INFORMAL NOTES ON HYPERKÄHLER GEOMETRY

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Abstract. Informal notes prepared for the CAMGSD working seminar on Symplectic/Contact Geometry/Topology.

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Contents

1. Lecture: Introduction - JM

Refs: [HKLR, Kr1, Kr2, Ve1, Fe1, Fe2, HT, GHJ, Pr1, Pr3, GW, Ve2, Wi1, Go, GM]
Background reading: Symplectic reduction and Kähler geometry [Ca1, Ca2]

1.1. Introduction - motivational.

As mentioned in the abstract this is an informal set of notes (in progress) prepared for the CAMGSD working seminar on Symplectic/Contact Geometry/Topology. We are all using this opportunity to learn more about the subject. Corrections/comments/suggestions are very much welcome.

A hyperkähler (HK) manifold is a (real) $4n$-dimensional Riemannian manifold with three **parallel**, **orthogonal** (which we call compatible with the metric) complex structures I, J, K[1], satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1,$$

or, in other words, it is a quintuple (M, γ, I, J, K). Recall that a Riemannian manifold is called Kähler if there is one (parallel & orthogonal) complex structure, (M, γ, I). A metric and a compatible complex structure define a symplectic structure (called Kähler form) so that a HK manifold has three symplectic structures, $\omega_1(\cdot,\cdot) = \gamma(I\cdot,\cdot)$, $\omega_2(\cdot,\cdot) = \gamma(J\cdot,\cdot)$, $\omega_3(\cdot,\cdot) = \gamma(K\cdot,\cdot)$, which are (non-degenerate closed) two-forms of type $(1,1)$ for the corresponding complex structure. Hyperkähler metrics are however much more rigid than Kähler structures [Hi4]. By adding $\partial\overline{\partial} f$, for a sufficiently small $f \in C^\infty(M)$, to a Kähler form we obtain a new Kähler form, which shows that the space Kähler metrics is infinite dimensional. Moreover it is easy to obtain examples of Kähler manifolds. Any complex submanifold of complex projective space has an induced Kähler structure. On the other hand is much more difficult to find examples of HK manifolds and even if one finds an HK metric on a compact manifold, than one can show that there is, up to isometry, only a finite dimensional space of HK metrics. An efficient way of constructing HK manifolds consists in using holomorphic symplectic reduction, which we will discuss below in section 1.2.3 and in Lecture 2.

1.1.1. Why would a differential/symplectic/algebraic geometer care?

There are many reasons for a geometer to care about HK manifolds. Let us comment briefly on some of them.

1.1.1.a) **Interesting manifolds/varieties.**

Many interesting manifolds and algebraic varieties have HK structure [Hi4, Kr3]:

(i) Resolutions of rational surface singularities and links with the McKay correspondence.

(ii) Cotangent bundles and coadjoint orbits of complex Lie groups.

[1] in fact these complex structures lead to a two-dimensional sphere of parallel complex structures, $aI + bJ + cK, a, b, c \in \mathbb{R} \ : \ a^2 + b^2 + c^2 = 1$

(iii) Spaces of representations of a Riemann surface group in a complex Lie group (related to point 1.1.1.d) below).
(iv) Space of based rational maps $f : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ of degree k.
(v) The space of based loops in a complex Lie group.

1.1.1.b) **Moduli spaces.**
Many interesting moduli spaces in geometry have hyperkähler structure and the properties of their metrics are important. For instance, non-abelian magnetic monopole moduli spaces are hyperkähler and their geodesics describe (low energy) monopole scattering [AH]. Several of these moduli spaces arise as moduli spaces of solutions of nonlinear partial differential equations related to the self-dual Yang-Mills equations. This bridges hyperkähler geometry to the world of integrable systems and soliton equations.

1.1.1.c) **Mirror symmetry.**
HK manifolds can be considered as special cases of Calabi–Yau (CY) manifolds (see section 1.2.1) and therefore they have trivial canonical bundle and the HK metrics are always Ricci flat. As we will see in the Lecture 5, for them the mirror map has (if they are smooth and compact) an important simplification: the mirrors are diffeomorphic to each other. They are very useful to test mirror symmetry in simpler cases. Examples are even (over $\mathbb{C}$) dimensional complex torii and $K3$ surfaces.
A manifestation of this (relative) simplicity concerns special lagrangian fibrations. In the Strominger, Yau and Zaslow (SYZ) formulation [GHJ] of mirror symmetry it is conjectured that if X and $\check{X}$ are a pair of mirror CY manifolds, then X should have a special Lagrangian (or SpLag) fibration (see section 1.2.2), $f : X \longrightarrow B$ (with some singular fibers), such that the mirror $\check{X}$ would be equal to an appropriate compactification of the dualizing fibration. If proved true, this conjecture would give a much wanted geometric understanding of mirror symmetry. What happens for HK manifolds is that a fibration is SpLag with respect, say, to the complex structure K if and only if is holomorphic Lagrangian with respect to $(I, \omega_J + i\omega_K)$, so that in the HK context, finding SpLag fibrations is equivalent to finding holomorphic Lagragian fibrations. This turned out to be much simpler than finding SpLag fibrations and holomorphic Lagrangian fibrations in HK manifolds give in fact, to date, the only nontrivial known SpLag fibrations in compact CY manifolds.

1.1.1.d) **Mirror symmetry, Hitchin's equations, and Langlands duality.**
The title of this point coincides with the title of Witten's paper [Wi1]. Witten and coauthors interpret the geometric Langlands duality as a particular instance of the mirror symmetry mentioned in the previous point,

for a pair of HK manifolds

$$\begin{aligned} X &= \mathcal{M}({}^LG, C) \longrightarrow B \\ \check{X} &= \mathcal{M}(G, C) \longrightarrow B, \end{aligned} \tag{1}$$

where $\mathcal{M}(G, C)$ denotes the moduli space of solutions of the Hitchin equations on the Riemann surface C with gauge group G (or equivalently the moduli space of $G_{\mathbb{C}}$–Higgs bundles on C [Go]) and ${}^LG$ denotes the Langlands dual of the group G. These moduli spaces have known holomorphic Lagrangian fibrations, the Hitchin fibrations. For the pair of compact Lie groups G and ${}^LG$, these fibrations have the same base and are dual to each other, making them an instance of SYZ mirror symmetry. The relation of the pairs of HK varieties (1) with mirror symmetry and with the geometric Langlands correpondence appeared before [KW] in eg [HT]. The novelty of [KW] for the geometric Langlands correspondence was to identify the quantum field theory (QFT) *behind* this correspondence. Besides giving a new perspective into the correspondence, this QFT interpretation leads to a significative extension of the correspondence. It is amazing that the physical theory involved is a close relative of the theory behind the 1994 revolution in four-manifold geometry: Seiberg-Witten theory. In the later case $N = 2$ super-Yang-Mills theory with group G in the (real four-)manifold M leads to the definition of Seiberg-Witten invariants of M and to their relation with the G-Donaldson invariants of M [Ig]. In the former case geometric Langlands correspondence for the Riemann surface C is "derived" from the (more rigid) $N = 4$ super-Yang-Mills theory with group G in the (real four-)manifold $M = \mathbb{R}^2 \times C$ (see [KW], specially Sect. 4). As Kapustin and Witten phrase in their paper:

Our focus in the present paper is on the geometric Langlands program for complex Riemann surfaces. We aim to show how this program can be understood as a chapter in quantum field theory.

We will expand a bit on some on these things in Lectures 4 and 5.

1.1.2. **Why would a"quantum" geometer care?** Motivated again by results in supersymmetric Yang-Mills theories, Gukov and Witten [GW, Wi2] proposed a new approach to the quantization of a symplectic manifold X. They suggest that the quantization of X can be reformulated in terms of a 2d conformal field theory (CFT): The A sigma model with target a complexification Y of X. Then, they argue that the conditions for this theory to have a good A model are equivalent to Y having a complete HK metric. Furthermore, if Y is an affine variety than they claim in the p.4 of [GW]:

Interestingly, the conditions under which deformation quantization of an affine variety gives an actual deformation of the ring of functions on Y are very similar *to the conditions for Y to admit a complete hyperkähler metric along the lines of the complete Calabi-Yau metrics constructed in [4].*

We will expand on this in Lecture 6.

1.2. Introduction - technical.

1.2.1. **Special holonomy.** We start with some classification results. Let (X, γ) be a Riemannian manifold (X, γ) and $x \in X$. For every loop $\alpha \in \mathcal{L}_x(X)$ based at x, denote by φ_α the holonomy along α, $\varphi_\alpha \in O(T_xX)$. The holonomy group, $H_\gamma = \{\varphi_\alpha, \alpha \in \mathcal{L}_x(X)\}$, acts on T_xX, $H_\gamma \subset O(T_xX)$, and the Riemannian manifold (X, γ), is called irreducible if the holonomy representation is irreducible.

Theorem 1.1 (De Rham)**.** *A compact simply connected Riemannian manifold is metrically a product of irreducible Riemannian manifolds.*

The full list of compact, simply connected, irreducible Riemannian manifolds is remarkably simple and given by Berger's theorem.

Theorem 1.2. *[Berger] Let (X, γ) be an irreducible, compact, simply connected Riemannian manifold, which is not a symmetric space. Then, the holonomy group H_γ of (X, γ) is one of the following groups:*

H_γ	$\dim_{\mathbb{R}}(X)$	Type
$SO(n)$	n	general Riemannian
$U(m)$	$2m$	Kähler
$SU(m)$	$2m$	Calabi–Yau
$Sp(k) \cdot Sp(1)$	$4k$	Quaternion–Kähler
$Sp(k)$	$4k$	hyperkähler
G_2	7	G_2–manifold
$Spin(7)$	8	$Spin(7)$–manifold

Fact. Irreducible manifolds have the additional structure making them of the type indicated in the 3d column of the table above. So, in particular, for a irreducible $Sp(k)$–manifold (X, γ), there exist 3 orthogonal, parallel complex structures, I, J, K, satisfying the quanternionic relations and making the quintuple, (X, γ, I, J, K) a HK manifold. On the other hand, for an irreducible real $4k$–dimensional H–manifold (X, γ), with $H \subset SU(2k)$ (as is the case of HK manifolds as $Sp(k) \subset SU(2k)$), the metric is Ricci-flat and the canonical bundle of a Kähler triple (X, γ, I) is trivial.

1.2.2. **Hyperkähler SYZ conjecture.** The main reference in this section is [Ve2]. Let (X, γ, I, J, K) be a HK manifold and $\omega_1, \omega_2, \omega_3$ be the corresponding symplectic forms. The following simple result is very important.

Proposition 1.3. *The (non-degenerate closed) 2–form $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$ is I–holomorphic.*

Proof. Let us first show that $\omega_{\mathbb{C}}(IY,Z) = i\omega_{\mathbb{C}}(Y,Z)$, for all vector fields $Y, Z \in \Gamma(TX^{\mathbb{C}})$.

$$\begin{aligned}\omega_{\mathbb{C}}(IY,Z) &= \omega_2(IY,Z) + i\omega_3(IY,Z) = \gamma(JIY,Z) + i\gamma(KIY,Z) = \\ &= -\omega_3(Y,Z) + i\omega_2(Y,Z) = i\omega_{\mathbb{C}}(Y,Z).\end{aligned}$$

Since for $Y \in \Gamma(T_I^{(0,1)}X)$, $IY = -iY$, we conclude that $\omega_{\mathbb{C}} \in \Omega^{(2,0)}(X)$. Being closed implies that $\overline{\partial}_I \omega_{\mathbb{C}} = 0$ and therefore $\omega_{\mathbb{C}}$ is I–holomorphic. □

Thus a HK manifold is holomorphic symplectic. Conversely, from the Calabi-Yau theorem one concludes:

Theorem 1.4. *A compact, Kähler, holomorphically symplectic manifold admits a unique HK metric in the cohomology class of its Kähler metric.*

Definition 1.5. *Let (X, ω) be a Calabi-Yau (CY) manifold with holomorphic volume form Ω. A real submanifold $Z \subset X$ is called special Lagrangian (SpLag) if $\Re\Omega|_Z$ is proportional to the Riemannian volume form.*

As mentioned in section **1.1.1.c** SpLag fibrations of CY manifolds play a crucial role in mirror symmetry. The following simple fact is then very important.

Proposition 1.6. *An I–holomorphic $\omega_{\mathbb{C}}$–Lagrangian subvariety of a HK manifold X is J–SpLag.*

Definition 1.7. *A compact HK manifold X is called simple if $H^1(X) = 0$ and $H^{(2,0)}(X) \cong \mathbb{C}$.*

Then, from Theorem 1.2 and Theorem 1.4 one obtains the Bogomolov decomposition of a compact HK manifold.

Theorem 1.8. *A compact HK manifold X admits a finite covering which is a product of a torus with several simple HK manifolds.*

The following amazing theorem is crucial.

Theorem 1.9 (Matsushita, 1997)**.** *Let X be a simple HK manifold and $\pi : X \longrightarrow B$ be a surjective, holomorphic map, with $0 < \dim(B) < \dim(X)$. Then $\dim B = \frac{1}{2}\dim X$ and the fibers of π are holomorphic Lagrangian.*

We see then that proving the existence of a SpLag fibration on a simple HK manifold X is reduced to proving the existence of a surjective holomorphic map such that the image has dimension smaller than that of X.

This leads to the HK SYZ conjecture or Huybrechts-Sawon conjecture:

Conjecture 1.10. *Let X be a compact HK manifold. Then X can be deformed to a HK manifold admitting a holomorphic Lagrangian fibration.*

Holomorphic line bundles with base-point free linear systems on a variety X are in bijective correspondence with holomorphic maps from X to projective space.

The line bundle $L \longrightarrow X$ is ample if and only if its first Chern class is in the Kähler cone. Of particular interest to the HK SYZ conjecture 1.10 are *nef* line bundles, which are those for which the first Chern class is in the closure of the Kähler cone.

Also the target of recent works is the following Beauville conjecture [GLR1]:

Conjecture 1.11. *Let X be a compact simply connected HK manifold and L a Lagrangian submanifold biholomorphic to a complex torus. Then L is a fiber of a meromorphic Lagrangian fibration, $f : X \longrightarrow B$.*

If true, this would simplify even further the task of proving the existence of SpLag fibration in a HK manifold: it would be sufficient to find a single complex Lagrangian submanifold, biholomorphic to a complex torus. The conjecture was proven in [GLR1] for non-projective X. For projective X the result is weaker.

We will return to this point in Lecture 5.

1.2.3. **Hyperkähler vs holomorphic symplectic quotient.** Let (X, ω) be a symplectic manifold with hamiltonian action of the Lie group G such that the moment map $\mu : X \longrightarrow \mathfrak{g}^*$ is equivariant. Then, for ζ a regular value $\zeta \in \mathfrak{z} \subset \mathfrak{g}^*$, where $\mathfrak{z}$ denotes the space of G–invariants, the quotient $\mu^{-1}(\zeta)/G$ has a unique symplectic form $\tilde{\omega}$, which pulls back to $\omega_{|_{\mu^{-1}(\zeta)}}$. This

$$(X, \omega, \mu) \longrightarrow (\tilde{X}, \tilde{\omega})$$
$$\tilde{X} = \mu^{-1}(\zeta)/G = X//G$$

is the usual Marsden-Weinstein symplectic quotient construction.

Suppose now X is a HK manifold with symplectic action of G, which is equivariantly hamiltonian with respect to the three symplectic forms $\omega_1, \omega_2, \omega_3$. We then have three hamiltonian maps, or equivalently a vector valued moment map

$$\mu : X \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3.$$

The structural result is then the following theorem [HKLR]

Theorem 1.12. *If $\zeta \in \mathfrak{z} \otimes \mathbb{R}^3 \subset \mathfrak{g}^* \otimes \mathbb{R}^3$ is a regular value of the HK moment map μ then*

$$M(\zeta_1, \zeta_2, \zeta_3) = X//G := \mu^{-1}(\zeta)/G = \left(\mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)\right)/G, \tag{2}$$

where $\zeta = \zeta_1 \otimes e_1 + \zeta_2 \otimes e_2 + \zeta_3 \otimes e_3$ is a HK manifold.

The analogous result in the Kähler case states that if the hamiltonian action of G preserves also the metric and thus the complex structure, then $X//G$ is Kähler and its complex structure is identified with the holomorphic quotient $X^s/G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the complexification of G with a holomorphic action extending that of G and X^s is the set of stable points, ie those whose $G_{\mathbb{C}}$ orbit intersects $\mu^{-1}(\zeta)$.

In the HK case, since the symplectic forms define the metric, a three-hamiltonian action is necessarily isometric also and preserves the three complex structures. Let us concentrate on the holomorphic structure defined by I and let

$$\mu_{\mathbb{C}} = \mu_2 + i\mu_3 \ : \ X \longrightarrow \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} \ .$$

From the proof of proposition 1.3 we obtain that

$$Y(\mu_{\mathbb{C}}(\xi)) = 0 \ , \ \forall Y \in \Gamma(T_I^{(0,1)}X)$$

and therefore $\mu_{\mathbb{C}}$ is holomorphic. It is in fact the moment map of the $\omega_{\mathbb{C}} = \omega_2 + i\omega_3-$ hamiltonian action of $G_{\mathbb{C}}$. Then, we see that

$$\mu^{-1}(\zeta) = \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3) = \mu_1^{-1}(\zeta) \cap \mu_{\mathbb{C}}^{-1}(\tilde{\zeta}),$$

where $\tilde{\zeta} = \zeta_2 + i\zeta_3 \in \mathfrak{z} \otimes \mathbb{C}$, and therefore, by taking appropriately care of stability, the complex structure on the HK quotient (2) is given by applying the holomorphic symplectic quotient construction to X,

$$(\mu_{\mathbb{C}}^s)^{-1}(\tilde{\zeta})/G_{\mathbb{C}},$$

or equivalently

$$M(\zeta_1, \zeta_2, \zeta_3) = X///G = X^s//G_{\mathbb{C}} = (\mu_{\mathbb{C}}^s)^{-1}(\tilde{\zeta})/G_{\mathbb{C}}.$$

The holomorphic symplectic description gives an efficient way of obtaining interesting HK manifolds. As Proudfoot puts it [Pr3]:

Let H [$= G_{\mathbb{C}}$] be a reductive algebraic group acting on a smooth variety V. The cotangent bundle T^*V admits a canonical holomorphic symplectic structure, and the induced action of H on T^*V is hamiltonian, that is, it admits a natural equivariant moment map $\mu : T^*V \longrightarrow \mathfrak{h}$ [$= \mathfrak{g} \otimes \mathbb{C}$]. Over the past ten years, a guiding principle has emerged that says that if M is an interesting variety which may be naturally presented as a GIT (geometric invariant theory) quotient of V by H, then the [holomorphic] symplectic quotient $\mu^{-1}(\zeta)/H$ of T^*V by H is also interesting. This mantra has been particularly fruitful on the level of cohomology, as we describe below. Over the complex numbers, a GIT quotient may often be interpreted as a Kähler quotient by the compact form [G] of H, and an algebraic quotient as a hyperkähler quotient.

Most known interesting examples of HK manifolds are in fact constructed in this way with V a complex (finite or infinite-dimensional) complex vector (or affine) space with a linear (or affine) Hamiltonian action of G. Then the action of $G_{\mathbb{C}}$ on T^*V is $h(v, w) = (hv, (h^*)^{-1}w)$, which reduces to $h(v, w) = (hv, h^{-1}w)$ if G is abelian as is the case of hypertoric varieties, which will be studied in Lecture 2.

In the non-abelian case the standard finite dimensional examples are given by quivers so that V_Q/H_Q is the moduli space of representations of a quiver Q and $T^*V_Q//H_Q$ is the HK moduli space of representations of the Nakajima double quiver associated with Q. Examples are the hyperpolygon spaces, corresponding to star-shaped quivers and also to moduli spaces of Higgs bundles on a punctured sphere [GM], $\mathcal{M}(U(n), \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \ldots, p_r\})$. The moduli spaces of Higgs bundles $\mathcal{M}(G, C)$ in general can be described by a HK reduction of a infinite dimensional space of pairs $(A, \phi) \in T^*\mathcal{A}$, where A is a connection on a complex vector bundle E on C and $\phi \in \Gamma(End(E) \otimes K_C)$. The Hitchin equations correspond to the simultaneous vanishing of all three moment maps [Go].

1.2.4. **Cotangent bundles of Kähler manifolds.** The standard references here are [Fe1, Fe2].

Let V be a Kähler manifold. On T^*V there is a canonical holomorphic symplectic structure for which the fibers of $T^*V \longrightarrow V$ and the zero section are holomorphic Lagrangian. Feix then proves.

Theorem 1.13 (Feix)**.** *Let V be a real-analytic Kähler manifold. Then, there exists a HK metric in a neighbourhood of the zero section of the cotangent bundle T^*V, which is compatible with the canonical holomorphic symplectic structure.*

She also proves no-go global theorems, like:

Proposition 1.14. *There is no complete HK metric in the cotangent bundle of a Riemann surface of genus > 1.*

Proposition 1.15. *If V is a compact complex surface admitting a complete HK metric on its cotangent bundle then there exists a finite cover $\hat{V}$ of V with abelian fundamental group.*

Proposition 1.16. *If V is a compact complex manifold of dimension n admitting a complete HK metric on its cotangent bundle then $b_1(X) \leq 4n - 2$.*

Notice that Proposition 1.14 is a simple consequence of Proposition 1.16. Feix also describes the very nice construction by Stenzel of a complete Ricci-flat Kähler metric on T^*S^n by giving its Kähler potential (p. 71).

2. Lecture: Hypertoric varieties - RSD

Background reading: Toric manifolds [Ca3, Gu, Ab]

2.1. **Preliminaries.** Recall that a toric Kähler manifolds is a Kähler manifold with a Hamiltonian action of a torus with maximal dimension. Hypertoric manifolds are the Hyperkähler analogue of this. We start with a definition.

Definition 2.1. *A hypertoric manifold is a hyperkähler manifold* (X, g, J_1, J_2, J_3) *of dimension* $4n$ *admitting an effective tri-hamiltonian* $\mathbb{T}^n$ *action.*

The fundamental building block of hypertoric manifolds is the following fundamental example.

Example 2.2. Let G be a subgroup of $U(d) = \mathrm{GL}(d,\mathbb{C}) \cap \mathrm{Sp}(2d,\mathbb{R})$ and denote its Lie algebra by $\mathcal{G}$. The group G acts on $\mathbb{C}^d$ Hamiltonianally. Let $\mu : \mathbb{C}^d \to \mathcal{G}^*$ be the moment map of the action. It is a standard fact in symplectic geometry that for "good" values of λ the manifold

$$\mathbb{C}^d//G = \mu^{-1}(\lambda)/G$$

is a symplectic manifold. In fact in "good" cases we have

$$X_{G,\lambda} = \mathbb{C}^d//G = \mathbb{C}^d_s/G^{\mathbb{C}}$$

where $\mathbb{C}^d_s$ is the set of stable points $\mathbb{C}^d$ with respect to the G action and $\mathcal{G}^{\mathbb{C}}$ is the complexification of G. This is a deep and interesting fact but we will not say more about it.

The group G also acts on $T^*\mathbb{C}^d = \mathbb{C}^d \oplus \mathbb{C}^d$ by the formula

$$g \cdot (z, w) = (gz, (g^T)^{-1}w),\ g \in G,\ (z, w) \in \mathbb{C}^d \oplus \mathbb{C}^d.$$

Now $\mathbb{C}^d \oplus \mathbb{C}^d$ is in fact hyperkähler with hyperkähler forms given by

$$\omega_{\mathbb{R}} = \frac{i}{2}\sum_k dz_k \wedge d\bar{z}_k + \frac{i}{2} dw_k \wedge d\bar{w}_k$$

and

$$\omega_{\mathbb{C}} = \sum_k dz_k \wedge dw_k.$$

Suppose the G action preserves the hyperkähler structure. It is tri-Hamiltonian and its moment maps are given by

$$\mu_{\mathbb{R}}(z, w) = \mu(z) - \mu(w),$$

and

$$\mu_{\mathbb{C}}(z, w)(v) = w(\hat{v}_z),\ v \in \mathcal{G}^{\mathbb{C}}$$

where $\hat{v}_z$ is the tangent vector induced by v on $T_z\mathbb{C}^d$. We can form the hyperkähler quotient and define the hyperkähler manifold

$$\mathcal{M}_{G,\lambda^{\mathbb{R}},\lambda^{\mathbb{C}}} = T^*\mathbb{C}^d///G = (\mu_{\mathbb{R}}^{-1}(\lambda^{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\lambda^{\mathbb{C}}))/G$$

There is a relation between the symplectic quotient and the hyperkähler quotient in this setting. Namely we have the following proposition

Proposition 2.3. *If λ is a regular value for μ and $(\lambda, 0)$ is a regular value for $(\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$ the manifold $T^*X_{G,\lambda}$ is contained as an open dense set of $\mathcal{M}_{G,\lambda,0}$ whenver it is non empty.*

Proof. This is just a stetchy sketch. We have that $X_G = \mathbb{C}^d_s / G^{\mathbb{C}}$ and therefore

$$T^*X_G = \{(z, w) : z \in \mathbb{C}^d_s, w(\hat{v}_z) = 0, \forall v \in \mathcal{G}^{\mathbb{C}}\}/G^{\mathbb{C}},$$

i.e.

$$T^*X_G = ((\mathbb{C}^d_s \oplus \mathbb{C}^d) \cap \mu_{\mathbb{C}}^{-1}(0))/G^{\mathbb{C}}.$$

Now points in $(\mathbb{C}^d_s \oplus \mathbb{C}^d) \cap \mu_{\mathbb{C}}^{-1}(0)$ are actually stable for the G action on $T^*\mathbb{C}^d$ and thus $(\mathbb{C}^d_s \oplus \mathbb{C}^d) \cap \mu_{\mathbb{C}}^{-1}(0) \subset (\mu_{\mathbb{C}}^{-1}(0))_s$. □

2.2. **Construction of hypertoric manifolds.** The construction we give here is due to Bielawski and Dancer and mimics the construction of toric manifolds as quotients of $\mathbb{C}^d$ by subtori of $\mathbb{T}^d$. We will assume some familarity with the toric version of the construction. For more details see [Gu]. Let $\mathcal{U}$ be a set of d vectors $u_1, \cdots u_d$ in $\mathbb{R}^n$.

$$\mathcal{U} = \{u_1, \cdots u_d\}.$$

Define the map $\beta : \mathbb{R}^d \to \mathbb{R}^n$ by $\beta(e_i) = u_i$ and assume there is a subtorus N of $\mathbb{T}^d$ whose lie algebra is the kernel of β. Let β^* be the dual of β, let $\iota : \ker(\beta) \to \mathbb{R}^d$ be the inclusion map and ι^* be its dual. The torus $\mathbb{T}^d$ acts Hamiltonially on $\mathbb{C}^d$ with moment map

$$\phi(z) = \sum_k |z_k|^2 e_k.$$

where we have identified $\mathbb{R}^d$ with its dual. We can restrict the $\mathbb{T}^d$ action to N. The resulting action will be Hamiltonian as well with moment map $\iota^* \circ \phi$. Set $\mu = \iota^* \circ \phi$. We denote

$$X_{\mathcal{U},\lambda} = \mathbb{C}^d//N = \mu^{-1}(\lambda)/N$$

to be the symplectic quotient of $\mathbb{C}^d$ by N. This admits a Hamiltonian $\mathbb{T}^d/N \simeq \mathbb{T}^n$ action where $n = d - \dim N$. When λ is a regular value for μ, we have

$$\dim(X_{\mathcal{U},\lambda}) = 2d - 2\dim N$$

so when $X_{\mathcal{U},\lambda}$ is a manifold it is toric. This will be the case if we take $\mathcal{U}$ to be the set of normals to a so called Delzant polytope.

If fact we can carry out this construction in the hyperkähler setting. The torus $\mathbb{T}^d$ acts on $T^*\mathbb{C}^d = \mathbb{C}^d \oplus \mathbb{C}^d$ as in the fundamental example of the previous section. The action is tri-Hamiltonian with respect to the symplectic forms on $T^*\mathbb{C}^d$. The moment maps are

$$\phi_{\mathbb{R}}(z, w) = \sum_k (|z_k|^2 - |w_k|^2) e_k,$$

and

$$\phi_{\mathbb{C}}(z, w) = \sum_k z_k w_k e_k.$$

The action restricts to N and the restricted action is again tri-Hamiltonian with moment maps $\mu_{\mathbb{R}} = \iota^* \circ \phi_{\mathbb{R}}$ and $\mu_{\mathbb{C}} = \iota^* \circ \phi_{\mathbb{C}}$. Let $\mathcal{M}_{\mathcal{U},\lambda^{\mathbb{R}},\lambda^{\mathbb{C}}}$ denote the hyperkähler quotient of $T^*\mathbb{C}^d$ by this action i.e.

$$\mathcal{M}_{\mathcal{U},(\lambda^{\mathbb{R}},\lambda^{\mathbb{C}})} = T^*\mathbb{C}^d///N.$$

This manifold admits a Hamiltonian action of $\mathbb{T}^d/N \simeq \mathbb{T}^n$ and its dimension is $4n$ hence this is a hypertoric manifold when it is smooth. Again we will need some conditions on $\mathcal{U}$ to ensure smoothness or at least some reasonable behaviour.

2.3. **Hyperplane arrangements.** Given a set $\mathcal{U}$ and a pair $(\lambda^{\mathbb{R}}, \lambda^{\mathbb{C}})$, we denote by

$$H_k^{\mathbb{R}} = \{x \in \mathbb{R}^n : x \cdot u_k = \lambda^{\mathbb{R}}\}$$

and

$$H_k^{\mathbb{C}} = \{x \in \mathbb{C}^n : x \cdot u_k = \lambda^{\mathbb{C}}\}.$$

We also set $H_k = H_k^{\mathbb{R}} \times H_k^{\mathbb{C}} \subset \mathbb{R}^n \times \mathbb{C}^n \simeq \mathbb{R}^{3n}$.

Proposition 2.4. *Let $\psi = (\psi_{\mathbb{R}}, \psi_{\mathbb{C}})$ denote the moment map for the $\mathbb{T}^n$ action on $\mathcal{M}_{\mathcal{U},\lambda}$ with respect to $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$. Then ψ is surjective and given p in $\psi^{-1}(a,b)$ the Lie algebra of the stabilizer of p is spanned by $\{u_k : (a,b) \in H_k\}$*

Proof. Again this is sketchy. We will make use of the maps $\mu = \iota^* \circ \phi$, $\mu_{\mathbb{R}} = \iota^* \circ \phi_{\mathbb{R}}$ and $\mu_{\mathbb{C}} = \iota^* \circ \phi_{\mathbb{C}}$. As in the toric case we have $\beta^* \circ \psi \circ \pi = \phi \circ \mathrm{inc}$ where

- inc is the inclusion map from $Z = \mu_{\mathbb{R}}^{-1}(\lambda^{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\lambda^{\mathbb{C}})$ in $T^*\mathbb{C}^d$ and
- $\pi : Z \to Z/N$ is the projection.

This translates into the following relations:

$$\langle \psi_{\mathbb{R}}[z,w], u_k \rangle = |z_k|^2 - |w_k|^2$$

and

$$\langle \psi_{\mathbb{C}}[z,w], u_k \rangle = z_k w_k.$$

where $[z,w] = \pi(z,w)$. Given (a,b) in $\mathbb{R}^n \times \mathbb{C}^n$ there is (z,w) such that

$$\langle a, u_k \rangle + \lambda_k^{\mathbb{R}} = |z_k|^2 - |w_k|^2$$

and

$$\langle b, u_k \rangle + \lambda_k^{\mathbb{C}} = z_k w_k.$$

This is the crucial point about this construction of hypertoric manifolds. It is a simple consequence of the fact that the triple moment map $(\phi_{\mathbb{R}}, \phi_{\mathbb{C}})$ itself is surjective. What remains to be seen is that (z,w) is in $\mu_{\mathbb{R}}^{-1}(\lambda^{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\lambda^{\mathbb{C}})$. This is just as in toric case.

$$(\phi_{\mathbb{R}}, \phi_{\mathbb{C}})(z,w) = \left(\sum_k (|z_k|^2 - |w_k|^2) e_k, \sum_k z_k w_k e_k \right)$$

i.e.

$$(\phi_{\mathbb{R}}, \phi_{\mathbb{C}})(z,w) = \left(\sum_k \langle a, u_k \rangle e_k, \sum_k \langle b, u_k \rangle e_k \right) + \left(\sum_k \lambda_k^{\mathbb{R}} e_k, \sum_k \lambda_k^{\mathbb{C}} e_k \right),$$

which gives

$$(\phi_{\mathbb{R}}, \phi_{\mathbb{C}})(z, w) = (\beta^*(a), \beta^*(b)) + \left(\sum_k \lambda_k^{\mathbb{R}} e_k, \sum_k \lambda_k^{\mathbb{C}} e_k \right),$$

thus

$$(\mu_{\mathbb{R}}(z, w), \mu_{\mathbb{C}}(z, w)) = \iota^* \circ \phi(z, w) = (\iota^* \circ \beta^*(a), \iota^* \circ \beta^*(b)) + (\lambda^{\mathbb{R}}, \lambda^{\mathbb{C}}).$$

Because $\beta \circ \iota = 0$ this shows that

$$(\mu_{\mathbb{R}}(z, w), \mu_{\mathbb{C}}(z, w)) = (\lambda^{\mathbb{R}}, \lambda^{\mathbb{C}}),$$

and the result follows.

As for the second statement notice that $(z_k, w_k) = 0$ if and only if $\langle \psi_{\mathbb{R}}[z, w], u_k \rangle - \lambda^{\mathbb{R}} = 0$ and if $\langle \psi_{\mathbb{C}}[z, w], u_k \rangle - \lambda^{\mathbb{C}} = 0$ (here we denote i.e. if and only if $\psi[z, w]$ is in H_k. Now

$$\text{Stab}[z, w] = \text{Stab}_{\mathbb{T}^d}(z, w)/\text{Stab}_N(z, w).$$

Because the Lie algebra of $\text{Stab}_{\mathbb{T}^d}(z, w)$ is $\text{span}\{e_i : z_i = w_i = 0\}$ the result follows. □

From this proposition we see that hyperplane arrangements play the part of the moment polytope. We would like to list a couple of important properties of the above construction without proof. Assume the vectors $u_1, \cdots u_d$ are primitive and that they generate $\mathbb{R}^n$.

- The manifold $\mathcal{M}_{\mathcal{U},\lambda}$ is an orbifold if and only if every intersection of $n+1$ hyperplanes H_k is empty.
- The manifold $\mathcal{M}_{\mathcal{U},\lambda}$ is smooth if it is an orbifold and in addition every non empty intersection of n hyperplanes H_k corresponds to normal vectors that generate $\mathbb{Z}^n$ i.e. if $\int_I H_i \neq$ and I has n elements then $\{u_i, i \in I\}$ is a basis of $\mathbb{Z}^n$.
- Every hypertoric manifold is equivariantly diffeomorphic to one of the $\mathcal{M}_{\mathcal{U},\lambda}$ constructed above. This result is due to Bielawski (see [Bi2]). In fact Bielawski proves more, a complete metric on the hypertoric manifold is Taub-NUT deformation equivalent to the standard metric coming from the flat metric on $T^*\mathbb{C}^d$ and descending to $\mathcal{M}_{\mathcal{U},\lambda}$. See [Bi2] for a definition of Taub-NUT deformation equivalence.

3. Lecture: Hypertoric manifolds from polytopes - RSD

Assume that $\mathcal{U}$ is the set of normals of a compact rational simple polytope P. We can then construct $X_{P,\lambda}$ the toric orbifold associated to P and a choice of λ as well as $\mathcal{M}_{P,\lambda}$.

Proposition 3.1. *T^*X_P is a open and dense in $\mathcal{M}_P$ and the two are equal if and only if X_P is a product of projective spaces.*

Proof. Sketchy sktetch again. We have seen the first part of the claim before (see 2.3). Note that in this setting it known that $\mathbb{C}^d_s = \mathbb{C}^d_P$ where

$$\mathbb{C}^d_P = \cup_F \mathbb{C}^d_F,$$

where the union is taken among all faces F of P and

$$\mathbb{C}^d_F = \{z \in \mathbb{C}^d : z_k = 0, u_k \perp F\}.$$

From what we saw in the proof of proposition 2.3, $\mathbb{C}^d_P \times \mathbb{C}^d \mu_{\mathbb{C}}^{-1}(0) \subset (\mu_{\mathbb{C}}^{-1}(0))_s$ and the inclusion is open and dense.

Next we state the following fact

Lemma 3.2. *X_P is a product of projective spaces if and only if every subset of n facets of P which intersect do so at a vertex of P.*

We will not show this in general (see [BD] for more details) but this actually easy to understand in dimension 2. In fact 2 dimensional rational simple polytopes are obtained from chopping right angle triangles or rectangles (this is equivalent to saying that compact toric surfaces with orbifold singularities are blow ups of $\mathbb{C}\mathcal{P}^2$ of of $\mathbb{C}\mathcal{P}^1 \times \mathbb{C}\mathcal{P}^1$). If the polytope actually has a chopping then two of its facets will meet outside the polytope so the only 2 dimensional polytopes that meet the condition of the lemma are (right angle) triangles and rectangles hence we need X_P to be $\mathbb{C}\mathcal{P}^2$ of of $\mathbb{C}\mathcal{P}^1 \times \mathbb{C}\mathcal{P}^1$.

It is actually easy to see that if P does not satisfy the property in lemma 3.2 then $\mathcal{M}_P$ cannot equal T^*X_P. In fact the fixed points under the torus action in T^*X_P correspond to fixed points of the torus action in X_P hence to vertices in P. As for the fixed points of $\mathcal{M}_P$ these correspond to intersections of n facets of P (for which the stabilizers have dimension n). If there are intersections of n facets which are not vertices, $\mathcal{M}_P$ will have more fixed points than T^*X_P.

Assume now that the P satisfies the property in the lemma 3.2. We will also assume that $0 \in P$ i.e. the λ_i are positive. Let $(z, w) \in \mu_{\mathbb{C}}^{-1}(0)$ then

$$\sum_k |z_k|^2 \iota^*(e_k) = \sum_k |w_k|^2 \iota^*(e_k) + \lambda^{\mathbb{R}}$$

and $\mu(z) = \sum_k |z_k|^2 \iota(e_k) = \lambda'$ where is "bigger" than λ'. What we really want to say is that z is in a polytope P' that is obtained as a dilation of P and contains P as a subset. Therefore it follows that $z \in \mathbb{C}^d_{P'}$. The point here is that $\mathbb{C}^d_{P'}$ is equal to $\mathbb{C}^d_P$ because the polytope P satisfies the property in the lemma. Hence $z \in \mathbb{C}^d_P$ and we have showed that

$$\mu_{\mathbb{C}}^{-1}(0) \subset \mathbb{C}^d_P \times \mathbb{C}^d.$$

This finishes the proof. □

3.1. **The Kähler potential in moment map coordinates.** The flat metric $T^*\mathbb{C}^d$ is $\mathbb{T}^d$ invariant thus it descends to a metric on $\mathcal{M}_{\mathcal{U}}$ which is invariant under the $\mathbb{T}^d/N \simeq \mathbb{T}^n$ action. Because of this the metric can be expressed in terms of the coordinates on the moment map image $\mathbb{R}^n \times \mathbb{C}^n$. There is an explicit formula for the Kähler potential of this metric in terms of these coordinates. Namely we have

Proposition 3.3. *Let $\mathcal{M}_{P,\lambda}$ be a hypertoric manifold from the construction above. Denote the moment map for the $\mathbb{T}^n$ action by $\psi = (\psi_{\mathbb{R}}, \psi_{\mathbb{C}})$. Let s_k and v_k be defined as $s_k = \langle \psi_{\mathbb{R}}, u_k \rangle + \lambda_k^{\mathbb{R}}$ and $v_k = \langle \psi_{\mathbb{C}}, u_k \rangle + \lambda_k^{\mathbb{C}}$. Define also r_k by the formula $r_k^2 = s_k^2 + 4 v_k \bar{v}_k$. Then*

$$\omega_{\mathbb{R}} = \bar{\partial}\partial(r_k + \lambda_k \log(r_k + s_k))$$

where ∂ and $\bar{\partial}$ are with respect to the complex structure J_1.

When X_P is a product of projective spaces this should restrict to the Guillemin metric on $X_P \subset T^* X_P$. In fact taking $w = 0$ in the above formula $v_k = \langle \psi_{\mathbb{C}}, u_k \rangle + \lambda_k^{\mathbb{C}} = z_k w_k = 0$ for all k and therefore $r_k = s_k = l_k$. Thus the Kähler potential becomes $l_k + \lambda_k \log(l_k)$. Refs: [BD, Pr1, Pr2]

4. Lecture: Nahm equations and hyperkähler manifolds - JPN

Refs: [AH, Kr2, Kr3, Hi1, Hi2, Hi3, Hi5, AB, Bi, N]

We will describe Nahm's equations, along with other interesting systems of "geometrical" PDE's, from dimensional reduction of the self-dual Yang-Mills equations. We will describe the corresponding HK moduli spaces of solutions. In particular, the moduli space of magnetic monopoles can be described in 3 equivalent ways. One of them is as a moduli space of spectral curves. The other two moduli spaces are naturally connected to the first: one is the moduli space of solutions of the Bogomolny equations in three dimensions; the other, is the moduli space of solutions of Nahm's equations with particular boundary conditions. The spectral curves appear from the side of Nahm's equations in a natural way in terms of Lax pairs. On the side of the Bogomolny equations, the spectral curves have a less common origin; they appear as a subset of the space of oriented lines in 3-dimensional space, namely corresponding to lines where certain naturally defined differential operators obtained form the Bogomolny equations have L^2 solutions in the kernel.

We will also try to cover Hitchin's description of the HK Atiyah-Hitchin metric on the moduli space of monopoles in terms of the geometry of theta functions on the spectral curves, given in [Hi5].

5. Lecture: Mirror symmetry for hyperkähler manifolds - JM

Refs: [Ve1, Hu, GHJ, GLR1, GLR2, Ve2, Ve2]

6. Lecture: Gukov-Witten quantization - JE

Refs: [GW, Wi2]

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