that the boundary of the surface does not change). Let us take $f(u, v) = \varphi(u, v)H(u, v)$ in the above expression. Assuming that S has minimal area implies

$$0 = \frac{dA}{d\varepsilon}(0) = \iint_U -2\varphi H^2 \sqrt{EG - F^2}.$$

This implies that $H \equiv 0$ on the support of φ; since φ is arbitrary, we have $H \equiv 0$, as required. ☐

In the same spirit we prove the following proposition.

Proposition 3.22. If S has minimal area while bounding a fixed volume then H is constant.

Proof. Let $f : U \to \mathbb{R}$ be a smooth function. Let $h : U \times \mathbb{R} \to \mathbb{R}^3$ be the function

$$h(u, v, w) = g(u, v) + wf(u, v)\vec{n}(u, v).$$

Let $A(\varepsilon)$ be the area of the surface parametrized by the function $h_\varepsilon(u, v)$ given by

$$h_\varepsilon(u, v) := h(u, v, \varepsilon).$$

We want to prove that if $A(\varepsilon) = 0$ has a minimum at $\varepsilon = 0$ for a fixed volume then H must be constant. We know from the method of Lagrange multipliers that we must have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(A(\varepsilon) - \lambda V(\varepsilon)) = 0 \Rightarrow \frac{dA}{d\varepsilon}(0) - \lambda\frac{dV}{d\varepsilon}(0) = 0$$

for some $\lambda \in \mathbb{R}$.

We have

$$V(\varepsilon) - V(0) = \int_0^\varepsilon \iint_U \det Dh(u, v, w)\, dudvdw,$$

and therefore

$$\frac{dV}{d\varepsilon}(\varepsilon) = \iint_U \det Dh(u, v, \varepsilon)\, dudv.$$

Furthermore,

$$Dh(u, v, 0) = \begin{bmatrix} | & | & | \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & f\vec{n} \\ | & | & | \end{bmatrix},$$

and thus

$$\begin{aligned}\frac{dV}{d\varepsilon}(0) = \iint_U \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) \cdot (f\vec{n})\, dudv &= \iint_U f \left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\| dudv \\ &= \iint_U f\sqrt{EG - F^2}\, dudv.\end{aligned}$$

Combining this formula with the formula already obtained for $\frac{dA}{d\varepsilon}(0)$ gives that, for any f,

$$\frac{dA}{d\varepsilon}(0) - \lambda\frac{dV}{d\varepsilon}(0) = 0 \Rightarrow -\iint_U (2H + \lambda)f\sqrt{EG - F^2}\, dudv = 0.$$

Again taking φ to be a smooth positive function with support away from the boundary and choosing $f(u, v) = \varphi(u, v)(2H(u, v) + \lambda)$ in the above expression we obtain

$$H \equiv -\frac{\lambda}{2},$$

which completes the proof. $\square$

3.3 Method of orthonormal frames

Let $g : U \to \mathbb{R}^3$ be a parametrization of some surface. The method of orthonormal frames gives as an alternative way to compute curvatures. Let us start by defining an orthonormal basis for the tangent space at a point in the surface. An easy way to do this is by simply defining

$$\vec{e_1} = \frac{\frac{\partial g}{\partial u}}{\left\| \frac{\partial g}{\partial u} \right\|}$$

and

$$\vec{e_2} = \frac{\frac{\partial g}{\partial v} - \left(\frac{\partial g}{\partial v} \cdot \vec{e_1} \right) \vec{e_1}}{\left\| \frac{\partial g}{\partial v} - \left(\frac{\partial g}{\partial v} \cdot \vec{e_1} \right) \vec{e_1} \right\|},$$

but this is just an example. We can then define $\vec{e_3} := \vec{e_1} \times \vec{e_2}$ as the vector normal to the surface at the point, and so $\vec{e_1}, \vec{e_2}, \vec{e_3}$ form an orthonormal basis of the space $\mathbb{R}^3$.

Now we define $a_1^1, a_1^2, a_2^1, a_2^2$ such that

$$\begin{aligned} \frac{\partial g}{\partial u} &= a_1^1 \vec{e_1} + a_1^2 \vec{e_2}, \\ \frac{\partial g}{\partial v} &= a_2^1 \vec{e_1} + a_2^2 \vec{e_2}. \end{aligned}$$

Let us interpret $g : U \to \mathbb{R}^3$ as three 0-forms in $\mathbb{R}^2$, and so dg will be a triple of 1-forms. We have

$$\begin{aligned} dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv &= (a_1^1 \vec{e_1} + a_1^2 \vec{e_2}) du + (a_2^1 \vec{e_1} + a_2^2 \vec{e_2}) dv \\ &= (a_1^1 du + a_2^1 dv) \vec{e_1} + (a_1^2 du + a_2^2 dv) \vec{e_2}. \end{aligned}$$

Let us define the forms $\theta^1, \theta^2 \in \Omega^1(\mathbb{R}^2)$ as

$$\begin{aligned} \theta^1 &= a_1^1 du + a_2^1 dv, \\ \theta^2 &= a_1^2 du + a_2^2 dv, \end{aligned}$$

that is,

$$\begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} = S \begin{bmatrix} du \\ dv \end{bmatrix},$$

where S is the matrix of functions

$$S = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}.$$

Note that

$$S^T S = \begin{bmatrix} (a_1^1)^2 + (a_1^2)^2 & a_1^1 a_2^1 + a_1^2 a_2^2 \\ a_2^1 a_1^1 + a_2^2 a_1^2 & (a_2^1)^2 + (a_2^2)^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} \\ \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

and thus

$$\det(S)^2 = \det(S^T S) = \det(\mathbb{I}) \neq 0 \Rightarrow \det(S) \neq 0.$$

Consequently, since du and dv are a basis for $(\mathbb{R}^2)^*$, so are the tensors $\theta^1(u,v), \theta^2(u,v)$ for each $(u,v) \in U$. Now, since $dg = \theta^1 \vec{e_1} + \theta^2 \vec{e_2}$, we have

$$\mathbb{I} = dg \cdot dg = (\theta^1)^2 + (\theta^2)^2.$$

Interpreting again $\vec{e_i} : U \to \mathbb{R}^3$ as triples of 0-forms in $\mathbb{R}^2$, we may write, for some forms $\omega_i^j \in \Omega^1(U)$,

$$\begin{aligned} d\vec{e_1} &= \omega_1^1 \vec{e_1} + \omega_1^2 \vec{e_2} + \omega_1^3 \vec{e_3}, \\ d\vec{e_2} &= \omega_2^1 \vec{e_1} + \omega_2^2 \vec{e_2} + \omega_2^3 \vec{e_3}, \\ d\vec{e_3} &= \omega_3^1 \vec{e_1} + \omega_3^2 \vec{e_2} + \omega_3^3 \vec{e_3}. \end{aligned}$$

To be precise, for each $(u,v) \in U$ the vectors $\vec{e_1}(u,v), \vec{e_2}(u,v), \vec{e_3}(u,v)$ form a orthonormal basis for $\mathbb{R}^3$ and so, for any $w \in \mathbb{R}^2$, we define the form ω_i^j by

$$\omega_i^j(u,v)(w) := d\vec{e_i}(u,v)(w) \cdot \vec{e_j}(u,v).$$

We will prove that these forms satisfy $\omega_i^j + \omega_j^i = 0$ for all $i, j \in \{1,2,3\}$. In particular, $w_k^k = 0$ for all $k \in \{1,2,3\}$. If we write each $\vec{e_k}$ as (e_k^1, e_k^2, e_k^3), we have

$$d(\vec{e_i} \cdot \vec{e_j}) = d\left(\sum_{k=1}^{3} e_i^k e_j^k\right) = \sum_{k=1}^{3} \Big((de_i^k) e_j^k + e_i^k (de_j^k)\Big) = (d\vec{e_i}) \cdot \vec{e_j} + \vec{e_i} \cdot (d\vec{e_j}),$$

and so, since $\vec{e_i} \cdot \vec{e_j}$ is a constant (either 0 or 1), we get

$$0 = d(\vec{e_i} \cdot \vec{e_j}) = (d\vec{e_i}) \cdot \vec{e_j} + \vec{e_i} \cdot (d\vec{e_j}) = w_i^j + w_j^i.$$

We proceed by studying the second fundamental form. We have

$$\begin{aligned}\mathbb{II} = -dg \cdot d\overrightarrow{n} = -dg \cdot d\overrightarrow{e_3} &= -(\theta^1 \overrightarrow{e_1} + \theta^2 \overrightarrow{e_2}) \cdot (\omega_3^1 \overrightarrow{e_1} + \omega_3^2 \overrightarrow{e_2} + \omega_3^3 \overrightarrow{e_3}) \\ &= -\theta^1 \omega_3^1 - \theta^2 \omega_3^2 \\ &= \theta^1 \omega_1^3 + \theta^2 \omega_2^3.\end{aligned}$$

Note that, for each $(u, v) \in U$, the object $\omega_1^3(u, v)$ is a tensor in $\Lambda^1(\mathbb{R}^2)$. Therefore, since $\theta^1(u, v)$ and $\theta^2(u, v)$ are a basis of the 2-dimensional space $\Lambda^1(\mathbb{R}^2)$, we can write $\omega_1^3(u, v)$ in the basis $\{\theta^1(u, v), \theta^2(u, v)\}$, that is, we can find functions $b_{11}, b_{12}, b_{21}, b_{22} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned}\omega_1^3 &= b_{11}\theta^1 + b_{12}\theta^2, \\ \omega_2^3 &= b_{21}\theta^1 + b_{22}\theta^2.\end{aligned}$$

Now we have, using the formulas in (3),

$$L du + M dv = -\frac{\partial g}{\partial u} \cdot \frac{\partial \overrightarrow{n}}{\partial u} du - \frac{\partial g}{\partial u} \cdot \frac{\partial \overrightarrow{n}}{\partial v} dv = -\frac{\partial g}{\partial u} \cdot d\overrightarrow{n} = -\frac{\partial g}{\partial u} \cdot d\overrightarrow{e_3},$$

and so

$$L du + M dv = -(a_1^1 \overrightarrow{e_1} + a_1^2 \overrightarrow{e_2})(\omega_3^1 \overrightarrow{e_1} + \omega_3^2 \overrightarrow{e_2}) = a_1^1 \omega_1^3 + a_1^2 \omega_2^3.$$

Similarly,

$$M du + N dv = a_2^1 \omega_1^3 + a_2^2 \omega_2^3.$$

Let us define the matrix of functions

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

We have

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix} \begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = S^T B \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} = S^T B S \begin{bmatrix} du \\ dv \end{bmatrix}$$

and so

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = S^T B S \Leftrightarrow B = S^{-T} \begin{bmatrix} L & M \\ M & N \end{bmatrix} S^{-1}.$$

Furthermore, we have seen that

$$S^T S = \begin{bmatrix} E & F \\ F & G \end{bmatrix},$$

and so we obtain

$$\det(B) = \det(S^{-T})(LN - M^2)\det(S^{-1}) = \frac{LN - M^2}{\det(S^T S)} = \frac{LN - M^2}{EG - F^2} = K,$$

which gives us a new way to compute the Gauss curvature K.

Moreover, we have

$$\begin{aligned}
\operatorname{tr}(B) = \operatorname{tr}\left(S^{-T}\begin{bmatrix} L & M \\ M & N \end{bmatrix} S^{-1}\right) &= \operatorname{tr}\left(S^{-1}S^{-T}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) \\
&= \operatorname{tr}\left((S^T S)^{-1}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) \\
&= \operatorname{tr}\left(\frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) \\
&= \frac{GL - FM - FM + EN}{EG - F^2} \\
&= \frac{EN + GL - 2FM}{EG - F^2} \\
&= 2H,
\end{aligned}$$

which gives us a way to compute the mean curvature.

Finally, this also implies, if we let k_1, k_2 denote the principal curvatures, that

$$\det(B) = k_1 k_2 \qquad \text{and} \qquad \operatorname{tr}(B) = k_1 + k_2$$

and so k_1, k_2 are in fact the eigenvalues of B.

3.4 Structure equations

Let us keep using the notation of the previous subsection. We will prove the **first structure equations**,

$$\begin{cases} d\theta^1 = \theta^2 \wedge \omega_2^1 \\ d\theta^2 = \theta^1 \wedge \omega_1^2 \end{cases},$$

and the **second structure equation**,

$$d\omega_2^1 = K\theta^1 \wedge \theta^2.$$

We have $dg = \theta^1 \vec{e_1} + \theta^2 \vec{e_2}$ and so

$$\begin{aligned} 0 = d(dg) &= d\theta^1 \wedge \vec{e_1} - \theta^1 \wedge d\vec{e_1} + d\theta^2 \wedge \vec{e_2} - \theta^2 \wedge d\vec{e_2} \\ &= d\theta^1 \vec{e_1} - \theta^1 \wedge \sum_{k=1}^{3} \omega_1^k \vec{e_k} + d\theta^2 \vec{e_2} - \theta^2 \wedge \sum_{k=1}^{3} \omega_2^k \vec{e_k} \\ &= (d\theta^1 - \theta^1 \wedge \omega_1^1 - \theta^2 \wedge \omega_2^1)\vec{e_1} + (d\theta^2 - \theta^1 \wedge \omega_1^2 - \theta^2 \wedge \omega_2^2)\vec{e_2} + (-\theta^1 \wedge \omega_1^3 - \theta^2 \wedge \omega_2^3)\vec{e_3} \\ &= (d\theta^1 - \theta^2 \wedge \omega_2^1)\vec{e_1} + (d\theta^2 - \theta^1 \wedge \omega_1^2)\vec{e_2} + (-\theta^1 \wedge \omega_1^3 - \theta^2 \wedge \omega_2^3)\vec{e_3}. \end{aligned}$$

This gives us

$$\begin{cases} d\theta^1 - \theta^2 \wedge \omega_2^1 = 0 \\ d\theta^2 - \theta^1 \wedge \omega_1^2 = 0 \end{cases},$$

as desired.

Let us prove now the second structure equation. By using the properties of the exterior derivative we easily see that

$$d(\omega_i^j \vec{e_j}) = d(\omega_i^j)\vec{e_j} - \omega_i^j \wedge d\vec{e_j}.$$

Now, we have

$$0 = d(d\overrightarrow{e_i}) = d\left(\sum_{j=1}^{3}\omega_i^j\overrightarrow{e_j}\right) = \sum_{j=1}^{3}\left(d(\omega_i^j)\overrightarrow{e_j} - \omega_i^j \wedge d\overrightarrow{e_j}\right)$$
$$= \sum_{j=1}^{3} d(\omega_i^j)\overrightarrow{e_j} - \sum_{j,k=1}^{3}\omega_i^j \wedge \omega_j^k\overrightarrow{e_k}$$
$$= \sum_{k=1}^{3}\left(d(\omega_i^k) - \sum_{j=1}^{3}\omega_i^j \wedge \omega_j^k\right)\overrightarrow{e_k}$$

and so

$$d(\omega_i^k) = \sum_{j=1}^{3}\omega_i^j \wedge \omega_j^k.$$

In particular,

$$d(\omega_2^1) = \sum_{j=1}^{3}\omega_2^j \wedge \omega_j^1 = \omega_2^1 \wedge \omega_1^1 + \omega_2^2 \wedge \omega_2^1 + \omega_2^3 \wedge \omega_3^1 = \omega_2^3 \wedge \omega_3^1.$$

Hence, we get

$$\begin{aligned} d(\omega_2^1) = -\omega_2^3 \wedge \omega_1^3 &= -(b_{21}\theta^1 + b_{22}\theta^2) \wedge (b_{11}\theta^1 + b_{12}\theta^2) \\ &= -b_{21}b_{12}\theta^1 \wedge \theta^2 - b_{22}b_{11}\theta^2 \wedge \theta^1 \\ &= (b_{11}b_{22} - b_{21}b_{12})\theta^1 \wedge \theta^2 \\ &= \det(B)\theta^1 \wedge \theta^2 \\ &= K\theta^1 \wedge \theta^2, \end{aligned}$$

as desired.

4 Intrinsic geometry of Riemannian surfaces

We will sometimes write ds^2 instead of $\mathbb{I}$.

Definition 4.1. (Riemannian Surface) A **Riemannian Surface** is a pair (U, ds^2) where U is an open subset of $\mathbb{R}^2$ and $\mathbb{I} = ds^2$ is a first fundamental form (or metric) defined on U.

Proposition 4.2. Given a Riemannian surface (U, ds^2), there exist two 1-forms θ^1, θ^2 such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2.$$

Furthermore, if θ^1, θ^2 are such that the above relation is verified, then $\theta^1(p)$ and $\theta^2(p)$ are linearly independent tensors for all $p \in U$.

Proof. We want to find two 1-forms θ^1, θ^2 such that

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 = (\theta^1)^2 + (\theta^2)^2.$$

Let

$$\begin{cases} \theta^1 = a^1_1 du + a^1_2 dv \\ \theta^2 = a^2_1 du + a^2_2 dv \end{cases}.$$

We want

$$Edu^2 + 2Fdudv + Gdv^2 = \Big((a^1_1)^2 + (a^2_1)^2\Big)du^2 + 2\Big(a^1_1 a^1_2 + a^2_1 a^2_2\Big)dudv + \Big((a^1_2)^2 + (a^2_2)^2\Big)dv^2,$$

which corresponds to solving the system

$$\begin{cases} E = (a^1_1)^2 + (a^2_1)^2 \\ F = a^1_1 a^1_2 + a^2_1 a^2_2 \\ G = (a^1_2)^2 + (a^2_2)^2 \end{cases}.$$

We can find explicit solutions to the above system, namely

$$a^1_1 = \sqrt{E}, \qquad a^2_1 = 0, \qquad a^1_2 = \frac{F}{\sqrt{E}}, \qquad a^2_2 = \sqrt{\frac{EG - F^2}{E}}.$$

Now let $\{\theta^1, \theta^2\}$ be a solution of

$$ds^2 = (\theta^1)^2 + (\theta^2)^2,$$

and let

$$S = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}.$$

We know that

$$Edu^2 + 2Fdudv + Gdv^2 = \begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \theta^1 & \theta^2 \end{bmatrix} \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} = \begin{bmatrix} du & dv \end{bmatrix} S^T S \begin{bmatrix} du \\ dv \end{bmatrix},$$

which implies that

$$S^T S = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \Rightarrow \det(S) \neq 0$$

(since $\det(\mathbb{I}) \neq 0$), and therefore $\theta^1(p)$ and $\theta^2(p)$ are linearly independent for all $p \in U$. □

The structure equations we proved in the previous subsection give us a way to define ω_2^1 and the Gauss curvature K for general Riemannian surfaces. That is what we will be doing in this subsection.

Proposition 4.3. Given a Riemannian surface (U, ds^2) and two 1-forms θ^1, θ^2 such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2$$

there exists an unique 1-form ω_2^1 such that

$$\begin{cases} d\theta^1 = \theta^2 \wedge \omega_2^1 \\ d\theta^2 = -\theta^1 \wedge \omega_2^1. \end{cases}$$

Proof. We have seen before that, given any 1-form ω_2^1, it is possible to find functions $c_1, c_2 : U \to \mathbb{R}$ such that

$$\omega_2^1 = c_1\theta^1 + c_2\theta^2.$$

The conditions we want ω_2^1 to satisfy become

$$\begin{cases} d\theta^1 = c_1\theta^2 \wedge \theta^1 \\ d\theta^2 = -c_2\theta^1 \wedge \theta^2 \end{cases}$$

If we let $\alpha_1, \alpha_2, \beta : U \to \mathbb{R}$ be such that

$$d\theta^1 = \alpha_1 du \wedge dv, \qquad d\theta^2 = \alpha_2 du \wedge dv, \qquad \theta^1 \wedge \theta^2 = \beta du \wedge dv,$$

then, because $\theta^1(p)$ and $\theta^2(p)$ are linearly independent for all $p \in U$, we know that $\beta \neq 0$ and therefore the functions c_1, c_2 are the unique functions such that, for each $(u, v) \in U$,

$$c_1(u, v) = -\frac{\alpha_1(u, v)}{\beta(u, v)} \qquad \text{and} \qquad c_2(u, v) = -\frac{\alpha_2(u, v)}{\beta(u, v)}.$$

□

Definition 4.4. (Connection form) Given a Riemannian surface (U, ds^2) and two 1-forms θ^1, θ^2 such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2$$

we define the **connection form** associated to $\{\theta^1, \theta^2\}$ as the unique 1-form ω_2^1 such that

$$\begin{cases} d\theta^1 = \theta^2 \wedge \omega_2^1 \\ d\theta^2 = -\theta^1 \wedge \omega_2^1. \end{cases}$$

Proposition 4.5. Given a Riemannian surface (U, ds^2), let θ^1, θ^2 be two 1-forms such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2$$

and let ω_2^1 denote the connection form associated to $\{\theta^1, \theta^2\}$.
Then, the function $K : U \to \mathbb{R}$ such that

$$d(\omega_2^1) = K\theta^1 \wedge \theta^2$$

does not depend on the choice of the forms θ^1, θ^2.

Proof. Let $\{\theta^1, \theta^2\}$ and $\{\bar{\theta}^1, \bar{\theta}^2\}$ be two pairs of 1-forms such that

$$(\theta^1)^2 + (\theta^2)^2 = \mathbb{I} = (\bar{\theta}^1)^2 + (\bar{\theta}^2)^2.$$

We know that there exists an invertible matrix S such that

$$\begin{bmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{bmatrix} = S \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix},$$

and so

$$\begin{aligned} \begin{bmatrix} \theta^1 & \theta^2 \end{bmatrix} \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} = (\theta^1)^2 + (\theta^2)^2 = (\bar{\theta}^1)^2 + (\bar{\theta}^2)^2 &= \begin{bmatrix} \bar{\theta}^1 & \bar{\theta}^2 \end{bmatrix} \begin{bmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{bmatrix} \\ &= \begin{bmatrix} \theta^1 & \theta^2 \end{bmatrix} S^T S \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix}. \end{aligned}$$

This implies that

$$S^T S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, let ω_2^1 and $\bar{\omega}_2^1$ denote the connection forms associated to $\{\theta^1, \theta^2\}$ and $\{\bar{\theta}^1, \bar{\theta}^2\}$, respectively, and define

$$\theta := \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix}, \qquad \bar{\theta} := \begin{bmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{bmatrix}, \qquad \omega := \begin{bmatrix} 0 & -\omega_2^1 \\ \omega_2^1 & 0 \end{bmatrix}, \qquad \bar{\omega} := \begin{bmatrix} 0 & -\bar{\omega}_2^1 \\ \bar{\omega}_2^1 & 0 \end{bmatrix}.$$

If we let the d and $\wedge$ apply to these matrices, with forms as entries, as would be expected, we can easily see that the identities we've been using for d and $\wedge$ still apply. In particular, $d\theta = \omega \wedge \theta$ and

$$\begin{aligned} \bar{\omega} \wedge \bar{\theta} = d\bar{\theta} = d(S\theta) = dS \wedge \theta + S \wedge d\theta = dS \wedge \theta + S\omega \wedge \theta &= dS \wedge S^{-1}\bar{\theta} + S\omega \wedge S^{-1}\bar{\theta} \\ &= (dSS^{-1} + S\omega S^{-1}) \wedge \bar{\theta}. \end{aligned}$$

Thus, we obtain

$$\bar{\omega} = S\omega S^{-1} + dSS^{-1}.$$

Since $S^T S = Id$ we know that, for each point $x \in U$, there is neighborhood of x such that

$$S = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \qquad \text{or} \qquad S = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$$

for some function φ defined on that neighborhood. Whether we are in the first case or in the second case depends on whether $\det(S) = -1$ or $\det(S) = 1$.

Let $S = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and let $\delta = \det(S) \in \{-1, 1\}$. Note that

$$d\left(\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\right) = \begin{bmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix} d\varphi \quad \text{and} \quad d\left(\begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}\right) = \begin{bmatrix} -\sin\varphi & \cos\varphi \\ \cos\varphi & \sin\varphi \end{bmatrix} d\varphi,$$

and so

$$dS = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} d\varphi.$$

Now,

$$S\omega S^{-1} = \omega^1_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \omega^1_2 \begin{bmatrix} 0 & -\delta \\ \delta & 0 \end{bmatrix}.$$

and

$$dSS^{-1} = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} d\varphi \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -ac - bd & -c^2 - d^2 \\ a^2 + b^2 & ac + bd \end{bmatrix} d\varphi.$$

We can check that in both possible expressions for S, we have

$$ac + bd = 0 \quad \text{and} \quad a^2 + b^2 = c^2 + d^2 = 1,$$

and so

$$dSS^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\varphi.$$

Hence, we have

$$\begin{aligned} \begin{bmatrix} 0 & -\bar{\omega}^1_2 \\ \bar{\omega}^1_2 & 0 \end{bmatrix} = \bar{\omega} = S\omega S^{-1} + dSS^{-1} = \delta\omega^1_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\varphi \\ \Rightarrow \bar{\omega}^1_2 = \delta\omega^1_2 + d\varphi. \end{aligned}$$

Finally, let K and $\bar{K}$ denote the Gauss curvatures which respect to $\{\theta^1, \theta^2\}$ and $\{\bar{\theta}^1, \bar{\theta}^2\}$, respectively. We have

$$\bar{\theta}^1 = a\theta^1 + b\theta^2 \quad \text{and} \quad \bar{\theta}^2 = c\theta^1 + d\theta^2$$

which implies

$$\bar{\theta}^1 \wedge \bar{\theta}^2 = (ad - bc)\theta^1 \wedge \theta^2 = \delta\theta^1 \wedge \theta^2$$

and therefore, since $\delta^2 = 1$,

$$
\begin{aligned}
K\theta^1 \wedge \theta^2 = d(\omega^1_2) = \delta d(\delta\omega^1_2) = \delta d(\delta\omega^1_2 + d\varphi) = \delta d(\bar{\omega}^1_2) &= \delta \bar{K}\bar{\theta}^1 \wedge \bar{\theta}^2 \\
&= \delta^2 \bar{K}\theta^1 \wedge \theta^2 \\
&= \bar{K}\theta^1 \wedge \theta^2 \\
&\Rightarrow K = \bar{K},
\end{aligned}
$$

as desired. □

Remark 4.6. The equation

$$\bar{\omega} = S\omega S^{-1} + dSS^{-1}.$$

is an equation between matrices of 1-forms. It is the same as saying that for every $(u, v) \in U$ and every $(v^1, v^2) \in \mathbb{R}^2$ we have

$$\bar{\omega}(u, v)(v^1, v^2) = S(u, v)\omega(u, v)(v^1, v^2)S^{-1}(u, v) + dS(u, v)(v^1, v^2)S^{-1}(u, v).$$

Definition 4.7. (Gaussian curvature) Given a Riemannian surface (U, ds^2), let $\{\theta^1, \theta^2\}$ be any pair of 1-forms such that

$$ds^2 = (\theta^1)^2 + (\theta^2)^2,$$

and let ω^1_2 denote the connection form associated to $\{\theta^1, \theta^2\}$. We define the **Gaussian curvature** of the surface as the function $K : U \to \mathbb{R}$ such that

$$d\omega^1_2 = K\theta^1 \wedge \theta^2.$$

Corollary 4.8. (Gauss's Egregium Theorem) The Gauss curvature of a surface in $\mathbb{R}^3$ depends only on its first fundamental form.

4.1 Tangent vectors on abstract Riemannian surfaces

In this section, when we talk about coordinates that parametrize some Riemannian surface (U, ds^2) we mean the coordinates used to parametrize the open set U. However, it is important to keep in mind that, when we define some set U, we are using, by default, the canonical coordinates of $\mathbb{R}^2$.

Definition 4.9. (Tangent vector) Let $S = (U, \mathbb{I})$ be a Riemannian surface and let (t^1, t^2) be a coordinate system for the surface. We define the **tangent vector** at $(t^1, t^2) \in U$ with components $(v^1, v^2) \in \mathbb{R}^2$ to be the function $\vec{v} : C^1(U, \mathbb{R}) \to \mathbb{R}$ given by the derivative operator

$$\vec{v} = v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2}$$

computed at the point $(t^1, t^2) \in U$.
Furthermore, for a point p in the surface, we define the tangent space of surface S at p as the set T_pS of all tangent vectors at p to S.

Proposition 4.10. Let $(U, \mathbb{I})$ be a Riemannian surface and let (t^1, t^2) be a coordinate system for the surface. Moreover, let $(t^1, t^2) \in U$ be a point,

$$\vec{v} = v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2}$$

be a tangent vector at (t^1, t^2) and $f \in C^1(U, \mathbb{R})$ be a function. Then

$$\vec{v}(f) = df(t^1, t^2)(v^1, v^2), \qquad \forall (v^1, v^2) \in \mathbb{R}^2.$$

Proof. We have

$$df(t^1, t^2) = \frac{\partial f}{\partial t^1}(t^1, t^2) dt^1 + \frac{\partial f}{\partial t^2}(t^1, t^2) dt^2$$

and so

$$df(t^1, t^2)(v^1, v^2) = \frac{\partial f}{\partial t^1}(t^1, t^2) v^1 + \frac{\partial f}{\partial t^2}(t^1, t^2) v^2 = \left(v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2} \right)(f)\Big|_{(t^1, t^2)}$$

as required. ☐

Remark 4.11. When we say a point p of the Riemannian surface $(U, \mathbb{I})$, we mean some point p in the manifold U.

Now fix some $p \in U$ and let us, as usual, write just $\mathbb{I}$ instead of $\mathbb{I}(p)$. Moreover, consider the coordinates $(v^1, v^2) \in \mathbb{R}^2$ of some tangent vector at p. It is very important to note that the value of $\mathbb{I}(v^1, v^2)$ depends on the coordinate system being used.

Let us, for example, consider the Riemannian surface $(H, \mathbb{I})$ where

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \mathbb{R} \times \mathbb{R}^+$$

and

$$\mathbb{I} = \frac{dx^2}{y^2} + \frac{dy^2}{y^2}.$$

Here, $\mathbb{I}$ was written with respect to the coordinate system (x, y), and so we should in fact write $\mathbb{I}^{\{x,y\}}$.

Now let us consider a different coordinate system, say (u, v), such that

$$(u, v) = (x, \log(y)).$$

We have

$$dx^2 = du^2$$

and

$$dy^2 = d(e^v)^2 = e^{2v} dv^2,$$

and so

$$\mathbb{I}^{\{u,v\}} = \frac{du^2}{(e^v)^2} + \frac{e^{2v} dv^2}{(e^v)^2} = \frac{du^2}{e^{2v}} + dv^2.$$

If we fix a point $(x, y) = (0, e) \in H$, or equivalently the point $(u, v) = (0, 1)$, we have

$$\mathbb{I}^{\{x,y\}} = \frac{dx^2}{e^2} + \frac{dy^2}{e^2}$$

and

$$\mathbb{I}^{\{u,v\}} = \frac{du^2}{e^2} + dv^2$$

and clearly, if we take $(1, 1) \in \mathbb{R}^2$,

$$\mathbb{I}^{\{x,y\}}(1, 1) = \frac{1}{e^2} + \frac{1}{e^2} \neq \frac{1}{e^2} + 1 = \mathbb{I}^{\{u,v\}}(1, 1).$$

However, what we will see next is that we can define what it means to apply $\mathbb{I}$ to a tangent vector, and that definition will not depend on the coordinate system.

Proposition 4.12. Let $(U, \mathbb{I})$ be a Riemannian surface and let (t^1, t^2) and (s^1, s^2) be two coordinate systems. Furthermore, let $\mathbb{I}^{\{t\}}$ and $\mathbb{I}^{\{s\}}$ be the first fundamental forms written with respect to (t^1, t^2) and (s^1, s^2), respectively. If we have a tangent vector $\vec{u}$ written as

$$v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2} = \vec{u} = w^1 \frac{\partial}{\partial s^1} + w^2 \frac{\partial}{\partial s^2}$$

then

$$\mathbb{I}^{\{t\}}(v^1, v^2) = \mathbb{I}^{\{s\}}(w^1, w^2).$$

Proof. Let us write $\mathbb{I}^{\{t\}}$ as

$$\mathbb{I}^{\{t\}} = \sum_{i,j=1}^{2} g_{ij} dt^i dt^j.$$

By thinking of t^i as a function of s^1, s^2 we have

$$dt^i = \sum_{k=1}^{2} \frac{\partial t^i}{\partial s^k} ds^k.$$

Therefore, we can write $\mathbb{I}^{(s)}$ as

$$\mathbb{I}^{\{s\}} = \sum_{i,j,k,l=1}^{2} g_{ij} \frac{\partial t^i}{\partial s^k} \frac{\partial t^j}{\partial s^l} ds^k ds^l.$$

Moreover, by the chain rule,

$$\sum_{i=1}^{2} v^i \frac{\partial}{\partial t^i} = \sum_{i,j=1}^{2} v^i \frac{\partial s^j}{\partial t^i} \frac{\partial}{\partial s^j} = \sum_{j=1}^{2} \left[\left(\sum_{i=1}^{2} v^i \frac{\partial s^j}{\partial t^i} \right) \frac{\partial}{\partial s^j} \right]$$

and so

$$w^j = \sum_{i=1}^{2} v^i \frac{\partial s^j}{\partial t^i}.$$

Thus, we have

$$\mathbb{I}^{\{s\}}(w^1, w^2) = \sum_{i,j,k,l=1}^{2} g_{ij} \frac{\partial t^i}{\partial s^k} \frac{\partial t^j}{\partial s^l} w^k w^l = \sum_{i,j,k,l,m,n=1}^{2} g_{ij} \frac{\partial t^i}{\partial s^k} \frac{\partial t^j}{\partial s^l} \frac{\partial s^k}{\partial t^m} \frac{\partial s^l}{\partial t^n} v^m v^n.$$

Since

$$\sum_{k=1}^{2} \frac{\partial t^i}{\partial s^k} \frac{\partial s^k}{\partial t^m} = \frac{\partial t^i}{\partial t^m} = \delta_{im},$$

we have

$$\mathbb{I}^{\{s\}}(w^1, w^2) = \sum_{i,j,m,n=1}^{2} g_{ij} \delta_{im} \delta_{jn} v^m v^n = \sum_{i,j=1}^{2} g_{ij} v^i v^j = \mathbb{I}^{\{t\}}(v^1, v^2)$$

as required. □

Definition 4.13. Let $(U, \mathbb{I})$ be a Riemannian surface and let (t^1, t^2) be a coordinate system. Let $\vec{v}$ is a vector tangent to some point of the surface, written as

$$\vec{v} = v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2}$$

then we define

$$\mathbb{I}(\vec{v}) = \mathbb{I}^{\{t\}}(v^1, v^2).$$

By Proposition 7.11., this is well defined.

Remark 4.14. Given some tangent vector

$$\vec{v} = v^1 \frac{\partial}{\partial t^1} + v^2 \frac{\partial}{\partial t^2}$$

it also makes sense to use

$$du(\vec{v}) := du(v^1, v^2)$$

and other similar notations.

4.2 Inner product and dual frames

Proposition 4.15. Let $S = (U, \mathbb{I})$ be a Riemannian surface and let p be a point in the surface. If we let

$$\langle v, w\rangle = \frac{1}{2}\Big[\mathbb{I}(v+w) - \mathbb{I}(v) - \mathbb{I}(w)\Big].$$

for all $v, w \in T_pS$, then $\langle\cdot,\cdot\rangle$ is an inner product on T_pS.

Proof. Let us write

$$\mathbb{I} = Edu^2 + 2Fdudv + Gdv^2$$

and, by abuse of notation, let us use v and w both for the vectors themselves and for their representation in the basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$.

Then

$$\begin{aligned}
\langle v, w\rangle = \frac{1}{2}\Big[\mathbb{I}(v+w) - \mathbb{I}(v) - \mathbb{I}(w)\Big] &= \frac{1}{2}\left[(v^T + w^T)\begin{bmatrix} E & F \\ F & G \end{bmatrix}(v+w) - v^T\begin{bmatrix} E & F \\ F & G \end{bmatrix}v - w^T\begin{bmatrix} E & F \\ F & G \end{bmatrix}w\right] \\
&= \frac{1}{2}\left[v^T\begin{bmatrix} E & F \\ F & G \end{bmatrix}w + w^T\begin{bmatrix} E & F \\ F & G \end{bmatrix}v\right] \\
&= v^T\begin{bmatrix} E & F \\ F & G \end{bmatrix}w,
\end{aligned}$$

and therefore, since the matrix $\left[\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right]$ is a symmetric, positive-definite matrix, $\langle\cdot,\cdot\rangle$ is indeed an inner product. □

As usual, this lets us define the length of a vector tangent to the surface. We have

$$\|v\| := \sqrt{\langle v, v\rangle} = \sqrt{\mathbb{I}(v)}.$$

Let $\{\theta^1, \theta^2\}$ be a pair of forms such that

$$\mathbb{I} = (\theta^1)^2 + (\theta^2)^2.$$

where $\mathbb{I}, \theta^1, \theta^2$ are written in some frame $\{du, dv\}$ with respect to the coordinate system (u, v).

Let $\{\vec{e_1}, \vec{e_2}\}$ be a basis for the tangent space of the surface at the point p and write

$$\begin{cases} \vec{e_1} = b_1^1 \frac{\partial}{\partial u} + b_1^2 \frac{\partial}{\partial v} \\ \vec{e_2} = b_2^1 \frac{\partial}{\partial u} + b_2^2 \frac{\partial}{\partial v} \end{cases}.$$

We say that $\{\vec{e_1}, \vec{e_2}\}$ is the **dual frame** of $\{\theta^1, \theta^2\}$ if

$$\theta^i(p)(b_j^1, b_j^2) = \delta_{ij}.$$

If we write

$$\begin{cases} \theta^1 = a_1^1 du + a_2^1 dv \\ \theta^2 = a_1^2 du + a_2^2 dv \end{cases}$$

then the dual frame is easily found by solving the system

$$\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $b_1^1, b_1^2, b_2^1, b_2^2$. We have seen already that $\theta^1(p)$ and $\theta^2(p)$ are independent, and so the above system always has a solution.

It is also clear, from the definition of dual frame, that

$$\mathbb{I}(\vec{e_1}) = 1, \qquad \mathbb{I}(\vec{e_2}) = 1, \qquad \mathbb{I}(\vec{e_1} + \vec{e_2}) = 2$$

and so

$$\langle \vec{e_1}, \vec{e_2} \rangle = \frac{1}{2}\Big(\mathbb{I}(\vec{e_1} + \vec{e_2}) - \mathbb{I}(\vec{e_1}) - \mathbb{I}(\vec{e_2})\Big) = 0.$$

Hence, the dual frame is in fact an orthonormal basis for the tangent space. Therefore, if

$$\vec{v} = v^1 \vec{e_1} + v^2 \vec{e_2} \qquad \text{and} \qquad \vec{w} = w^1 \vec{e_1} + w^2 \vec{e_2}$$

are two tangent vectors then

$$\langle \vec{v}, \vec{w} \rangle = v^1 w^1 + v^2 w^2.$$

4.3 Vector fields and covariant derivatives

We will first give the more intuitive definition of vector fields along curves and their covariant derivatives for surfaces embedded in $\mathbb{R}^2$, and then extend these notions to general Riemannian surfaces.

Definition 4.16. (Vector field) Let $S \in \mathbb{R}^3$ be a surface and $c : I \subset \mathbb{R} \to S$ be a curve. A **vector field along** c is a function $\vec{V} : I \to \mathbb{R}^3$ such that $\vec{V}(t) \in T_{c(t)}S$ for all $t \in I$.

Definition 4.17. (Covariant derivative) Let $S \in \mathbb{R}^3$ be a surface, let $c : I \subset \mathbb{R} \to S$ be a curve and let $\vec{V}$ be a vector field along c. The **covariant derivative** of $\vec{V}$ is the new vector field along c given by

$$\frac{D\vec{V}}{dt}(t) := \frac{d\vec{V}}{dt}(t) - \left(\frac{d\vec{V}}{dt}(t) \cdot \vec{n}(t)\right)\vec{n}(t),$$

where $\vec{n}(t)$ is the unit normal to S at $c(t)$.

The covariant derivative of the vector field $\vec{V}$ is then simply the projection of $\frac{d\vec{V}}{dt}(t)$ onto $T_{c(t)}S$, for each $t \in I$.

Because we defined the covariant derivative using the unit normal $\vec{n}$, this definition does not make sense for general Riemannian surfaces. However, we will see that we can rewrite this definition without using $\vec{n}$.

Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of S and let $\vec{e_1}, \vec{e_2} : U \to \mathbb{R}^3$ be an orthonormal basis for $T_{g(u,v)}S$ for each point $(u, v) \in U$. Furthermore, let us use $\vec{e_3} := \vec{n}$.

Let $u, v : I \to \mathbb{R}$ be the functions such that

$$c(t) = g(u(t), v(t)),$$

and let $V^1, V^2 : I \to \mathbb{R}$ be the functions such that

$$\vec{V}(t) = V^1(t)\vec{e_1}(u(t), v(t)) + V^2(t)\vec{e_2}(u(t), v(t)).$$

In the following computations we will be dropping the argument t in the functions u and v, to make everything easier to read.

If we let $\vec{e_1} = (f_1, f_2, f_3)$ we have

$$\left[\frac{d}{dt}(\vec{e_1}(u,v))\right]_i = \frac{\partial f_i}{\partial u}(u,v)u' + \frac{\partial f_i}{\partial v}(u,v)v'.$$

On the other hand,

$$[d(\vec{e_1})]_i = d(f_i) = \frac{\partial f_i}{\partial u}du + \frac{\partial f_i}{\partial v}dv$$

and so

$$[d(\vec{e_1})]_i(u,v)(u',v') = \frac{\partial f_i}{\partial u}(u,v)u' + \frac{\partial f_i}{\partial v}(u,v)v' = \left[\frac{d}{dt}(\vec{e_1}(u,v))\right]_i.$$

Hence we have

$$\frac{d}{dt}(\vec{e_1}(u,v)) = d(\vec{e_1})(u,v)(u',v') = \omega_1^2(u,v)(u',v')\vec{e_2}(u,v) + \omega_1^3(u,v)(u',v')\vec{e_3}(u,v)$$

Similarly,

$$\frac{d}{dt}(\vec{e_2}(u,v)) = d(\vec{e_2})(u,v)(u',v') = \omega_2^1(u,v)(u',v')\vec{e_1}(u,v) + \omega_2^3(u,v)(u',v')\vec{e_3}(u,v).$$

Let us now drop the arguments (u, v) as well; we already know that we are working on the plane tangent to the surface at $g(u(t), v(t))$.

We have

$$\begin{aligned}\frac{d\vec{V}}{dt}(t) &= \frac{dV^1}{dt}\vec{e_1} + V^1\frac{d}{dt}(\vec{e_1}) + \frac{dV^2}{dt}\vec{e_2} + V^2\frac{d}{dt}(\vec{e_2}) \\ &= \frac{dV^1}{dt}\vec{e_1} + V^1\Big(\omega_1^2(u',v')\vec{e_2} + \omega_1^3(u',v')\vec{e_3}\Big) + \frac{dV^2}{dt}\vec{e_2} + V^2\Big(\omega_2^1(u',v')\vec{e_1} + \omega_2^3(u',v')\vec{e_3}\Big),\end{aligned}$$

and so the covariant derivative is given by

$$\frac{d\vec{V}}{dt}(t) = \left(\frac{dV^1}{dt} + V^2\omega_2^1(u',v')\right)\vec{e_1} + \left(\frac{dV^2}{dt} + V^1\omega_1^2(u',v')\right)\vec{e_2}.$$

We found a way of expressing the covariant derivative using only objects we have already defined for general Riemannian surfaces. All we need to do now is to show that this definition indeed works.

Definition 4.18. (Vector field) Let (U, ds^2) be a Riemannian surface and $c : I \subset \mathbb{R} \to U$ be a curve. A **vector field along** c is a function $\vec{V}$ with domain I and such that $\vec{V}(t)$ is a tangent vector for all $t \in I$.

Definition 4.19. (Covariant derivative) Let (U, ds^2) be a Riemannian surface, θ^1 and θ^2 be two 1-forms such that $ds^2 = (\theta^1)^2 + (\theta^2)^2$ and $\{\vec{e_1}, \vec{e_2}\}$ be the dual frame to $\{\theta^1, \theta^2\}$. Moreover, let $(u, v) : I \subset \mathbb{R} \to U$ be a curve. Given a vector field $\vec{V}$ along c such that, for two given smooth functions $V^1, V^2 : I \to \mathbb{R}$,

$$\vec{V}(t) = V^1(t)\vec{e_1}(u(t), v(t)) + V^2(t)\vec{e_2}(u(t), v(t)),$$

we define the **covariant derivative** of $\vec{V}$ as the new vector field along c, written as $\frac{D\vec{V}}{dt}$, given by

$$\frac{D\vec{V}}{dt} = \left(\frac{dV^1}{dt} + V^2\omega_2^1(u', v')\right)\vec{e_1} + \left(\frac{dV^2}{dt} - V^1\omega_2^1(u', v')\right)\vec{e_2}.$$

Here, ω_2^1 is the connection form associated to $\{\theta^1, \theta^2\}$.

Proposition 4.20. The covariant derivative is well defined, i.e., it does not depend on the choice of $\{\theta^1, \theta^2\}$.

Proof. Let $\{\theta^1, \theta^2\}$ and $\{\bar{\theta}^1, \bar{\theta}^2\}$ be two pairs of 1-forms such that

$$(\theta^1)^2 + (\theta^2)^2 = \mathbb{I} = (\bar{\theta}^1)^2 + (\bar{\theta}^2)^2.$$

Let $\{\vec{e_1}, \vec{e_2}\}$ and $\{\vec{\bar{e}_1}, \vec{\bar{e}_2}\}$ be dual frames to $\{\theta^1, \theta^2\}$ and $\{\bar{\theta}^1, \bar{\theta}^2\}$, respectively, and define

$$\theta := \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix}, \qquad \bar{\theta} := \begin{bmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{bmatrix}, \qquad \vec{e} := \begin{bmatrix} \vec{e_1} & \vec{e_2} \end{bmatrix}, \qquad \vec{\bar{e}} := \begin{bmatrix} \vec{\bar{e}_1} & \vec{\bar{e}_2} \end{bmatrix}.$$

We know that there exists an invertible matrix S with $S^T S = I$ such that

$$\bar{\theta} = S\theta.$$

Since we must have

$$\theta\vec{e} = I \Leftrightarrow S^{-1}\bar{\theta}\vec{e} = I \Leftrightarrow \bar{\theta}\vec{e} = S \Leftrightarrow \bar{\theta}\vec{e}S^{-1} = I,$$

we see from the definition of dual frames that

$$\vec{\bar{e}} = \vec{e}S^{-1}.$$

Now, let

$$V := \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \qquad \text{and} \qquad \bar{V} := \begin{bmatrix} \bar{V}^1 \\ \bar{V}^2 \end{bmatrix}.$$

We have

$$\vec{\bar{e}}\bar{V} = \vec{V} = \vec{e}V = \vec{\bar{e}}SV \Rightarrow \bar{V} = SV.$$

Moreover, we have also seen before that if we set

$$\omega := \begin{bmatrix} 0 & -\omega_2^1 \\ \omega_2^1 & 0 \end{bmatrix}, \qquad \bar{\omega} := \begin{bmatrix} 0 & -\bar{\omega}_2^1 \\ \bar{\omega}_2^1 & 0 \end{bmatrix}$$

then

$$\bar{\omega} = S\omega S^{-1} + dSS^{-1}.$$

Note that again

$$dS(u', v') = \frac{d}{dt}\Big[S(u(t), v(t))\Big].$$

Furthermore, writting the expression for the covariant derivative in matrix form gives us

$$\frac{D\vec{V}}{dt} = \vec{e}\left(\frac{dV}{dt} - \omega(u', v')V\right).$$

Finally, we have

$$\begin{aligned} \vec{\bar{e}}\left(\frac{d\bar{V}}{dt} - \bar{\omega}(u', v')\bar{V}\right) &= \vec{e}S^{-1}\left(\frac{d}{dt}(SV) - \Big(S\omega(u', v')S^{-1} + dS(u', v')S^{-1}\Big)SV\right) \\ &= \vec{e}S^{-1}\left(\frac{dS}{dt}V + S\frac{dV}{dt} - S\omega(u', v')V - \frac{dS}{dt}S^{-1}SV\right) \\ &= \vec{e}\left(\frac{dV}{dt} - \omega(u', v')V\right) \\ &= \frac{D\vec{V}}{dt}, \end{aligned}$$

as desired. □

Proposition 4.21. If $\vec{V}, \vec{W}$ are vector fields along a curve then

$$\frac{d}{dt}\left\langle \vec{V}(t), \vec{W}(t) \right\rangle = \left\langle \frac{D\vec{V}}{dt}(t), \vec{W}(t) \right\rangle + \left\langle \vec{V}(t), \frac{D\vec{W}}{dt}(t) \right\rangle.$$

Proof. Let

$$\vec{V} = V^1\vec{e_1} + V^2\vec{e_2} \qquad \text{and} \qquad \vec{W} = W^1\vec{e_1} + W^2\vec{e_2}$$

for given smooth functions $V^1, V^2, W^1, W^2 : I \to \mathbb{R}^2$. We have

$$\begin{aligned}
\left\langle \frac{D\vec{V}}{dt}(t), \vec{W}(t) \right\rangle + \left\langle \vec{V}(t), \frac{D\vec{W}}{dt}(t) \right\rangle &= \left(\frac{dV^1}{dt} + V^2\omega_2^1(u', v') \right) W^1 + \left(\frac{dV^2}{dt} - V^1\omega_2^1(u', v') \right) W^2 \\
&\quad + \left(\frac{dW^1}{dt} + W^2\omega_2^1(u', v') \right) V^1 + \left(\frac{dW^2}{dt} - W^1\omega_2^1(u', v') \right) V^2 \\
&= \frac{dV^1}{dt}W^1 + \frac{dV^2}{dt}W^2 + \frac{dW^1}{dt}V^1 + \frac{dW^2}{dt}V^2 \\
&= \frac{d}{dt}(V^1W^1 + V^2W^2) \\
&= \frac{d}{dt}\left\langle \vec{V}(t), \vec{W}(t) \right\rangle,
\end{aligned}$$

as desired. □

Definition 4.22. (Parallel vector field) We say that a vector field $\vec{V}$ is **parallel** along a curve if $\frac{D\vec{V}}{dt} = 0$.

Corollary 4.23. If $\vec{V}, \vec{W}$ are two vector fields parallel along a curve then $\langle \vec{V}, \vec{W} \rangle$ is constant along a curve.
In particular, $\mathbb{I}(\vec{V})$, $\mathbb{I}(\vec{W})$ and $\measuredangle(\vec{V}, \vec{W})$ are constant.

4.4 Geodesics

Let $S \subset \mathbb{R}^3$ be a surface and $c : I \to S$ be a curve parametrized by arclength. We have previously written

$$c''(s) = \overrightarrow{k_g}(s) + \overrightarrow{k_n}(s)$$

where $\overrightarrow{k_g}$, the geodesic curvature vector, is tangent to S, and $\overrightarrow{k_n}$, the normal curvature vector, is orthogonal to S.

Note that

$$\overrightarrow{k_g}(s) = \vec{c}'' - \overrightarrow{k_n}(s) = \frac{dc'}{ds}(s) - \left(\frac{dc'}{ds}(s) \cdot \vec{n}\right) \vec{n} = \frac{Dc'}{ds}(s).$$

Therefore we make the following definition.

Definition 4.24. (Geodesic) A curve on a Riemannian surface is called a **geodesic** if its velocity is parallel.

Remark 4.25. The above definition might seem strange, since if $c : I \to U$ is a curve written as $c(t) = (u(t), v(t))$ then its velocity $c'(t) = (u'(t), v'(t))$ is a function from I to $\mathbb{R}^2$, which is not really a vector field. In fact, when we say "if its velocity is parallel" we really mean "if the vector field

$$u'(t)\frac{\partial}{\partial x} + v'(t)\frac{\partial}{\partial y}$$

is parallel along the curve". We will be denoting this vector field by $\mathbf{c}'$.

If $c : I \subset \mathbb{R} \to U$ is a geodesic then

$$\frac{d}{dt}(\|\mathbf{c}'(t)\|^2) = \frac{d}{dt}\langle \mathbf{c}'(t), \mathbf{c}'(t)\rangle = 2\left\langle \frac{D\mathbf{c}'}{dt}(t), \mathbf{c}'(t)\right\rangle = 0,$$

and so $\mathbf{c}'$ has constant length. This implies that the arclength parameter is given by

$$s(t) = \int_{t_0}^{t} \|\mathbf{c}'(u)\| du = (t - t_0)\|\mathbf{c}'\|,$$

and so it is an affine function of t (and reciprocally).

Proposition 4.26. Let $S \subset \mathbb{R}^3$ be a surface. If $c : [0, l] \to S$ is a curve with minimal length connecting the points $c(0)$ and $c(l)$, then c is a geodesic (up to reparametrization).

Proof. Suppose that c is parametrized by arclength. For some $\varepsilon > 0$ let $H : [0, l] \times [-\varepsilon, \varepsilon] \to S$

be any smooth function such that

$$H(s,0) = c(s), \qquad H(0,\lambda) = c(0), \qquad H(l,\lambda) = c(l).$$

So, for each λ, $H(s,\lambda)$ is a curve from $c(0)$ to $c(l)$.

Define $L(\lambda)$ as the length of $H(s,\lambda)$, i.e.,

$$L(\lambda) := \int_0^l \left\| \frac{\partial H}{\partial s}(s,\lambda) \right\| ds.$$

We have

$$\begin{aligned} L'(\lambda) = \int_0^l \frac{\partial}{\partial \lambda} \left[\left(\frac{\partial H}{\partial s} \cdot \frac{\partial H}{\partial s} \right)^{1/2} \right] ds &= \int_0^l \frac{1}{2} \cdot 2 \cdot \frac{\partial H}{\partial s} \cdot \frac{\partial^2 H}{\partial \lambda \partial s} \left(\frac{\partial H}{\partial s} \cdot \frac{\partial H}{\partial s} \right)^{-1/2} ds \\ &= \int_0^l \frac{\partial H}{\partial t} \cdot \frac{\partial^2 H}{\partial \lambda \partial s} \left\| \frac{\partial H}{\partial s} \right\|^{-1} ds. \end{aligned}$$

Because c is the curve with minimal length we have $L'(0) = 0$. Furthermore, we know that $\left\| \frac{\partial H}{\partial s}(s,0) \right\| = 1$ and so

$$\begin{aligned} 0 = L'(0) = \int_0^l \frac{\partial H}{\partial s} \cdot \frac{\partial^2 H}{\partial \lambda \partial s} ds &= \int_0^l \left[\frac{\partial}{\partial s} \left(\frac{\partial H}{\partial s}(s,0) \cdot \frac{\partial H}{\partial \lambda}(s,0) \right) - \frac{\partial^2 H}{\partial s^2}(s,0) \cdot \frac{\partial H}{\partial \lambda}(s,0) \right] ds \\ &= \left[\frac{\partial H}{\partial s}(s,0) \cdot \frac{\partial H}{\partial \lambda}(s,0) \right]_{s=0}^{s=l} - \int_0^l c''(s) \cdot \frac{\partial H}{\partial \lambda}(s,0) ds. \end{aligned}$$

Since $H(0,\lambda) = c(0)$ and $H(l,\lambda) = c(l)$, we have

$$\frac{\partial H}{\partial \lambda}(0,0) = \frac{\partial H}{\partial \lambda}(l,0) = 0 \Rightarrow \left[\frac{\partial H}{\partial s}(s,0) \cdot \frac{\partial H}{\partial \lambda}(s,0) \right]_{s=0}^{s=l} = 0.$$

Moreover, for any given s the function $H(s,\lambda)$ is also a curve in S, and so $\frac{\partial H}{\partial \lambda}(s,\lambda)$ is tangent to S at $H(s,\lambda)$. In particular,

$$\frac{\partial H}{\partial \lambda}(s,0) \in T_{c(s)}S,$$

and so, if $\overrightarrow{k_n}(s)$ is the normal curvature vector of $c(s)$,

$$c''(s) \cdot \frac{\partial H}{\partial \lambda}(s,0) = (c''(s) - \overrightarrow{k_n}(s)) \cdot \frac{\partial H}{\partial \lambda}(s,0) = \frac{Dc'}{ds}(s) \cdot \frac{\partial H}{\partial \lambda}(s,0).$$

Hence,

$$\int_0^l \frac{Dc'}{ds}(s) \cdot \frac{\partial H}{\partial \lambda}(s,0) ds = 0.$$

Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of S and let $u, v : [0, l] \to \mathbb{R}$ be the smooth functions such that

$$c(s) = g(u(s), v(s)).$$

Moreover, let $a, b : [0, l] \to \mathbb{R}$ be the smooth functions such that

$$\frac{Dc'}{dt}(s) = a(s)\frac{\partial g}{\partial u}(u(s), v(s)) + b(s)\frac{\partial g}{\partial v}(u(s), v(s)).$$

Let $\varphi : [0, l] \to \mathbb{R}$ be any smooth function with $\varphi(0) = \varphi(l) = 0$ and $\varphi(s) > 0$ for all $s \in]0, l[$. We choose H to be the smooth function given by

$$H(s, \lambda) = g\big(u(s) + \lambda a(s)\varphi(s), v(s) + \lambda b(t)\varphi(s)\big).$$

It is easy to see that this function satisfies the requirements we set for the function H. We have

$$\frac{\partial H}{\partial \lambda}(s, 0) = \frac{\partial g}{\partial u}(u(s), v(s))a(s)\varphi(s) + \frac{\partial g}{\partial v}(u(s), v(s))b(s)\varphi(s) = \varphi(s)\frac{Dc'}{ds}(s),$$

and therefore

$$0 = \int_0^l \frac{Dc'}{ds}(s) \cdot \frac{\partial H}{\partial \lambda}(s, 0)ds = \int_0^l \varphi(s) \left\| \frac{Dc'}{ds}(s) \right\|^2 ds,$$

which implies, since $\varphi(s) > 0$ on $(0, l)$, that

$$\frac{Dc'}{ds}(s) = 0 \text{ for all } s \in [0, l],$$

as desired. □

Let us now introduce **Gauss' equations**. The vectors $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \vec{n}$, at each point, form a basis for $\mathbb{R}^3$, and so we can make the following definition, where we again make use of the formulas (3).

Definition 4.27. (Gauss' equations) Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of a surface $S \subset \mathbb{R}^3$. We define the **Christoffel symbols** $\Gamma^\gamma_{\alpha\beta} : U \to \mathbb{R}$, for $\alpha, \beta, \gamma \in \{u, v\}$, as the 8 smooth functions such that

$$\begin{cases} \frac{\partial^2 g}{\partial u^2} &= \Gamma^u_{uu}\frac{\partial g}{\partial u} + \Gamma^v_{uu}\frac{\partial g}{\partial v} + L\vec{n} \\ \frac{\partial^2 g}{\partial u \partial v} &= \Gamma^u_{uv}\frac{\partial g}{\partial u} + \Gamma^v_{uv}\frac{\partial g}{\partial v} + M\vec{n} \\ \frac{\partial^2 g}{\partial v \partial u} &= \Gamma^u_{vu}\frac{\partial g}{\partial u} + \Gamma^v_{vu}\frac{\partial g}{\partial v} + M\vec{n} \\ \frac{\partial^2 g}{\partial v^2} &= \Gamma^u_{vv}\frac{\partial g}{\partial u} + \Gamma^v_{vv}\frac{\partial g}{\partial v} + N\vec{n} \end{cases}.$$

Remark 4.28. It is clear that $\Gamma^u_{uv} = \Gamma^u_{vu}$ and $\Gamma^v_{uv} = \Gamma^v_{vu}$.

Let $c(t) = g(u(t), v(t))$ be a curve. We have

$$c' = u'\frac{\partial g}{\partial u} + v'\frac{\partial g}{\partial v},$$

and so

$$c'' = u''\frac{\partial g}{\partial u} + (u')^2\frac{\partial^2 g}{\partial u^2} + u'v'\frac{\partial^2 g}{\partial u \partial v} + v''\frac{\partial g}{\partial v} + v'u'\frac{\partial^2 g}{\partial u \partial v} + (v')^2\frac{\partial^2 g}{\partial v^2}.$$

If we use Gauss' equations in the expression above and remove the term multiplying by $\vec{n}$, we obtain an expression for the covariant derivative of c':

$$\begin{aligned}\frac{Dc'}{dt} &= \left(u'' + \Gamma^u_{uu}(u')^2 + 2\Gamma^u_{uv}u'v' + \Gamma^u_{vv}(v')^2\right)\frac{\partial g}{\partial u} \\ &\quad + \left(v'' + \Gamma^v_{uu}(u')^2 + 2\Gamma^v_{uv}u'v' + \Gamma^v_{vv}(v')^2\right)\frac{\partial g}{\partial v}.\end{aligned}$$

This gives us the equations for finding the geodesics of the surface:

$$\begin{cases} u'' + \Gamma^u_{uu}(u')^2 + 2\Gamma^u_{uv}u'v' + \Gamma^u_{vv}(v')^2 = 0 \\ v'' + \Gamma^v_{uu}(u')^2 + 2\Gamma^v_{uv}u'v' + \Gamma^v_{vv}(v')^2 = 0 \end{cases}.$$

We now want to find a way to compute the Chirstoffel symbols. We have

$$\begin{aligned}\Gamma^u_{uu}E + \Gamma^v_{uu}F &= \Gamma^u_{uu}\frac{\partial g}{\partial u}\cdot\frac{\partial g}{\partial u} + \Gamma^v_{uu}\frac{\partial g}{\partial v}\cdot\frac{\partial g}{\partial u} \\ &= \left(\Gamma^u_{uu}\frac{\partial g}{\partial u} + \Gamma^v_{uu}\frac{\partial g}{\partial v}\right)\cdot\frac{\partial g}{\partial u} \\ &= \frac{\partial^2 g}{\partial u^2}\cdot\frac{\partial g}{\partial u} \\ &= \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\cdot\frac{\partial g}{\partial u}\right) \\ &= \frac{1}{2}\frac{\partial E}{\partial u}\end{aligned}$$

and

$$\begin{aligned}
\Gamma^u_{uu}F + \Gamma^v_{uu}G &= \Gamma^u_{uu}\frac{\partial g}{\partial u}\cdot\frac{\partial g}{\partial v} + \Gamma^v_{uu}\frac{\partial g}{\partial v}\cdot\frac{\partial g}{\partial v} \\
&= \left(\Gamma^u_{uu}\frac{\partial g}{\partial u} + \Gamma^v_{uu}\frac{\partial g}{\partial v}\right)\cdot\frac{\partial g}{\partial v} \\
&= \frac{\partial^2 g}{\partial u^2}\cdot\frac{\partial g}{\partial v} \\
&= \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\cdot\frac{\partial g}{\partial v}\right) - \frac{\partial g}{\partial u}\cdot\frac{\partial^2 g}{\partial u\partial v} \\
&= \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\cdot\frac{\partial g}{\partial u}\right) \\
&= \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}.
\end{aligned}$$

This gives us

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial u} \\ \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} \end{bmatrix}.$$

By similar computations, we get

$$\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial u} & \frac{1}{2}\frac{\partial E}{\partial v} & \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} & \frac{1}{2}\frac{\partial G}{\partial u} & \frac{1}{2}\frac{\partial G}{\partial v} \end{bmatrix}.$$

Remark 4.29. The Christoffel symbols only depend on the first fundamental form, which implies that we can define them for general Riemannian surfaces as well. In fact, it can be shown that the above results for the geodesics of a surface are also true for Riemannian surfaces.

5 Gauss-Bonnet Theorem

To define integrals on manifolds we have to assume some ordering of the variables of positive parameterizations. This ordering corresponds to **positive coordinate systems** for our manifold. For a general a Riemannian surface $(U, \mathbb{I})$ we assume that the usual coordinate system (u, v) is positive. In this case, we call the Riemannian surface **oriented** if we only accept coordinate systems whose Jacobian with respect to (u, v) is positive. For example, the coordinate system (v, u) would not be accepted in an oriented Riemannian surface.

Definition 5.1. Let $(U, \mathbb{I})$ be a oriented Riemannian surface. We say that $\{\vec{e_1}, \vec{e_2}\}$ is a **positive** orthonormal frame if

$$du \wedge dv(\vec{e_1}, \vec{e_2}) > 0$$

for a positive coordinate system (u, v).

Proposition 5.2. Let $(U, \mathbb{I})$ be a Riemannian surface and let θ^1, θ^2 be such that

$$\mathbb{I} = (\theta^1)^2 + (\theta^2)^2.$$

Furthermore, write

$$\begin{cases} \theta^1 = a_1^1 du + a_2^1 dv \\ \theta^2 = a_1^2 du + a_2^2 dv \end{cases}$$

for a positive coordinate system (u, v). The following are equivalent:

(1) The dual frame to $\{\theta^1, \theta^2\}$ is positive;

(2) $\det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} > 0$;

(3) For any 2-manifold $A \subset U$ with compact closure we have

$$\int_A \theta^1 \wedge \theta^2 > 0.$$

Proof. (1) ⇔ (2). Write

$$\begin{cases} \vec{e_1} = b_1^1 \frac{\partial}{\partial u} + b_1^2 \frac{\partial}{\partial v} \\ \vec{e_2} = b_2^1 \frac{\partial}{\partial u} + b_2^2 \frac{\partial}{\partial v} \end{cases}.$$

We have

$$du \wedge dv(\vec{e_1}, \vec{e_2}) = \det \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix},$$

and since

$$\det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} = \det \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix}^{-1},$$

we have

$$du \wedge dv(\vec{e_1}, \vec{e_2}) > 0 \Leftrightarrow \det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} > 0.$$

(2) $\Leftrightarrow$ (3). We have

$$\int_A \theta^1 \wedge \theta^2 = \int_A \int (a_1^1 a_2^2 - a_1^2 a_2^1) du \wedge dv = \int_A \det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} du \wedge dv.$$

Since 2-manifolds of U are just open sets of U, it is clear that

$$\int_A \theta^1 \wedge \theta^2 > 0 \text{ for all } A \subset U \Leftrightarrow \det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} > 0.$$

□

Hence, it also makes sense to define a positive pair $\{\theta^1, \theta^2\}$. It is clear that if $\{\theta^1, \theta^2\}$ is positive then $\{\theta^2, \theta^1\}$ is negative.

The Gauss Bonnet theorem states that, for domains $A \subset U$,

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} k_g(s) ds = 2\pi.$$

To understand this theorem, we first need to say what a domain is and what the function k_g is. We first introduce the function k_g; in order to to that, we need the notion of a unit normal to a curve on an oriented Riemannian surface.

Definition 5.3. Let $(U, \mathbb{I})$ be an oriented Riemannian surface with a positive coordinate system (u, v) and let $\{\vec{e_1}, \vec{e_2}\}$ be a positive orthonormal frame. For a curve $c : I \to U$, parameterized by arclength and written as $c(s) = (u(s), v(s))$, we define its **unit normal** $\vec{n}$, with respect to $\{\vec{e_1}, \vec{e_2}\}$, as

$$\vec{n}(s) := -v'(s)\vec{e_1} + u'(s)\vec{e_2}.$$

In other words, the unit normal $\vec{n}$ is the unique unit vector such that $\{c'(s), \vec{n}(s)\}$ is a positive orthonormal frame.

Definition 5.4. (Geodesic curvature) Let $(U, \mathbb{I})$ be an oriented Riemannian surface with a positive coordinate system (u, v) and let $\{\vec{e_1}, \vec{e_2}\}$ be a positive orthonormal frame. For a curve $c : I \to U$, let $\vec{n}$ be its unit normal with respect to $\{\vec{e_1}, \vec{e_2}\}$. We define the (scalar) **geodesic curvature** $k_g : I \to \mathbb{R}$ as the function given by

$$k_g(s) = \left\langle \frac{Dc'}{ds}(s), \vec{n}(s) \right\rangle.$$

Definition 5.5. (Domains) A **domain** on $\mathbb{R}^2$ is a compact 2-dimensional manifold with boundary, that is, a compact set $A \subset R^2$ whose boundary ∂A is a 1-dimensional manifold. Informally, a domain with corners is a generalization where we allow ∂A to have a finite number of vertices (that is, it can be parameterized by a sectionally smooth regular curve).

Theorem 5.6. *(Gauss-Bonnet for domains) Let A is a simply connected domain on an oriented Riemannian surface with metric $ds^2 = (\theta^1)^2 + (\theta^2)^2$, with $\{\theta^1, \theta^2\}$ dual to a positive orthonormal frame. If k_g is the geodesic curvature on the boundary of ∂A and K is the Gauss curvature of the surface, then*

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} k_g(s) ds = 2\pi$$

where ∂A has the induced orientation.

Proof. Let $\{\vec{e_1}, \vec{e_2}\}$ be the dual frame to $\{\theta^1, \theta^2\}$.

We have, by the Stokes Theorem,

$$\int_A K\theta^1 \wedge \theta^2 = \int_A d\omega_2^1 = \int_{\partial A} \omega_2^1.$$

Let $c : [s_0, s_1] \to \partial A$ be a parameterization with arclength of ∂A, written as $c(s) = (u(s), v(s))$.

Moreover, let $\vec{n}(s)$ be the unit normal with respect to $\{\vec{e_1}, \vec{e_2}\}$. We have

$$\begin{aligned}
k_g(s) &= \left\langle \frac{Dc'}{ds}(s), \vec{n}(s) \right\rangle \\
&= \Big(u'' + v'\omega_2^1(u', v')\Big)(-v') + \Big(v'' - u'\omega_2^1(u', v')\Big)u' \\
&= -u''v' + v''u' - ((v')^2 + (u')^2)\omega_2^1(u', v') \\
&= -u''v' + v''u' - \omega_2^1(u', v').
\end{aligned}$$

Therefore,

$$\int_{\partial A} \omega_2^1 = \int_{s_0}^{s_1} \omega_2^1(u', v')ds = -\int_{s_0}^{s_1} k_g(s)ds + \int_{s_0}^{s_1} (-u''(s)v'(s) + v''(s)u'(s))ds.$$

Since $u'(s)^2 + v'(s)^2 = 1$, there exists a smooth function $\varphi : [s_0, s_1] \to [0, 2\pi]$ such that

$$u'(s) = \cos(\varphi(s)) \qquad \text{and} \qquad v'(s) = \sin(\varphi(s)).$$

We have

$$-u''(s)v'(s) + v''(s)u'(s) = \sin^2(\varphi(s))\varphi'(s) + \cos^2(\varphi(s))\varphi'(s) = \varphi'(s),$$

and thus, using the fact that ∂A is homotopic to a circle (and the invariance of the rotation index under homotopy by regular curves),

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} k_g(s)ds = \int_{\partial A} \omega_2^1 + \int_{\partial A} k_g(s)ds = \int_{s_0}^{s_1} \varphi'(s)ds = \varphi(s_1) - \varphi(s_0) = 2\pi,$$

as required. □

Remark 5.7. Recall that $\theta^1 \wedge \theta^2$ does not depend on the choice of positive orthonormal frame (as would be expected from the statement of the Gauss-Bonnet Theorem for domains). In fact, we saw that a possible choice is

$$\theta^1 = \sqrt{E}\,du + \frac{F}{\sqrt{E}}\,dv, \qquad \theta^2 = \sqrt{\frac{EG - F^2}{E}}\,dv,$$

so that

$$\theta^1 \wedge \theta^2 = \sqrt{EG - F^2}\,du \wedge dv.$$

Hence, $\int_A K\theta^1 \wedge \theta^2$ can be interpreted as the surface integral of the Gauss curvature.

A stronger version of the Gauss-Bonnet Theorem is available for domains with corners.

Theorem 5.8. *(Gauss-Bonnet for domains with corners) Let A is a simply connected domain with $n \geq 0$ corners on an oriented Riemannian surface with metric $ds^2 = (\theta^1)^2 + (\theta^2)^2$, with $\{\theta^1, \theta^2\}$ dual to a positive orthonormal frame. If k_g is the geodesic curvature on the boundary of ∂A and K is the Gauss curvature of the surface, then*

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} k_g(s)ds + \sum_{i=1}^{n} \varepsilon_i = 2\pi$$

where ∂A has the induced orientation and $\varepsilon_1, \ldots, \varepsilon_n$ are the angles by which the velocity vector rotates at each corner.

Proof. Assume that the i-th corner corresponds to the value $s = s_i$ of the arclength parameter along ∂A. If we approximate ∂A by a smooth curve C_δ obtained by "rounding the corners" in small intervals of the form $(s_i - \delta, s_i + \delta)$, then we see that along this curve C_δ

$$\begin{aligned}\int_{s_i-\delta}^{s_i+\delta} k_g(s)ds &= \int_{s_i-\delta}^{s_i+\delta} \varphi'(s)ds - \int_{s_i-\delta}^{s_i+\delta} \omega_2^1(c'(s))ds \\ &= \varphi(s_i+\delta) - \varphi(s_i-\delta) - \int_{s_i\delta}^{s_i+\delta} \omega_2^1(c'(s))ds \\ &\to \varepsilon_i + 0\end{aligned}$$

as $\delta \to 0$. Consequently,

$$\lim_{\delta\to 0} \int_{C_\delta} k_g(s)ds = \int_{\partial A} k_g(s)ds + \sum_{i=1}^{n} \varepsilon_i.$$

Applying the Gauss-Bonnet for domains to the domain D_δ with boundary $\partial D_\delta = C_\delta$, and then taking the limit as $\delta \to 0$, yields the result. ☐

5.1 Euler characteristic

Definition 5.9. (Triangulation) A triangle on a compact 2-manifold (surface) $S \subset \mathbb{R}^n$ is the image of an Euclidean triangle by a parameterization $g : U \subset \mathbb{R}^2 \to S$. A triangulation of S is a decomposition of S into a finite number of triangles such that the intersection of any two triangles is precisely a common edge, a common vertex or empty.

The following theorem will not be proved.

Theorem 5.10. *Let $S \subset \mathbb{R}^n$ be a compact surface and consider some triangulation of S with V vertices, E edges and F triangles. The number $V - E + F$ does not depend on the choice of the triangulation.*

Hence, we make the following definition.

Definition 5.11. (Euler characteristic) Let $S \subset \mathbb{R}^n$ be a compact surface. The **Euler characteristic** of S is the integer $\chi(S) = V - E + F$, where V, E and F are the total numbers of vertices, edges and triangles on any triangulation, respectively.

For embedded compact surfaces $S \subset \mathbb{R}^n$ we can prove a new version of the Gauss-Bonnet theorem.

Theorem 5.12. *(Gauss-Bonnet for compact surfaces) If $S \subset \mathbb{R}^n$ is a compact surface and K is its Gauss curvature then*

$$\int_S K = 2\pi\chi(S).$$

Proof. □

The following propositions will also be left without a proof.

Proposition 5.13. The Euler characteristic is invariant under homeomorphisms.

Proposition 5.14. We consider compact surfaces up to homeomorphism. The connected sum $S_1 \# S_2$ of two surfaces S_1 and S_2 is the surface obtained by removing a small disk on both surfaces and gluing them along the disk's boundary. We have

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proposition 5.15. Any orientable surface is homeomorphic to either the sphere S^2 or a connected sum of $g \in \mathbb{N}$ tori T^2, and its Euler characteristic is $2 - 2g$ (with $g = 0$ for the sphere). The integer g is known as the **genus** of the surface.

6 Minimal surfaces

Recall that a surface is said to be minimal if $H \equiv 0$. We will start by deducing the conditions for the graph of a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ to be a minimal surface.

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and consider the function $g : U \to \mathbb{R}^3$ given by

$$g(x, y) = (x, y, f(x, y)).$$

It is clear that g is a parametrization of the surface $S := g(U) \subset \mathbb{R}^3$. We have

$$\frac{\partial g}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial g}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

and so

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix}.$$

Moreover,

$$\frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

and so if we define

$$W := \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

then the unit normal $\vec{n}$ is given by

$$\vec{n} = \frac{1}{W}\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right).$$

Hence, we have

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} \cdot \vec{n} & \frac{\partial^2 g}{\partial x \partial y} \cdot \vec{n} \\ \frac{\partial^2 g}{\partial y \partial x} \cdot \vec{n} & \frac{\partial^2 g}{\partial y^2} \cdot \vec{n} \end{bmatrix} = \begin{bmatrix} \frac{1}{W}\frac{\partial^2 f}{\partial x^2} & \frac{1}{W}\frac{\partial^2 f}{\partial x \partial y} \\ \frac{1}{W}\frac{\partial^2 f}{\partial y \partial x} & \frac{1}{W}\frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

The condition for S to be minimal is $EN - 2FM + GL = 0$, which is equivalent to

$$0 = \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\frac{\partial^2 f}{\partial y^2} - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right)\frac{\partial^2 f}{\partial x^2}.$$

Now note that

$$\frac{\partial}{\partial x}\left(\frac{1}{W}\right) = -\frac{1}{W^3}\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}\right)$$

and so

$$\frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right) = \frac{1}{W}\frac{\partial^2 f}{\partial x^2} - \frac{1}{W^3}\left(\left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}\right)$$
$$= \frac{1}{W^3}\left[\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}\right]$$
$$= \frac{1}{W^3}\left[\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right)\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}\right].$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right) = \frac{1}{W^3}\left[\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}\right]$$

and so we see that

$$EN - 2FM + GL = 0 \Leftrightarrow \frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right) = 0.$$

This expression will be very useful in the proofs of some theorems coming ahead in this section.

We now introduce the concept of isothermal coordinates.

Definition 6.1. (Isothermal coordinates) A coordinate system (u, v) for a Riemannian surface (U, ds^2) is called **isothermal** if the metric is given by

$$ds^2 = E(du^2 + dv^2)$$

for some function $E : U \to \mathbb{R}$.

[conforme]

Theorem 6.2. *If $S \subset \mathbb{R}^3$ is a minimal surface then there exists a isothermal coordinate system around any point in S.*

Proof. Let p be any point in S. Since S is a surface we know, by definition, that there exists an open neighborhood of p and a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$S \cap V = \text{Graph}(f) \cap V.$$

Let

$$W := \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

and define the forms $\alpha, \beta \in \Omega^1(U)$ given by

$$\alpha = \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) dx + \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} dy$$

$$\beta = \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} dx + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) dy.$$

We will prove that $d\alpha = 0$. We have

$$\begin{aligned}
&\frac{\partial}{\partial y}\left[\frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\right] - \frac{\partial}{\partial x}\left[\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right] \\
=&\frac{\partial}{\partial y}\left(\frac{1}{W}\left(W^2 - \left(\frac{\partial f}{\partial y}\right)^2\right)\right) - \frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y} - \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y} \\
=&\frac{\partial W}{\partial y} - \frac{1}{W}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} - \frac{\partial}{\partial y}\left(\frac{1}{W}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y} - \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

which, using the equation we got for a minimal surface, simplifies to

$$-\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y} - \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial W}{\partial y}.$$

Moreover, we have

$$\frac{\partial W}{\partial y} = \frac{1}{W}\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2}\right)$$

and so we obtain

$$\frac{\partial}{\partial y}\left[\frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\right] - \frac{\partial}{\partial x}\left[\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right] = 0.$$

Thus,

$$d\alpha = \frac{\partial}{\partial y}\left[\frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\right] dy \wedge dx + \frac{\partial}{\partial x}\left[\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right] dx \wedge dy = 0.$$

Similarly, $d\beta = 0$.

Assume U is star-shaped; if it was not, we could just consider some smaller neighborhood $U' \subset U$ of p that is star-shaped.

By Poincaré's lemma (2.38) we know that there are 0-forms $\varphi, \psi \in \Omega^0(U)$, i.e., smooth functions $\varphi, \psi : U \to \mathbb{R}$, such that $\alpha = d\varphi$ and $\beta = d\psi$.

Define

$$u = x + \varphi(x, y) \qquad \text{and} \qquad v = y + \psi(x, y).$$

We will prove that (u, v) is a isothermal coordinate system around p.

We have

$$\begin{aligned}\frac{\partial u}{\partial x}dx + \frac{\partial v}{\partial y}dy &= du = dx + d\varphi = dx + \alpha \\ &= \left(1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)\right)dx + \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}dy\end{aligned}$$

and similarly

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}dx + \left(1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right)\right)dy$$

and therefore

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) & \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) \end{bmatrix}.$$

Let J denote the determinant of the matrix in the right-hand-side above. We have

$$\begin{aligned}J = 1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) + \frac{1}{W^2}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial f}{\partial y}\right)^2\right) \\ - \frac{1}{W^2}\left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial f}{\partial y}\right)^2\end{aligned}$$

which simplifies to

$$J = 1 + \frac{1}{W}(1 + W^2) + \frac{1}{W^2}(W^2) = \frac{1}{W} + 2 + W = \left(\frac{1}{\sqrt{W}} + \sqrt{W}\right)^2 > 0.$$

The fact that this matrix has nonzero determinant implies that (u, v) is a coordinate system around p.

Furthermore, the Inverse Function Theorem gives us

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} 1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) & -\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ -\frac{1}{W}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \frac{1}{W}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) \end{bmatrix}$$

$$= \frac{1}{JW} \underbrace{\begin{bmatrix} 1 + W + \left(\frac{\partial f}{\partial y}\right)^2 & -\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + W + \left(\frac{\partial f}{\partial x}\right)^2 \end{bmatrix}}_{=:A}$$

and so

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = \frac{1}{JW} A \begin{bmatrix} du \\ dv \end{bmatrix}.$$

The metric

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

then becomes (noting that $A^T = A$)

$$ds^2 = \begin{bmatrix} du & dv \end{bmatrix} \frac{1}{JW} A^T \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix} \frac{1}{JW} A \begin{bmatrix} du \\ dv \end{bmatrix}$$

$$= \begin{bmatrix} du & dv \end{bmatrix} \frac{1}{J^2W^2} A \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix} A \begin{bmatrix} du \\ dv \end{bmatrix}.$$

We have

$$A \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix} = \begin{bmatrix} W + W^2 + W\left(\frac{\partial f}{\partial x}\right)^2 & W\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ W\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & W + W^2 + W\left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix}$$

$$= W \begin{bmatrix} 1 + W + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + W + \left(\frac{\partial f}{\partial y}\right)^2 \end{bmatrix}$$

$$= W \det(A) A^{-1}$$

and so

$$ds^2 = \begin{bmatrix} du & dv \end{bmatrix} \frac{1}{J^2W^2} W \det(A) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}.$$

By the definition of A, we know that

$$\det\left(\frac{1}{JW}A\right) = \frac{1}{J} \Rightarrow \det(A) = JW^2$$

and hence we have

$$ds^2 = \frac{W}{J}(du^2 + dv^2) = \left(\frac{W}{W+1}\right)^2 (du^2 + dv^2)$$

as required. □

Proposition 6.3. Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of a surface $S \subset \mathbb{R}^3$ using isothermal coordinates. If H is the mean curvature of the surface and $\vec{n}$ is the unit normal with respect to g then

$$\frac{\partial^2 g}{\partial u} + \frac{\partial^2 g}{\partial v} = \Delta g = 2EH\vec{n}.$$

Proof. We have

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{E(N+L)}{2E^2} = \frac{L+N}{2E} \Rightarrow L + N = 2EH$$

and

$$\Delta g = \frac{\partial^2 g}{\partial u} \cdot \vec{n} + \frac{\partial^2 g}{\partial v} \cdot \vec{n} = L + N$$

and therefore

$$\Delta g = (L+N)\vec{n} = 2EH\vec{n}$$

as desired. □

In particular, the surface S is minimal if and only if

$$\Delta g = 0.$$

If we write $g(u, v) = (x(u, v), y(u, v), z(u, v))$ then this is equivalent to

$$\Delta x = \Delta y = \Delta z = 0$$

which is the same as saying that the functions x, y, z are harmonic.

This allows us to prove the following theorem.

Theorem 6.4. *There are no compact minimal surfaces with no boundary.*

FALTA. □

[REMARK]

Proposition 6.5. Let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of a surface $S \subset \mathbb{R}^3$ using isothermal coordinates. Then

$$\frac{\partial g}{\partial u} \cdot \Delta g = \frac{\partial g}{\partial v} \cdot \Delta g = 0.$$

Proof. We have

$$\frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial v} = F = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial u} = E = G = \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial v}.$$

and therefore

$$\frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u} \cdot \frac{\partial g}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial v}\right) \Rightarrow \frac{\partial g}{\partial u} \cdot \frac{\partial^2 g}{\partial u^2} = \frac{\partial g}{\partial v} \cdot \frac{\partial^2 g}{\partial u \partial v}.$$

On the other hand,

$$\frac{\partial g}{\partial v} \cdot \frac{\partial^2 g}{\partial u \partial v} = \frac{\partial}{\partial v} \underbrace{\left(\frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial u}\right)}_{=0} - \frac{\partial g}{\partial u} \cdot \frac{\partial^2 g}{\partial v^2}$$

and thus

$$\frac{\partial g}{\partial u} \cdot \frac{\partial^2 g}{\partial u^2} = -\frac{\partial g}{\partial u} \cdot \frac{\partial^2 g}{\partial v^2} \Rightarrow \frac{\partial g}{\partial u} \cdot \Delta g = 0.$$

In a similar way we obtain $\frac{\partial g}{\partial v} \cdot \Delta g = 0$. □

6.1 Complex analysis

We will now use tools of complex analysis to help us further study minimal surfaces.

Throughout this section we will often use the same letter for a set $V \subset \mathbb{R}^2$ and the corresponding set

$$\{u + iv : (u, v) \in V\} \subset \mathbb{C}.$$

We may also write equalities between functions with domain $\mathbb{R}^2$ and functions with domain $\mathbb{C}$. In all these instances it should be clear what we are doing.

Let $S \subset \mathbb{R}^3$ be a surface and let $g : U \subset \mathbb{R}^2 \to S$ be a parametrization of S written as

$$g(u, v) = (x(u, v), y(u, v), z(u, v)).$$

and define the functions $\varphi_1, \varphi_2, \varphi_3 : U^* \to \mathbb{C}$ given by

$$\begin{aligned} \varphi_1(u + iv) &= \frac{\partial x}{\partial u}(u, v) - i\frac{\partial x}{\partial v}(u, v) \\ \varphi_2(u + iv) &= \frac{\partial y}{\partial u}(u, v) - i\frac{\partial y}{\partial v}(u, v) \\ \varphi_3(u + iv) &= \frac{\partial z}{\partial u}(u, v) - i\frac{\partial z}{\partial v}(u, v). \end{aligned}$$

These functions will help us study the surface S.

Proposition 6.6.

i) The functions $\varphi_1, \varphi_2, \varphi_3$ are holomorphic if and only if x, y, z are harmonic.

ii) The coordinate system (u, v) is isothermal if and only if $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$.

Proof. *i*) The function φ_1 is holomorphic if and only if

$$\begin{cases} \frac{\partial^2 x}{\partial u^2} = -\frac{\partial^2 x}{\partial v^2} \\ \frac{\partial^2 x}{\partial x \partial y} = \frac{\partial^2 x}{\partial y \partial x} \end{cases} \Leftrightarrow \Delta x = 0.$$

Similarly, φ_2 is holomorphic if and only if $\Delta y = 0$ and φ_3 is holomorphic if and only if $\Delta z = 0$.

ii) We have

$$\varphi_1^2 = \left(\frac{\partial x}{\partial u}\right)^2 - \left(\frac{\partial x}{\partial v}\right)^2 - 2i\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}$$
$$\varphi_2^2 = \left(\frac{\partial y}{\partial u}\right)^2 - \left(\frac{\partial y}{\partial v}\right)^2 - 2i\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}$$
$$\varphi_3^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2 - 2i\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

and so it is clear that

$$\begin{aligned} E - G &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 - \left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) \\ &= \mathrm{Re}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2) \end{aligned}$$

and

$$-2F = -2\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right) = \mathrm{Im}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2).$$

Hence,

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0 \qquad \Leftrightarrow \qquad E = G \quad \wedge \quad F = 0.$$

□

Note that, if $w = u + iv$, then

$$dwd\overline{w} = d(u + iv)d(u - iv) = du^2 + dv^2$$

and so, when we write the first fundamental form in terms of complex functions, we use $dwd\overline{w}$ instead of $du^2 + dv^2$.

Theorem 6.7. *If the function g parametrizes a minimal surface by isothermal coordinates then*

- $\varphi_1, \varphi_2, \varphi_3$ *are holomorphic;*
- $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$*;*
- *the first fundamental form is given by*

$$\mathbb{I} = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2)dwd\overline{w}.$$

Conversely, if there exists a sufficiently small open set $U^* \subset \mathbb{C}$ such that the functions $\varphi_1, \varphi_2, \varphi_3 : U^* \to \mathbb{C}$

- *are holomorphic;*
- *satisfy $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$;*
- *satisfy $|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 > 0$*

then there exists a parametrization $g : U \to S$ of a minimal surface S such that

$$(\varphi_1, \varphi_2, \varphi_3) = \frac{\partial g}{\partial u} - i\frac{\partial g}{\partial v}.$$

Proof. ($\Rightarrow$) This is just Proposition 9.6..

($\Leftarrow$) [FALTA] □

These results allow us to prove the Weierstrass-Enneper Theorem.

Theorem 6.8. *Given any simply connected minimal surface, there exists a simply connected $U \subset \mathbb{C}$, a holomorphic function $f : U \to \mathbb{C}$ and a meromorphic function $g : U \to \mathbb{C}$ such that the function $\vec{g} : U \to \mathbb{R}^3$ given by*

$$\vec{g}(w) = \left(\mathrm{Re}\int \frac{1}{2}f(1-g^2)dw, \mathrm{Re}\int \frac{1}{2}f(1+g^2)dw, \mathrm{Re}\int fg\,dw\right)$$

is a parametrization of S. Moreover, the metric is given by

$$\mathbb{I} = \frac{1}{4}|f|^2(1+|g|^2)^2 dwd\overline{w}.$$

Proof. □

If we have a surface given by the parametrization

$$\vec{g}(w) = \left(\mathrm{Re}\int \frac{1}{2}f(1-g^2)dw, \mathrm{Re}\int \frac{1}{2}f(1+g^2)dw, \mathrm{Re}\int fg\,dw\right)$$

and we define, for each $\theta \in \mathbb{R}$, the new function f_θ given by

$$f_\theta(w) := e^{i\theta} f(w)$$

then we obtain a family of associated minimal surfaces given by the parametrizations

$$\vec{g}_\theta(w) := \left(\mathrm{Re} \int \frac{1}{2} f_\theta (1 - g^2) dw, \mathrm{Re} \int \frac{1}{2} f_\theta (1 + g^2) dw, \mathrm{Re} \int f_\theta g \, dw \right).$$

These surfaces are all isometric, i.e., have the same metric:

$$\mathbb{I}_\theta = \frac{1}{4} |f_\theta|^2 (1 + |g|^2)^2 dw d\overline{w} = \frac{1}{4} |f|^2 (1 + |g|^2)^2 dw d\overline{w} = \mathbb{I}.$$

6.2 Ricci's Theorem

Let $\vec{g} : U \to S$ be a parametrization of a minimal surface $S \subset \mathbb{R}^3$ and define the functions $\varphi_1, \varphi_2, \varphi_3 : U^* \to \mathbb{C}$ given by

$$\begin{aligned}
\varphi_1(u+iv) &= \frac{\partial x}{\partial u}(u,v) - i\frac{\partial x}{\partial v}(u,v) \\
\varphi_2(u+iv) &= \frac{\partial y}{\partial u}(u,v) - i\frac{\partial y}{\partial v}(u,v) \\
\varphi_3(u+iv) &= \frac{\partial z}{\partial u}(u,v) - i\frac{\partial z}{\partial v}(u,v).
\end{aligned}$$

Recall that if we let f, g be the functions

$$f := \varphi_1 - i\varphi - 2 \qquad \text{and} \qquad g := \frac{\varphi_3}{\varphi_1 - i\varphi_2}$$

then we have

$$\varphi_1 = \frac{1}{2}f(1-g^2), \qquad \varphi_2 = \frac{i}{2}f(1+g^2), \qquad \varphi_3 = fg.$$

Let us compute the normal

$$\vec{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|}$$

in terms of f and g. First, we have

$$\begin{aligned}
\frac{\partial g}{\partial u} &= \frac{1}{2}(\varphi_1 + \overline{\varphi}_1, \varphi_2 + \overline{\varphi}_2, \varphi_3 + \overline{\varphi}_3) \\
\frac{\partial g}{\partial v} &= \frac{i}{2}(\varphi_1 - \overline{\varphi}_1, \varphi_2 - \overline{\varphi}_2, \varphi_3 - \overline{\varphi}_3)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} &= \frac{i}{4}(-2\varphi_2\overline{\varphi}_3 + 2\overline{\varphi}_2\varphi_3, -2\varphi_3\overline{\varphi}_1 + 2\overline{\varphi}_3\varphi_1, -2\varphi_1\overline{\varphi}_2 + 2\overline{\varphi}_1\varphi_2) \\
&= \operatorname{Im}(\varphi_2\overline{\varphi}_3, \varphi_3\overline{\varphi}_1, \varphi_1\overline{\varphi}_2) \\
&= \operatorname{Im}\left(\frac{i}{2}f(1+g^2)\overline{fg}, \frac{1}{2}fg\overline{f}(1-\overline{g}^2), -\frac{i}{4}f(1-g^2)\overline{f}(1+\overline{g}^2)\right) \\
&= \frac{1}{4}|f|^2 \operatorname{Im}\Big(2i(1+g^2)\overline{g}, 2(1-\overline{g}^2)g, -i(1-g^2)(1+\overline{g}^2)\Big).
\end{aligned}$$

We have

$$\operatorname{Im}\Big(2i(1+g^2)\overline{g}\Big) = 2\operatorname{Re}\Big((1+g^2)\overline{g}\Big) = (1+g^2)\overline{g} + (1+\overline{g}^2)g$$
$$= (\overline{g}+g)(1+|g|^2)$$
$$= (1+|g|^2)\operatorname{Re} g.$$

Doing similar computations for the other terms yields

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \frac{1}{4}|f|^2(1+|g|^2)\Big(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2-1\Big).$$

Therefore,

$$\left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\|^2 = \frac{1}{16}|f|^4(1+|g|^2)^2\Big(4|g|^2 + (|g|^2-1)^2\Big) = \frac{1}{16}|f|^4(1+|g|^2)^4$$

which gives us

$$\left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\| = \frac{1}{4}|f|^2(1+|g|^2)^2$$

and therefore

$$\vec{n} = \left(\frac{2\operatorname{Re} g}{|g|^2+1}, \frac{2\operatorname{Im} g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right).$$

[FALTA]

Theorem 6.9. *(Ricci) Let (U, ds^2) be a simply connected Riemannian surface with negative Gauss curvature K. Then there exists a parametrization $\vec{g} : U \to \mathbb{R}^3$ of a minimal surface such that $ds^2 = d\vec{g} \cdot d\vec{g}$ if and only if the Gauss curvature of $(U, d\widetilde{s}^2)$ with $d\widetilde{s}^2 := \sqrt{-K}ds^2$ is zero.*

Proof. $(\Rightarrow)$ □