Differentiable manifolds and differential forms

1 Differentiable manifolds

Definition 1.1 *A set* $M \subset \mathbb{R}^n$ *is said to be a* **differentiable manifold of dimension** $m \in \{1, \ldots, n-1\}$ *if for any point* $\mathbf{a} \in M$ *there exists an open set* $U \ni \mathbf{a}$ *and a smooth function* $\mathbf{f} : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$ *such that*

$$M \cap U = \operatorname{Graph}(\mathbf{f}) \cap U$$

for some ordering of the Cartesian coordinates of $\mathbb{R}^n$*.*

Remark 1.2 *We also define a manifold of dimension* 0 *as a set of isolated points, and a manifold of dimension* n *as an open set.*

Theorem 1.3 $M \subset \mathbb{R}^n$ *is a differentiable manifold of dimension* m **iff** *for each point* $\mathbf{a} \in M$ *there exists an open set* $U \ni \mathbf{a}$ *and a smooth function* $\mathbf{F} : U \to \mathbb{R}^{n-m}$ *such that*

(i) $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$;

(ii) $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$.

Proof: Assume without loss of generality that $\det \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{a}) \neq 0$, where we write $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ with $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^{n-m}$. Then by the Implicit Function Theorem there exists an open set $V \subset U$ with $\mathbf{a} \in V$ such that $M \cap V$ is given by the graph of a smooth function $\mathbf{z} = \mathbf{f}(\mathbf{y})$. This shows that a set satisfying the conditions in the statement is indeed a differentiable manifold. To show that a differentiable manifold satisfies the conditions in the statement it suffices to take $\mathbf{F}(\mathbf{y}, \mathbf{z}) = \mathbf{f}(\mathbf{y}) - \mathbf{z}$. □

Definition 1.4 *A vector* $\mathbf{v} \in \mathbb{R}^n$ *is said to be* **tangent** *to a set* $M \subset \mathbb{R}^n$ *at the point* $\mathbf{a} \in M$ *if there exists a smooth curve* $\mathbf{c} : \mathbb{R} \to M$ *such that* $\mathbf{c}(0) = \mathbf{a}$ *and* $\dot{\mathbf{c}}(0) = \mathbf{v}$*. A vector* $\mathbf{v} \in \mathbb{R}^n$ *is said to be* **orthogonal** *to* M *at the point* $\mathbf{a}$ *if it is orthogonal to all vectors tangent to* M *at* $\mathbf{a}$*.*

Proposition 1.5 *If* $M \subset \mathbb{R}^n$ *is a manifold of dimension* m *then the set* $T_{\mathbf{a}}M$ *of all vectors tangent to* M *at the point* $\mathbf{a} \in M$ *is a vector space of dimension* m*, called the* **tangent space** *to* M *at the point* $\mathbf{a}$*.*

Proof: Assume without loss of generality that M is given by $\mathbf{z} = \mathbf{f}(\mathbf{y})$ in a neighborhood of the point $\mathbf{a}$, where we use the notation in the proof of the theorem above. Any curve $\mathbf{c} : \mathbb{R} \to M$ with $\mathbf{c}(0) = \mathbf{a}$ is given in this neighborhood by $\mathbf{c}(t) = (\mathbf{d}(t), \mathbf{f}(\mathbf{d}(t)))$, where $\mathbf{d} : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$. Therefore, $\dot{\mathbf{c}}(0) = (\dot{\mathbf{d}}(0), D\mathbf{f}(\mathbf{a}) \cdot \dot{\mathbf{d}}(0))$, and so any vector tangent to M at the point $\mathbf{a}$ is contained in the image of $\mathbb{R}^m$ by the injective linear map $\mathbf{u} \mapsto (\mathbf{u}, D\mathbf{f}(\mathbf{a}) \cdot \mathbf{u})$. On the other hand, given $\mathbf{u} \in \mathbb{R}^m$, its image by this map is the vector tangent to the curve $\mathbf{c}(t) = (\mathbf{b} + t\mathbf{u}, \mathbf{f}(\mathbf{b} + t\mathbf{u}))$, where we write $\mathbf{a} = (\mathbf{b}, \mathbf{c})$, and so it is tangent to M at the point $\mathbf{a}$. We conclude that $T_{\mathbf{a}}M$ is an m-dimensional vector subspace of $\mathbb{R}^n$. □

Definition 1.6 *The* **normal space** *to an m-manifold $M \subset \mathbb{R}^n$ at the point $\mathbf{a} \in M$ is the $(n-m)$-dimensional vector space $T_{\mathbf{a}}^{\perp}M$ obtained by taking the orthogonal complement of $T_{\mathbf{a}}M$.*

Proposition 1.7 *Let $M \subset \mathbb{R}^n$ be an m-manifold, $\mathbf{a} \in M$ a point in M, $U \ni \mathbf{a}$ an open set and $\mathbf{F} : U \to \mathbb{R}^{n-m}$ such that $M \cap U = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$ with $\operatorname{rank} D\mathbf{F}(\mathbf{a}) = n - m$. Then $T_{\mathbf{a}}M = \ker D\mathbf{F}(\mathbf{a})$.*

Proof: Since $\dim \ker D\mathbf{F}(\mathbf{a}) = m$, it suffices to show that $T_{\mathbf{a}}M \subset \ker D\mathbf{F}(\mathbf{a})$. For any smooth curve $\mathbf{c} : \mathbb{R} \to M$ satisfying $\mathbf{c}(0) = \mathbf{a}$, we have $\mathbf{F}(\mathbf{c}(t)) = \mathbf{0}$ whenever $\mathbf{c}(t) \in U$, and so $D\mathbf{F}(\mathbf{a}) \cdot \dot{\mathbf{c}}(0) = \mathbf{0}$. □

Definition 1.8 *A* **parameterization** *of a given m-manifold $M \subset \mathbb{R}^n$ is a smooth injective map $\mathbf{g} : U \to M$ (with $U \subset \mathbb{R}^m$ open) such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$.*

Proposition 1.9 *If $\mathbf{g} : U \to M$ is a parameterization of the m-manifold $M \subset \mathbb{R}^n$ then*

$$T_{\mathbf{g}(\mathbf{t})}M = \operatorname{span}\left\{\frac{\partial \mathbf{g}}{\partial t^1}(\mathbf{t}), \ldots, \frac{\partial \mathbf{g}}{\partial t^m}(\mathbf{t})\right\}.$$

Proof: Obvious. □

Theorem 1.10 *If $M \subset \mathbb{R}^n$ is an m-manifold and $\mathbf{a} \in M$ then there exists an open set $V \ni \mathbf{a}$ such that $M \cap V$ is the image of a parameterization $\mathbf{g} : U \subset \mathbb{R}^m \to M$. Conversely, given a smooth map $\mathbf{g} : U \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $\operatorname{rank} D\mathbf{g}(\mathbf{t}) = m$ for all $\mathbf{t} \in U$, and given any point $\mathbf{t}_0 \in U$, there exists an open set $U_0 \subset U$ with $\mathbf{t}_0 \in U_0$ such that $\mathbf{g}(U_0)$ is an m-manifold.*

Proof: Since there exists an open set $V \ni \mathbf{a}$ such that $M \cap V$ is the graph of a smooth function $\mathbf{f} : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$, we can choose the parameterization to be $\mathbf{g}(\mathbf{t}) = (\mathbf{t}, \mathbf{f}(\mathbf{t}))$. To prove the converse, assume that the first m lines of $D\mathbf{g}(\mathbf{t}_0)$ are linearly independent. Then, writing $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ with $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^{n-m}$, we have from the Inverse Function Theorem that the equations $(\mathbf{y}, \mathbf{z}) = \mathbf{g}(\mathbf{t})$ can be solved to yield $\mathbf{t} = \mathbf{h}(\mathbf{y})$ in some open neighborhood U_0 of $\mathbf{t}_0$, with $\mathbf{h}$ a smooth function. Therefore, $\mathbf{g}(\mathbf{h}(\mathbf{y})) = (\mathbf{y}, \mathbf{k}(\mathbf{y}))$, and so $\mathbf{g}(U_0)$ is the graph of the smooth function $\mathbf{k}$. □

2 Covectors and wedge product

Definition 2.1 *The* **dual vector space** *to* $\mathbb{R}^n$ *is*

$$(\mathbb{R}^n)^* = \{\alpha : \mathbb{R}^n \to \mathbb{R} : \alpha \text{ is linear}\}.$$

The elements of $(\mathbb{R}^n)^*$ *are called* **covectors***.*

By linearity, a covector is determined by its action on the canonical basis of $\mathbb{R}^n$. We define the covectors $dx^1, \ldots, dx^n \in (\mathbb{R}^n)^*$ through

$$dx^i(\mathbf{e}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(we will see later on the reason for this notation). Given $\alpha \in (\mathbb{R}^n)^*$, we have

$$\alpha(\mathbf{v}) = \alpha\left(\sum_{i=1}^{n} v^i \mathbf{e}_i\right) = \sum_{i=1}^{n} v^i \alpha\left(\mathbf{e}_i\right) = \sum_{i=1}^{n} v^i \alpha_i,$$

where $\alpha_i = \alpha(\mathbf{e}_i)$. In particular, $dx^i(\mathbf{v}) = v^i$, and so

$$\alpha(\mathbf{v}) = \sum_{i=1}^{n} \alpha_i \, dx^i(\mathbf{v}) \Leftrightarrow \alpha = \sum_{i=1}^{n} \alpha_i \, dx^i.$$

One easily concludes from this that $\{dx^1, \ldots, dx^n\}$ is a basis for $(\mathbb{R}^n)^*$, which is then a vector space of dimension n.

Definition 2.2 *A (covariant)* k-**tensor** T *is a multilinear map* $T : (\mathbb{R}^n)^k \to \mathbb{R}$*, i.e.*

(i) $T(\mathbf{v}_1, \ldots, \mathbf{v}_i + \mathbf{w}_i, \ldots, \mathbf{v}_k) = T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k) + T(\mathbf{v}_1, \ldots, \mathbf{w}_i, \ldots, \mathbf{v}_k)$*;*

(ii) $T(\mathbf{v}_1, \ldots, \lambda\mathbf{v}_i, \ldots, \mathbf{v}_k) = \lambda T(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k)$*.*

Example 2.3

(i) A covector is a 1*-tensor.*

(ii) $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *given by* $g(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ *is a* 2*-tensor (***metric tensor***).*

(iii) $T : (\mathbb{R}^n)^n \to \mathbb{R}$ *given by* $T(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \det(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *is an* **alternanting** n*-tensor.*

Definition 2.4 *A* k*-tensor* α *is said to be* **alternanting***, or a* k-**covector***, if*

$$\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j \ldots, \mathbf{v}_k) = -\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i \ldots, \mathbf{v}_k).$$

We denote by $\Lambda^k(\mathbb{R}^n)$ *the vector space of all* k*-covectors.*

Given $i_1, \ldots, i_k \in \{1, \ldots, n\}$, we define $dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Lambda^k(\mathbb{R}^n)$ as

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det \begin{bmatrix} dx^{i_1}(\mathbf{v}_1) & \ldots & dx^{i_1}(\mathbf{v}_k) \\ \ldots & \ldots & \ldots \\ dx^{i_k}(\mathbf{v}_1) & \ldots & dx^{i_k}(\mathbf{v}_k) \end{bmatrix}.$$

Note that

$$dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge \ldots \wedge dx^{i_q} \wedge \ldots \wedge dx^{i_k} = -dx^{i_1} \wedge \ldots \wedge dx^{i_q} \wedge \ldots \wedge dx^{i_p} \wedge \ldots \wedge dx^{i_k},$$

and so

$$dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge \ldots \wedge dx^{i_p} \wedge \ldots \wedge dx^{i_k} = 0.$$

Proposition 2.5 *$\left\{dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right\}_{1 \leq i_1 < \ldots < i_k \leq n}$ is a basis for $\Lambda^k(\mathbb{R}^n)$, whose dimension is therefore $\binom{n}{k}$.*

Proof: If

$$\sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} = 0$$

then by applying this k-covector to $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}$ (with $j_1 < \ldots < j_k$) we obtain $\alpha_{j_1 \ldots j_k} = 0$. This shows that the elements of $\left\{dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right\}_{1 \leq i_1 < \ldots < i_k \leq n}$ are linearly independent. To show that they generate $\Lambda^k(\mathbb{R}^n)$, take a k-covector T and consider

$$\alpha \;=\; \sum_{i_1 < \ldots < i_k} T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

It should be clear that

$$\alpha(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) = T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k})$$

for $i_1 < \ldots < i_k$; since both covectors are alternating, the equality holds for any ordering of the indices, and by multilinearity it holds for any vectors. □

Remark 2.6 *Since $\binom{n}{0} = 1$, we define $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$.*

Example 2.7 *$\Lambda^2(\mathbb{R}^3)$ has dimension 3, and a basis is for instance $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$. This makes it possible to identify $\mathbb{R}^3$ both with $\Lambda^1(\mathbb{R}^3) \equiv (\mathbb{R}^3)^*$ and with $\Lambda^2(\mathbb{R}^3)$: if $\mathbf{v} \in \mathbb{R}^3$, we define*

$$\omega_{\mathbf{v}} = v^1 dx + v^2 dy + v^3 dz$$

and

$$\Omega_{\mathbf{v}} = v^1 dy \wedge dz + v^2 dz \wedge dx + v^3 dx \wedge dy.$$

Note that actually $\omega_{\mathbf{v}}$ can be defined for $\mathbf{v} \in \mathbb{R}^n$. It is easy to see that

$$\omega_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w},$$

and that

$$\Omega_{\mathbf{u}}(\mathbf{v}, \mathbf{w}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Definition 2.8 *If $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta \in \Lambda^l(\mathbb{R}^n)$,*

$$\alpha \;=\; \sum_{i_1<\ldots<i_k} \alpha_{i_1\ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \qquad \beta \;=\; \sum_{j_1<\ldots<j_l} \beta_{j_1\ldots j_l}\, dx^{j_1} \wedge \ldots \wedge dx^{j_l},$$

we define their **wedge product** $\alpha \wedge \beta \in \Lambda^{k+l}(\mathbb{R}^n)$ *as*

$$\alpha \wedge \beta \;=\; \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} \alpha_{i_1\ldots i_k}\, \beta_{j_1\ldots j_l}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$

Remark 2.9 *If α is a 0-covetor (real number), its wedge product by α is simply the product by a scalar.*

Example 2.10 *The wedge product can be seen as a generalization of the cross product: if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ then*

$$\omega_{\mathbf{v}} \wedge \omega_{\mathbf{w}} = \Omega_{\mathbf{v}\times\mathbf{w}}.$$

It can also be seen as a generalization of the inner product:

$$\omega_{\mathbf{v}} \wedge \Omega_{\mathbf{w}} = (\mathbf{v} \cdot \mathbf{w})\, dx \wedge dy \wedge dz.$$

Proposition 2.11 Properties of the wedge product:

(i) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$;

(ii) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ *if* $\alpha \in \Lambda^k(\mathbb{R}^n), \beta \in \Lambda^l(\mathbb{R}^n)$;

(iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

Proof: Exercise. □

3 Differential forms, pull-back and exterior derivative

Definition 3.1 *A* **differential form of degree** k *in* $\mathbb{R}^n$ *is a smooth function* $\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$. *We denote by* $\Omega^k(\mathbb{R}^n)$ *the set of* k*-forms in* $\mathbb{R}^n$.

Example 3.2 *A* 0*-form is simply a smooth function* $\phi : \mathbb{R}^n \to \mathbb{R}$.

Definition 3.3 *If* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *is smooth and* $\omega \in \Omega^k(\mathbb{R}^m)$ *then the* **pull-back** *of* ω *by* $\mathbf{f}$ *is the* k*-form* $\mathbf{f}^*\omega \in \Omega^k(\mathbb{R}^n)$ *defined by*

$$(\mathbf{f}^*\omega)(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega(\mathbf{f}(\mathbf{x}))(D\mathbf{f}(\mathbf{x})\mathbf{v}_1, \ldots, D\mathbf{f}(\mathbf{x})\mathbf{v}_k).$$

Example 3.4 *The pull-back of a* 0*-form* ϕ *by* $\mathbf{f}$ *is simply the pre-composition* $\phi \circ \mathbf{f}$.

Proposition 3.5 Properties of the pull-back:

(i) $\mathbf{f}^*(\omega + \eta) = \mathbf{f}^*\omega + \mathbf{f}^*\eta$;

(ii) $\mathbf{f}^*(\omega \wedge \eta) = \mathbf{f}^*\omega \wedge \mathbf{f}^*\eta$;

(iii) $(\mathbf{g} \circ \mathbf{f})^*(\omega) = \mathbf{f}^*(\mathbf{g}^*\omega)$.

Proof: Exercise. □

Definition 3.6 *If* $\omega \in \Omega^k(\mathbb{R}^n)$,

$$\omega \;=\; \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}(\mathbf{x})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

then its **exterior derivative** *is the* $(k+1)$*-form* $d\omega \in \Omega^{k+1}(\mathbb{R}^n)$ *defined by*

$$d\omega \;=\; \sum_{i_1<\ldots<i_k} \sum_{i=1}^{n} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^i}\, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Example 3.7

1. *The exterior derivative of a* 0*-form* ϕ *is the* 1*-form*

$$d\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i}\, dx^i.$$

   At each point, this 1*-form is the linear transformation represented by the Jacobian matrix of* ϕ*, that is, the derivative of* ϕ*. In particular, the exterior derivative of the function* x^i *is* dx^i*, which explains our notation for the basis of* $(\mathbb{R}^n)^*$.

2. *Another way of thinking of the exterior derivative of a* 0*-form* $\phi : \mathbb{R}^n \to \mathbb{R}$ *is as the* 1*-form corresponding to the gradient of* ϕ:

$$d\phi = \omega_{\mathrm{grad}\,\phi}.$$

3. *If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth vector field then*

$$d\omega_{\mathbf{F}} = \Omega_{\mathrm{curl}\,\mathbf{F}}$$

and

$$d\Omega_{\mathbf{F}} = (\mathrm{div}\,\mathbf{F})\, dx \wedge dy \wedge dz.$$

Differential forms give a quick method for computing curls: for example, if $\mathbf{F} = (-y, x, z)$ then $\omega_{\mathbf{F}} = -ydx + xdy + zdz$, whence

$$\Omega_{\mathrm{curl}\,\mathbf{F}} = d\omega_{\mathbf{F}} = -dy \wedge dx + dx \wedge dy = 2dx \wedge dy,$$

and so $\mathrm{curl}\,\mathbf{F} = (0, 0, 2)$.

Proposition 3.8 Properties of the exterior derivative:

(i) $d(\omega + \eta) = d\omega + d\eta$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad$ *if* $\quad \omega \in \Omega^k(\mathbb{R}^n)$;

(iii) $d(d\omega) = 0$;

(iv) $\mathbf{f}^*(d\omega) = d(\mathbf{f}^*\omega)$.

Proof: (i) and (ii) are immediate. To prove (iii), note that

$$d(d\omega) = d \sum_{i_1 < \ldots < i_k} \sum_{i=1}^{n} \frac{\partial \omega_{i_1 \ldots i_k}}{\partial x^i}\, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$
$$\sum_{i_1 < \ldots < i_k} \sum_{i,j=1}^{n} \frac{\partial^2 \omega_{i_1 \ldots i_k}}{\partial x^j \partial x^i}\, dx^j \wedge dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

By Schwarz's Lemma,

$$\sum_{i,j=1}^{n} \frac{\partial^2 \omega_{i_1 \ldots i_k}}{\partial x^j \partial x^i}\, dx^j \wedge dx^i = \sum_{i,j=1}^{n} \frac{\partial^2 \omega_{i_1 \ldots i_k}}{\partial x^i \partial x^j}\, dx^j \wedge dx^i = -\sum_{i,j=1}^{n} \frac{\partial^2 \omega_{i_1 \ldots i_k}}{\partial x^i \partial x^j}\, dx^i \wedge dx^j.$$

Therefore this expression vanishes, and so $d(d\omega) = 0$.

To prove (iv), we note that for 0-forms ϕ we have

$$d(\mathbf{f}^*\phi)(\mathbf{v}) = d(\phi \circ \mathbf{f})(\mathbf{v}) = d\phi(D\mathbf{f}(\mathbf{v})) = (\mathbf{f}^* d\phi)(\mathbf{v}).$$

In particular, we have

$$d(\mathbf{f}^*(d\phi)) = d(d(\mathbf{f}^*\phi)) = 0 = \mathbf{f}^*(d(d\phi)),$$

and so (iv) holds for 1-forms of the type $d\phi$. Since any k-form can be built out of 0-forms and the 1-forms $dx^1, \ldots, dx^n$ by using wedge products and sums, it is easy to see that (iv) holds for any k-form. □

Example 3.9

(i) If $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field then $d(d\phi) = 0 \Leftrightarrow \mathrm{curl}(\mathrm{grad}\,\phi) = \mathbf{0}$.

(ii) If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *is a vector field then* $d(d\omega_{\mathbf{F}}) = 0 \Leftrightarrow \operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.

Definition 3.10 *We say that* $\omega \in \Omega^k(\mathbb{R}^n)$ *is:*

(i) **closed** *if* $d\omega = 0$*;*

(ii) **exact** *if* $\omega = d\eta$ *for some* $\eta \in \Omega^{k-1}(\mathbb{R}^n)$ *(called a* **potential** *for* ω*).*

Proposition 3.11 *If* $\omega \in \Omega^k(\mathbb{R}^n)$ *is exact then* ω *is closed.*

Proof: Obvious. □

Remark 3.12 *More generally, we can consider the set* $\Omega^k(U)$ *of* k*-forms whose domain is any open set* $U \subset \mathbb{R}^n$*. It turns out that the relation between closed and exact forms on* U *depends crucially on the topology of* U*.*

Theorem 3.13 (Poincaré Lemma) *If* $\omega \in \Omega^k(U)$ *is closed and the open set* U *is star-shaped then* ω *is exact.*

Proof: Assume without loss of generality that $\mathbf{0}$ is a center for U. If

$$\omega(\mathbf{x}) = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}(\mathbf{x})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

we define $I\Omega \in \Omega^{k-1}(U)$ as

$$I\omega(\mathbf{x}) = \sum_{i_1<\ldots<i_k} \sum_{l=1}^{n} (-1)^{l-1} \left(\int_0^1 t^{k-1} \omega_{i_1\ldots i_k}(t\mathbf{x})\, dt \right) x^{i_l}\, dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_l}} \wedge \ldots \wedge dx^{i_k}$$

(where $\widehat{\ }$ means omission). We have

$$\omega = d(I\omega) + I(d\omega),$$

and so if $d\omega = 0$ then $\omega = d(I\omega)$. □

Example 3.14

(i) If $\mathbf{F} : U \to \mathbb{R}^n$ *is a closed vector field and* $U \subset \mathbb{R}^n$ *is star-shaped then there exists* $\phi : U \to \mathbb{R}$ *such that* $\mathbf{F} = \operatorname{grad}\phi$*.*

(ii) If $\mathbf{F} : U \to \mathbb{R}^3$ *is a divergenceless vector field and* $U \subset \mathbb{R}^3$ *is star-shaped then there exists* $\mathbf{A} : U \to \mathbb{R}^3$ *such that* $\mathbf{F} = \operatorname{curl}\mathbf{A}$*.*

Differential forms give a quick method for computing vector potentials: for example, given $\mathbf{F} = (2y, 2z, 0)$ we have

$$\Omega_{\mathbf{F}} = 2y\, dy \wedge dz + 2z\, dz \wedge dx = d\left(y^2\right) \wedge dz + d\left(z^2\right) \wedge dx = d\left(y^2 dz + z^2 dx\right),$$

and so a vector potential for $\mathbf{F}$ is $\mathbf{A} = \left(z^2, 0, y^2\right)$.

4 Integration and Stokes Theorem

Proposition 4.1 *If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ are parameterizations of the m-manifold $M \subset \mathbb{R}^n$ then $\mathbf{h}^{-1} \circ \mathbf{g}$ is a* **diffeomorphism** *(smooth bijection with smooth inverse).*

Proof: Since $\mathbf{g}$ and $\mathbf{h}$ are injective, $\mathbf{h}^{-1} \circ \mathbf{g}$ is a bijection on its domain. Since the columns of $D\mathbf{g}$ and $D\mathbf{h}$ generate the same vector space at each point (tangent space), there exists a unique nonsingular $m \times m$ matrix A such that $D\mathbf{g} = D\mathbf{h} \cdot A$. It is easy to check that $A = D(\mathbf{h}^{-1} \circ \mathbf{g})$, and the Inverse Function Theorem then guarantees that $\mathbf{h}^{-1} \circ \mathbf{g}$ is a diffeomorphism. □

Definition 4.2 *We say that two parameterizations $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ of the m-manifold $M \subset \mathbb{R}^n$ induce the* **same orientation** *if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) > 0$, and* **opposite orientations** *if $\det D(\mathbf{h}^{-1} \circ \mathbf{g}) < 0$. The manifold M is called* **orientable** *if it is possible to choose parameterizations whose images cover M and induce the same orientation. An* **orientation** *on an orientable manifold is a choice of a maximal family of parameterizations under these conditions, which are said to be* **positive***. An orientable manifold with a choice of orientation is said to be* **oriented***.*

Definition 4.3 *If $\mathbf{g} : U \subset \mathbb{R}^m \to M$ is a positive parameterization of the oriented m-manifold $M \subset \mathbb{R}^n$ and $\omega \in \Omega^m(\mathbb{R}^n)$, we define the* **integral** *of ω along $\mathbf{g}(U)$ (assumed bounded) as*

$$\int_{\mathbf{g}(U)} \omega = \int_U \omega(\mathbf{g}(\mathbf{t})) \left(\frac{\partial \mathbf{g}}{\partial t^1}, \ldots, \frac{\partial \mathbf{g}}{\partial t^m} \right) dt^1 \ldots dt^m$$
$$= \int_U \mathbf{g}^* \omega \left(\mathbf{e}_1, \ldots, \mathbf{e}_m \right) dt^1 \ldots dt^m.$$

Remark 4.4 *If we think of an open set $U \subset \mathbb{R}^n$ as an n-manifold parameterized by the identity map (which we take to be positive), this definition implies*

$$\int_U f(\mathbf{x})\, dx^1 \wedge \ldots \wedge dx^n = \int_U f(\mathbf{x})\, dx^1 \ldots dx^n,$$

and so

$$\int_{\mathbf{g}(U)} \omega = \int_U \mathbf{g}^* \omega.$$

Example 4.5

1. *If $M \subset \mathbb{R}^n$ is a 1-manifold and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field then*
$$\int_M \omega_{\mathbf{F}} = \int_a^b \omega_{\mathbf{F}}(\mathbf{g}(t)) \left(\frac{d\mathbf{g}}{dt}(t) \right) dt = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \frac{d\mathbf{g}}{dt}(t)\, dt = \int_M \mathbf{F} \cdot d\mathbf{g}$$
is the line integral of $\mathbf{F}$ along M. Note that in this case the choice of orientation is the choice of the direction of the curve.

2. *If $M \subset \mathbb{R}^3$ is a 2-manifold and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then*
$$\int_M \Omega_{\mathbf{F}} = \int_U \Omega_{\mathbf{F}}(\mathbf{g}(u,v)) \left(\frac{\partial \mathbf{g}}{\partial u}, \frac{\partial \mathbf{g}}{\partial v} \right) du\, dv = \int_U \mathbf{F}(\mathbf{g}(u,v)) \cdot \left(\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right) du\, dv = \int_M \langle \mathbf{F}, \mathbf{n} \rangle$$
is the flux of $\mathbf{F}$ through M. Note that in this case the choice of orientation is the choice of the unit normal.

Proposition 4.6 *The integral of a m-form on the image of a positive parameterization of an m-manifold is well defined, that is, it is independent of the choice of parameterization.*

Proof: We start by checking that the definition is consistent for integrals on n-manifolds: if $U, V \subset \mathbb{R}^n$ are open sets, $\omega = f dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ and $\mathbf{g} : U \to V$ is a positive diffeomorphism then

$$\begin{aligned}
\int_U f(\mathbf{g}(\mathbf{t}))\, dx^1 \wedge \ldots \wedge dx^n \left(\frac{\partial \mathbf{g}}{\partial t^1}, \ldots, \frac{\partial \mathbf{g}}{\partial t^n} \right) dt^1 \ldots dt^n \\
= \int_U f(\mathbf{g}(\mathbf{t})) \det \left(\frac{\partial \mathbf{g}}{\partial t^1}, \ldots, \frac{\partial \mathbf{g}}{\partial t^n} \right) dt^1 \ldots dt^n \\
= \int_U f(\mathbf{g}(\mathbf{t}))(\det D\mathbf{g})\, dt^1 \ldots dt^n \\
= \int_U f(\mathbf{g}(\mathbf{t}))\, |\det D\mathbf{g}|\, dt^1 \ldots dt^n \\
= \int_V f(\mathbf{x})\, dx^1 \ldots dx^n
\end{aligned}$$

by the Theorem of Change of Variables.

Now let M be an oriented m-manifold, let $\mathbf{g} : U \subset \mathbb{R}^m \to M$ and $\mathbf{h} : V \subset \mathbb{R}^m \to M$ be two positive parameterizations with the same image (so that $\mathbf{g} = \mathbf{h} \circ \mathbf{k}$ for some diffeomorphism $\mathbf{k} : U \to V$), and suppose that $\omega \in \Omega^m(\mathbb{R}^n)$. Using the result above,

$$\int_U \mathbf{g}^* \omega = \int_U (\mathbf{h} \circ \mathbf{k})^* \omega = \int_U \mathbf{k}^*(\mathbf{h}^* \omega) = \int_V \mathbf{h}^* \omega.$$

□

Definition 4.7 *If $M \subset \mathbb{R}^n$ is a bounded, oriented m-manifold and $\omega \in \Omega^m(\mathbb{R}^n)$, we define*

$$\int_M \omega = \sum_{i=1}^{N} \int_{\mathbf{g}_i(U_i)} \omega,$$

where $\mathbf{g}_i : U_i \to M$ are positive parameterizations whose images are disjoint and cover M except for a finite number of manifolds of dimension smaller than m.

It can be shown that it is always possible to obtain a finite number of parameterizations of this kind, and that the definition above does not depend on the choice of these parameterizations.

Definition 4.8 *Informally, an m-**manifold with boundary** is a subset $M \subset N$ of an m-manifold $N \subset \mathbb{R}^n$ delimited by an $(m-1)$-manifold $\partial M \subset M$, called the **boundary** of M, such that $M \setminus \partial M$ is again an m-manifold. We say that M é **orientable** if N is orientable. If M is oriented, the **induced orientation** on ∂M is defined as follows: if $\mathbf{g} : U \cap \{t^1 \leq 0\} \to M$ is a positive parameterization of M such that $\mathbf{h}(t^2, \ldots, t^m) = \mathbf{g}(0, t^2, \ldots, t^m)$ is a parameterization of ∂M, then $\mathbf{h}$ is positive. Moreover, if $\omega \in \Omega^m(\mathbb{R}^n)$, we define*

$$\int_M \omega = \int_{M \setminus \partial M} \omega.$$

Remark 4.9 *A manifold is a particular case of a manifold with boundary, but a manifold with boundary is not a manifold in general.*

Theorem 4.10 (Stokes) *If $M \subset \mathbb{R}^n$ is a compact, oriented m-manifold with boundary and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M has the induced orientation.

Proof: We assume that M can be decomposed into images of cubes by positive parameterizations. Since the integrals along adjacent faces correspond to opposite orientations, and consequently cancel out, we can assume without loss of generality that M is the image of a cube.

Let $\mathbf{g} : [0,1]^m \to M$ be a parameterization. Since

$$\int_M d\omega = \int_{[0,1]^m} \mathbf{g}^*(d\omega) = \int_{[0,1]^m} d(\mathbf{g}^*\omega)$$

and

$$\int_{\partial M} \omega = \int_{\partial[0,1]^m} \mathbf{g}^*\omega,$$

it suffices to prove the Stokes Theorem in the case when $M = [0,1]^m$. If $\omega \in \Omega^{m-1}(\mathbb{R}^m)$ then

$$\omega = \sum_{i=1}^m \omega_i(\mathbf{x})\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m,$$

and so

$$d\omega = \sum_{i=1}^m (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \ldots \wedge dx^m.$$

Consequently,

$$\begin{aligned}
\int_{[0,1]^m} d\omega &= \sum_{i=1}^m (-1)^{i-1} \int_{[0,1]^m} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \ldots \wedge dx^m \\
&= \sum_{i=1}^m (-1)^{i-1} \int_{[0,1]^m} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \ldots dx^m \\
&= \sum_{i=1}^m (-1)^{i-1} \left(\int_{\{x^i=1\}} \omega_i\, dx^1 \ldots \widehat{dx^i} \ldots dx^m - \int_{\{x^i=0\}} \omega_i\, dx^1 \ldots \widehat{dx^i} \ldots dx^m \right) \\
&= \int_{\partial[0,1]^m} \omega,
\end{aligned}$$

where we used the definition of induced orientation (note that the orientation is reversed each time the coordinate functions are switched). □

Corollary 4.11 *If M is an oriented compact m-manifold (without boundary) and $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ then*

$$\oint_M d\omega = 0.$$

Example 4.12 *If M is a regular domain in $\mathbb{R}^2$ and $\omega = Pdx + Qdy \in \Omega^1(\mathbb{R}^2)$, we have*

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy,$$

and so the Stokes Theorem reduces to the Green Theorem:

$$\iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \oint_{\partial M} Pdx + Qdy.$$

Example 4.13 *The whole Vector Calculus in $\mathbb{R}^3$ can be reinterpreted in terms of differential forms:*

(i) Products:

$$\begin{aligned}
&\omega_{\phi\mathbf{F}} = \phi\,\omega_{\mathbf{F}};\\
&\Omega_{\phi\mathbf{F}} = \phi\,\Omega_{\mathbf{F}};\\
&\Omega_{\mathbf{F}\times\mathbf{G}} = \omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}};\\
&(\mathbf{F}\cdot\mathbf{G}) dx \wedge dy \wedge dz = \Omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}} = \omega_{\mathbf{F}} \wedge \Omega_{\mathbf{G}}.
\end{aligned}$$

(ii) Derivatives:

$$\begin{aligned}
&\omega_{\operatorname{grad}\phi} = d\phi;\\
&\Omega_{\operatorname{curl}\mathbf{F}} = d\omega_{\mathbf{F}};\\
&(\operatorname{div}\mathbf{F})\, dx \wedge dy \wedge dz = d\Omega_{\mathbf{F}}.
\end{aligned}$$

(iii) Integrals:

$$\begin{aligned}
\int_M \mathbf{F}\cdot d\mathbf{g} &= \int_M \omega_{\mathbf{F}};\\
\int_M \mathbf{F}\cdot \mathbf{n} &= \int_M \Omega_{\mathbf{F}}.
\end{aligned}$$

(iv) Theorems involving derivatives:

$$\begin{aligned}
\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0} \quad &\Leftrightarrow \quad d(d\phi) = 0;\\
\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0 \quad &\Leftrightarrow \quad d(d\omega_{\mathbf{F}}) = 0.
\end{aligned}$$

(v) Theorems involving integrals:

$$\begin{aligned}
\int_M \operatorname{grad}\phi\cdot d\mathbf{g} = \phi(\mathbf{b}) - \phi(\mathbf{a}) \quad &\Leftrightarrow \quad \int_M d\phi = \phi(\mathbf{b}) - \phi(\mathbf{a});\\
\iint_M \operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \oint_{\partial M} \mathbf{F}\cdot d\mathbf{g} \quad &\Leftrightarrow \quad \int_M d\omega_{\mathbf{F}} = \oint_{\partial M} \omega_{\mathbf{F}};\\
\iiint_M \operatorname{div}\mathbf{F} = \oiint_{\partial M} \mathbf{F}\cdot\mathbf{n} \quad &\Leftrightarrow \quad \int_M d\Omega_{\mathbf{F}} = \oint_{\partial M} \Omega_{\mathbf{F}}.
\end{aligned}$$

5 Vector Calculus in curvilinear coordinates

If $\mathbf{g} : U \to \mathbb{R}^n$ is a coordinate transformation then we can think of the new coordinates $t^1, \ldots, t^n$ as scalar fields on $\mathbb{R}^n$. We have

$$dt^i \left(\frac{\partial \mathbf{g}}{\partial t^j} \right) = \frac{\partial t^i}{\partial t^j} = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ (**Kronecker delta**). Defining the 1-forms

$$\omega_i = \sum_{j=1}^{n} g_{ij} dt^j,$$

where

$$g_{ij} = \frac{\partial \mathbf{g}}{\partial t^i} \cdot \frac{\partial \mathbf{g}}{\partial t^j}$$

are the components of the **metric matrix** G, we see that

$$\omega_i \left(\frac{\partial \mathbf{g}}{\partial t^j} \right) = g_{ij} = \frac{\partial \mathbf{g}}{\partial t^i} \cdot \frac{\partial \mathbf{g}}{\partial t^j},$$

that is, ω_i is the 1-form associated to $\dfrac{\partial \mathbf{g}}{\partial t^j}$.

Example 5.1 *Recall that the* **cylindrical coordinates** *in* $\mathbb{R}^3$ *correspond to the coordinate transformation* $\mathbf{g} : \mathbb{R}^+ \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ *given by*

$$\mathbf{g}(\rho, \varphi, z) = (\rho \cos \varphi, \rho \operatorname{sen} \varphi, z).$$

The metric matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so $\left\{ \dfrac{\partial \mathbf{g}}{\partial \rho}, \dfrac{\partial \mathbf{g}}{\partial \varphi}, \dfrac{\partial \mathbf{g}}{\partial z} \right\}$ *is an orthogonal basis associated to the* 1*-forms* $\left\{ d\rho, \rho^2 d\varphi, dz \right\}$*. The corresponding orthonormal basis* $\{\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{e}_z\}$ *satisfies*

$$\begin{aligned} \mathbf{e}_\rho &= \frac{\partial \mathbf{g}}{\partial \rho} \sim d\rho \sim \mathbf{e}_\varphi \times \mathbf{e}_z \sim \rho \, d\varphi \wedge dz; \\ \mathbf{e}_\varphi &= \frac{1}{\rho} \frac{\partial \mathbf{g}}{\partial \varphi} \sim \rho \, d\varphi \sim \mathbf{e}_z \times \mathbf{e}_\rho \sim dz \wedge d\rho; \\ \mathbf{e}_z &= \frac{\partial \mathbf{g}}{\partial z} \sim dz \sim \mathbf{e}_\rho \times \mathbf{e}_\varphi \sim \rho \, d\rho \wedge d\varphi \end{aligned}$$

(where we write $\mathbf{F} \sim \omega_{\mathbf{F}} \sim \Omega_{\mathbf{F}}$*). We also have*

$$\rho \, d\rho \wedge d\varphi \wedge dz = (\mathbf{e}_\rho \cdot (\mathbf{e}_\varphi \times \mathbf{e}_z)) \, dx \wedge dy \wedge dz = dx \wedge dy \wedge dz.$$

To compute, for instance,

$$\triangle \phi = \operatorname{div}(\operatorname{grad} \phi)$$

in cylindrical coordinates, we note that

$$\begin{aligned}
\operatorname{grad}\phi \sim d\phi &= \frac{\partial\phi}{\partial\rho}\,d\rho + \frac{\partial\phi}{\partial\varphi}\,d\varphi + \frac{\partial\phi}{\partial z}\,dz \\
&\sim \frac{\partial\phi}{\partial\rho}\,\rho\,d\varphi\wedge dz + \frac{\partial\phi}{\partial\varphi}\,\frac{1}{\rho}\,dz\wedge d\rho + \frac{\partial\phi}{\partial z}\,\rho\,d\rho\wedge d\varphi,
\end{aligned}$$

whence

$$\begin{aligned}
\triangle\phi\,\rho\,d\rho\wedge d\varphi\wedge dz &= d\left(\frac{\partial\phi}{\partial\rho}\,\rho\,d\varphi\wedge dz + \frac{\partial\phi}{\partial\varphi}\,\frac{1}{\rho}\,dz\wedge d\rho + \frac{\partial\phi}{\partial z}\,\rho\,d\rho\wedge d\varphi\right) \\
&= \left(\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho}\,\frac{\partial^2\phi}{\partial\varphi^2} + \rho\,\frac{\partial^2\phi}{\partial z^2}\right)d\rho\wedge d\varphi\wedge dz,
\end{aligned}$$

that is:

$$\triangle\phi = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Example 5.2 *Recall that the* **spherical coordinates** *in* $\mathbb{R}^3$ *correspond to the coordinate transformation* $\mathbf{g}:\mathbb{R}^+\times(0,\pi)\times(0,2\pi)\to\mathbb{R}^3$ *given by*

$$\mathbf{g}(r,\theta,\varphi) = (r\operatorname{sen}\theta\cos\varphi, r\operatorname{sen}\theta\operatorname{sen}\varphi, r\cos\theta).$$

The metric matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\operatorname{sen}^2\theta \end{bmatrix}$$

and so $\left\{\frac{\partial\mathbf{g}}{\partial r}, \frac{\partial\mathbf{g}}{\partial\theta}, \frac{\partial\mathbf{g}}{\partial\varphi}\right\}$ *is an orthogonal basis associated to the* 1*-forms* $\{dr, r^2 d\theta, r^2\operatorname{sen}^2\theta d\varphi\}$. *The corresponding orthonormal basis satisfies*

$$\begin{aligned}
\mathbf{e}_r &= \frac{\partial\mathbf{g}}{\partial r} \sim dr \sim \mathbf{e}_\theta\times\mathbf{e}_\varphi \sim r^2\operatorname{sen}\theta\,d\theta\wedge d\varphi; \\
\mathbf{e}_\theta &= \frac{1}{r}\frac{\partial\mathbf{g}}{\partial\theta} \sim r\,d\theta \sim \mathbf{e}_\varphi\times\mathbf{e}_r \sim r\operatorname{sen}\theta\,d\varphi\wedge dr; \\
\mathbf{e}_\varphi &= \frac{1}{r\operatorname{sen}\theta}\frac{\partial\mathbf{g}}{\partial\varphi} \sim r\operatorname{sen}\theta\,d\varphi \sim \mathbf{e}_r\times\mathbf{e}_\theta \sim r\,dr\wedge d\theta.
\end{aligned}$$

We also have

$$r^2\operatorname{sen}\theta\,dr\wedge d\theta\wedge d\varphi = (\mathbf{e}_r\cdot(\mathbf{e}_\theta\times\mathbf{e}_\varphi))\,dx\wedge dy\wedge dz = dx\wedge dy\wedge dz.$$

To compute, for instance,

$$\triangle\phi = \operatorname{div}(\operatorname{grad}\phi)$$

is spherical coordinates, we note that

$$\begin{aligned}
\operatorname{grad}\phi \sim d\phi &= \frac{\partial\phi}{\partial r}\,dr + \frac{\partial\phi}{\partial\theta}\,d\theta + \frac{\partial\phi}{\partial\varphi}\,d\varphi \\
&\sim \frac{\partial\phi}{\partial r}\,r^2\operatorname{sen}\theta\,d\theta\wedge d\varphi + \frac{\partial\phi}{\partial\theta}\,\operatorname{sen}\theta\,d\varphi\wedge dr + \frac{\partial\phi}{\partial\varphi}\,\frac{1}{\operatorname{sen}\theta}\,dr\wedge d\theta,
\end{aligned}$$

whence

$$\begin{aligned}\triangle\phi\, r^2 \operatorname{sen}\theta\, dr\wedge d\theta\wedge d\varphi &= d\left(\frac{\partial\phi}{\partial r}\, r^2 \operatorname{sen}\theta\, d\theta\wedge d\varphi + \frac{\partial\phi}{\partial\theta}\,\operatorname{sen}\theta\, d\varphi\wedge dr + \frac{\partial\phi}{\partial\varphi}\,\frac{1}{\operatorname{sen}\theta}\, dr\wedge d\theta\right)\\ &= \left(\operatorname{sen}\theta\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\,\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\operatorname{sen}\theta}\,\frac{\partial^2\phi}{\partial\varphi^2}\right) dr\wedge d\theta\wedge d\varphi,\end{aligned}$$

that is:

$$\triangle\phi = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\operatorname{sen}\theta}\,\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\,\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\operatorname{sen}^2\theta}\,\frac{\partial^2\phi}{\partial\varphi^2}.$$