# Imaginary time flow in geometric quantization, degeneration to real polarizations and tropicalization

**José Mourão**

Técnico Lisboa, U Lisboa

**WaGaRy – Workshop on Differential Geometry**

**Institute of Mathematics**
**Polish Academy of Sciences**

December, 18, 2013, Warsaw

work in collaboration with João Nunes (Técnico, Univ Lisboa)
& Thomas Baier (Univ Porto), Carlos Florentino (Técnico, Univ Lisboa), Will Kirwin (Univ Köln) and Siye Wu (Hong Kong Univ)

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## Summary

We will discuss three main topics:

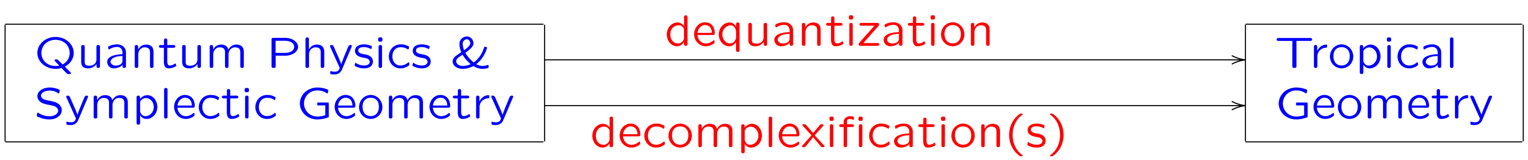
I. Hamiltonian complex time evolution in Kähler geometry and in quantum physics

II. Problem that motivated us initially: For a completely integrable system $(M, \omega, \mu = (H_1, \ldots, H_n) \ : \ M \to \mathbb{R}^n)$ define its quantization in the (usually singular) real polarization $\mathcal{P}_\mu$ defined by $\mu$,

$$\mathcal{H}^Q_{\mathcal{P}_\mu} = ??$$

## III. New relation:

[for (some?) completely integrable systems $(M, \omega, \mu \ : \ M \to \mathbb{R}^n)$]




Dequantization — $\hbar \to 0$

Decomplexifications — (Kähler) Geometric degenerations to tropical varieties
= geometric tropicalization:
=: Follow (to infinite time) a geodesic ray,
in the space of Kähler metrics,
generated by $H = ||\mu||^2 = H_1^2 + \cdots + H_n^2$,
= Do a Wick rotation, time $\rightsquigarrow it$, followed by $t \to \infty$
for $H = ||\mu||^2$

# 1. Preliminaries

## 1.1. Integrable systems: general

A Liouville (completely) integrable system is a symplectic manifold $(M^{2n}, \omega)$ plus a nice, free, in a dense subset, Hamiltonian action of $K = \mathbb{R}^k \times \mathbb{T}^{n-k}$ with moment map

$$\mu = (H_1, \ldots, H_n) \ : \ M^{2n} \to \mathbb{R}^n$$

defining, in a dense subset, a regular Lagrangian fibration.

If the regular level sets of $\mu$ are compact then they are $n$-dimensional torii or unions of $n$-dimensional torii.

**Theorem**[Arnold–Liouville]
Let $\eta_0 \in \mathbb{R}^n$ be a regular value of $\mu$ and $m_0 \in \mu^{-1}(\eta_0)$. Then there is a $K$–invariant neighborhood $U$ of $m_0$ and coordinates $\alpha = (\alpha_1, \ldots, \alpha_n)$ on the fibers of the moment map on $U$, canonically conjugate to $H = (H_1, \ldots, H_n)$, i.e. such that,

$$\omega = \sum_{j=1}^{n} dH_j \wedge d\alpha_j$$

Notice that $X_{H_j} = -\frac{\partial}{\partial \alpha_j}$ and $X_{\alpha_j} = \frac{\partial}{\partial H_j}$ so that, in particular, if

$h = ||\mu||^2/2 = \frac{1}{2}(H_1^2 + \cdots + H_n^2)$, then

$$X_h = -H_1 \frac{\partial}{\partial \alpha_1} - \cdots - H_n \frac{\partial}{\partial \alpha_n}$$

and

$$(\varphi_t^{X_h})^*(f)(H, \alpha) = f(H, \alpha - tH)$$

Naturally then, if $f$ is analytic on $\alpha$, we will be able to analytically continue this result and get

$$(\varphi_\tau^{X_h})^*(f)(H, \alpha) = f(H, \alpha - \tau H)$$

for sufficiently small $\tau \in \mathbb{C}$.

**Remark.** So some fibers of $\mu$ may be singular but $\mu$ is *usually* assumed to be a smooth map to $\mathbb{R}^n$ thus defining $n$ smooth global functions $H_1, \ldots, H_n$. Cases in which one drops smoothness of $\mu$ are nevertheless very interesting and have been the focus of much recent interest.

**Examples:**

**Ex 1.** $C^0$–case, i.e. $H_j \in C^0(M)$, as the Gel'fand-Cetlin systems [Guillemin-Sternberg, J. Funct. Anal. 1983] and the integrable systems associated with Okounkov bodies [Harada-Kaveh, arXiv:1205.5249].

**Ex 2.** Discontinuous examples appear in torii $\mathbb{T}^{2n}$ and in moduli spaces of flat connections on Riemann surfaces [Jeffrey-Weitsman, Comm. Math. Phys., 1992].

## 1.2. Integrable systems: Symplectic toric manifolds

Toric Kähler manifolds provide the simplest, yet quite rich examples of Liouville integrable systems $(X, \omega, \mu = (H_1, \ldots, H_n))$. These are the cases in which $\mu$ generates an effective action of $\mathbb{T}^n$. These manifolds correspond to (partial) equivariant compactifications of the complex torus $(\mathbb{C}^*)^n$ and a very convenient way of studying their infinite dimensional space of Kähler structures is with the help of the following diagram.

$$\begin{array}{rcccl} \psi_{\mathsf{st}} & : & T^*\mathbb{T}^n & \longrightarrow & T\mathbb{T}^n \longrightarrow (\mathbb{C}^*)^n \\ & & (\theta, H) & \mapsto & (\theta, H) \mapsto e^{i\theta} e^{H} \end{array} \tag{1}$$

Notice the following trivial, yet important facts:

**1.** $T^*\mathbb{T}^n$ has a standard symplectic structure and a free hamiltonian action of $\mathbb{T}^n$, $e^{i\beta} \cdot (\theta, H) = (\theta + \beta, H)$, with moment map $\mu(\theta, H) = H$.

**2.** $(\mathbb{C}^*)^n$ has a standard complex structure and a free action of $\mathbb{T}^n$ (as subgroup), $e^{i\beta} \cdot w = (e^{i\beta_1} w_1, \ldots, e^{i\beta_n} w_n)$.

**3.** The diffeomorphism $\psi_{\mathsf{st}}$ is $\mathbb{T}^n$–equivariant and induces a $\mathbb{T}^n$–invariant complex structure on $T^*\mathbb{T}^n$ and a $\mathbb{T}^n$–invariant symplectic structure on $(\mathbb{C}^*)^n$ making the $\mathbb{T}^n$ action Hamiltonian with moment map $\widehat{\mu}(w) = \mu \circ \psi_{\mathsf{st}}^{-1}(w) = (\log |w_1|, \ldots, \log |w_n|)$. The resulting structure on both sides is a $\mathbb{T}^n$–invariant Kähler structure.

$$\begin{array}{rcl}\psi_{\mathsf{st}} \; : \; (T^*\mathbb{T}^n, \omega_{\mathsf{st}}, \psi_{\mathsf{st}}^*(\widehat{J}_{\mathsf{st}})) & \longrightarrow & ((\mathbb{C}^*)^n, (\psi_{\mathsf{st}}^{-1})^*(\omega_{\mathsf{st}}), \widehat{J}_{\mathsf{st}}) \\ (\theta, H) & \mapsto & e^{i\theta}\, e^{H} \end{array} \tag{2}$$

While the second map in (1) is standard (the inverse of the polar decomposition of complex numbers) the first can be changed, preserving $\mathbb{T}^n$ equivariance. We thus change the Kähler structure on either side of (2) by changing the fiber part of the map $\psi_{\mathsf{st}}$. That freedom is equivalent to the choice of a $\mathbb{T}^n$–invariant Kähler potential on $(\mathbb{C}^*)^n$. The *standard* one corresponding to (2) reads

$$k_{\mathsf{st}}(w) = \frac{1}{2}\log^2|w_1| + \cdots + \frac{1}{2}\log^2|w_n|$$

The starting point to describe the other $\mathbb{T}^n$–invariant Kähler structures is then $((\mathbb{C}^*)^n, \widehat{J}_{\mathrm{st}})$ and a $\mathbb{T}^n$–invariant Kähler potential (i.e. a strictly plurisubharmonic function), $k$. Let $u = \mathrm{Log}(w) = (\log(|w_1|), \ldots, \log(|w_n|))$ so that, $k(w) = \widetilde{k}(u)$ and

$$\widehat{\omega} = \frac{i}{2}\partial\bar{\partial}\, k = \sum_{l,j} \mathrm{Hess}(\widetilde{k})_{lj}\ du_l \wedge d\theta_j$$

with $\mathrm{Hess}(\widetilde{k})$ positive definite. Introducing $H = (H_1, \ldots, H_n) = \frac{\partial \widetilde{k}}{\partial u}$, we obtain

$$\widehat{\omega} = \sum_j\ dH_j \wedge d\theta_j$$

and therefore the map

$$\begin{array}{rcl} \psi^{-1} \; : \; ((\mathbb{C}^*)^n, \widehat{\omega}) & \longrightarrow & (T^*\mathbb{T}^n, \omega_{\text{st}}) \\ w = e^{i\theta} e^{u} & \mapsto & (\theta, \dfrac{\partial \widetilde{k}}{\partial u}) \end{array} \tag{3}$$

is a symplectomorphism to its image and $\widehat{\mu}(w) = \frac{\partial \widetilde{k}}{\partial u} = H(w)$, is the moment map of the $\mathbb{T}^n$ action on $((\mathbb{C}^*)^n, \widehat{\omega})$. Having invariant divisors $D_j$ to which the Kähler structure $((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\text{st}})$ extends implies loss of surjectivity of the Legendre transform of the fibers in (3): we get $\ell_{F_j}(H) = \nu_{F_j} \cdot H + \lambda_{F_j} \geq 0$, where $\nu_{F_j}$ is the integral vector associated with $D_j$ in the fan of the resulting toric variety $\widehat{X}$.

If the corresponding Kähler manifold $(\widehat{X}, \widehat{\omega}, \widehat{J}_{\mathsf{st}}) \supset ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\mathsf{st}})$ is compact then the image of the fibers is a Delzant polytope P (Atiyah–Guillemin–Sternberg and Delzant theorems [Atiyah, Bull. LMS, 1982, Delzant, Bull. SMF, 1988, Guillemin-Sternberg, Invent. Math, 1982]

$$P = \{H \in \mathbb{R}^n \ : \ \ell_F(H) = \nu_F(H) \cdot H + \lambda_F \geq 0, F \text{ facet of } P\}$$

We get

$$(\widehat{X}_P, \hat{\omega}, \hat{J}_{\mathsf{st}}) \supset ((\mathbb{C}^*)^n, \hat{\omega}, \hat{J}_{\mathsf{st}}) \xrightarrow{\psi^{-1}} (\mathbb{T}^n \times \breve{P}, \omega_{\mathsf{st}}, J) \subset (X_P, \omega_{\mathsf{st}}, J)$$
$$w = e^{i\theta}e^u \quad \mapsto \quad (\theta, \frac{\partial \widetilde{k}}{\partial u}), \tag{4}$$

where $\breve{P}$ denotes the interior of $P$.
The map $\psi^{-1}$ extends to a Kähler isomorphism, $\psi^{-1} \ : \ (\widehat{X}_P, \hat{\omega}, \hat{J}_{\mathsf{st}}) \longrightarrow (X_P, \omega_{\mathsf{st}}, J)$. The inverse $\psi$ of the map in (4) is also given by a Legendre transform associated with the so called symplectic potential, $g(H) = u \cdot H - \widetilde{k}(u)$.

$$(X_P, \omega_{\mathsf{st}}, J) \supset (\mathbb{T}^n \times \breve{P}, \omega_{\mathsf{st}}, J) \xrightarrow{\psi} ((\mathbb{C}^*)^n, \hat{\omega}, \hat{J}_{\mathsf{st}}) \subset (\widehat{X}_P, \hat{\omega}, \hat{J}_{\mathsf{st}})$$
$$(\theta, H) \quad \mapsto \quad e^{i\theta} e^{\frac{\partial g}{\partial H}} \tag{5}$$

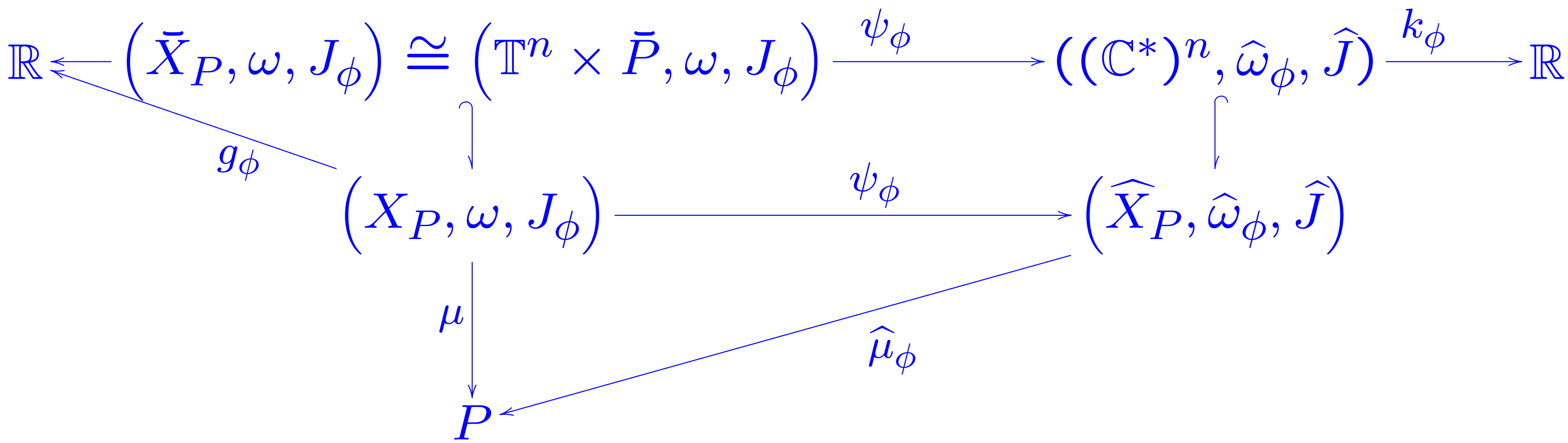
Then, Guillemin–Abreu theory [Abreu, Fields Inst. Commun., 2003] tells us that $\mathbb{T}^n$–invariant Kähler structures on $(X_P, \omega_{\mathrm{st}})$ are "parametrized" by symplectic potentials

$$g_\phi(H) = g_P(H) + \phi(H) = \sum_{F\subset P} \frac{1}{2}\ell_F(H)\log\ell_F(H) + \phi(H),$$

where $P = \{H \in \mathbb{R}^n \ : \ \ell_F(H) = \nu_F(H)\cdot H + \lambda_F \geq 0, F \text{ facet of } P\}$ and $\phi \in C^\infty(P) \ : \ \mathrm{Hess}(g_P + \phi) > 0$.

Notice that $g_P \in C^0(P)$.

We get the following diagram




$$
\begin{aligned}
\psi_\phi(\theta, H) &= e^{u_\phi + i\theta} = e^{\partial g_\phi / \partial H + i\theta} \\
g_\phi(H) &= g_P(H) + \phi(H) = \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi \\
\mu(\theta, H) &= H \; - \text{moment map in the } \textit{symplectic} \text{ picture} \\
\widehat{\mu}_\phi(w) &= \mu \circ \psi_\phi^{-1}(w) \; - \text{moment map in the } \textit{complex} \text{ picture}
\end{aligned}
$$

# 2. Complex time evolution

## 2.1 Imaginary time: introduction

Geometric quantization

$$(M,\omega), \qquad \frac{1}{\hbar}[\omega] \in H^2(M,\mathbb{Z})$$

Prequantum data: $(L,\nabla,h)$, $L \to M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M,L) = \overline{\left\{ s \in \Gamma^\infty(M,L) : ||s||^2 = \int_M h(s,s)\, \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f} = Q_\hbar(f) = -i\hbar\nabla_{X_f} + f$

Kähler quantization: fix a complex structure $I$ on $M$ such that $(M, \omega, I, \gamma)$ is a Kähler manifold.

$$\mathcal{H}_I^Q = \left\{ s \in \mathcal{H}^{\mathsf{prQ}} \ : \ \nabla_{\bar\partial}\, s = 0 \right\} = H^0(M, L_I)$$

Get the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

Need to study the dependence of quantization on the choice of the complex structure.

It is precisely to study the dependence of $Q_\hbar$ on the choice of the complex structure that evolution in imaginary time enters the scene.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

What we are studying is a new way of looking at imaginary time evolution in (some situations in) quantum mechanics giving it a precise geometric meaning.

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in order to transform the spin connection to the Ashtekar connections $\Gamma_\mu \mapsto A^{\mathbb{C}}_\mu$. [Thiemann, Class. Quant. Grav., 1996]

In non-Hermitian Quantum Mechanics one considers time evolution associated with complex valued observables [Moiseyev, Cambridge University Press, 2011]

In $\mathcal{PT}$–symmetric quantum mechanics one deals with complex valued observables which may sometimes have meaningful real spectrum – Bessis and Zinn-Justin example: $H = p^2 + ix^3$ [Dorey-Dunning-Tateo, J.Phys.A, 2007, Section 2]

In Geometric Quantization [Hall-Kirwin, Math. Ann. 2011, Kirwin-Mourão-Nunes, J. Funct. Anal. 2013, Burns-Upercio-Uribe, 2013, Mourão-Nunes, 2013]

In Kähler Geometry imaginary time evolution leads to geodesics and is used to study the stability of varieties [Semmes, American J. Math., 1992 and Donaldson, AMS Transl. (2), 1999].

## 2.2 Complex time evolution in Kähler geometry

Compact simple Lie groups have a natural unique bi-$G$-invariant metric of constant positive curvature.

Example: $G = SU(2) \cong (SU(2) \times SU(2))\,/SU(2) \cong S^3$

The dual is $G' = G_{\mathbb{C}}/G$. It has a natural unique $G$-invariant metric of constant negative curvature.

Example: $SU(2)' = SL(2,\mathbb{C})/SU(2) \cong \mathbb{H}^3$

sectional curvature

$\xrightarrow{G}$ $$K_G(\xi,\eta) = \tfrac{1}{4}||[\xi,\eta]||^2$$

$\xrightarrow{G'}$ $$K_{G'}(\xi,\eta) = -\tfrac{1}{4}||[\xi,\eta]||^2$$

$$T_e(G\times G) = \mathcal{G}\oplus\mathcal{G} \quad \Rightarrow \quad T_{[e]}(G) \cong \mathcal{G}$$
$$T_e(G_{\mathbb{C}}) = \mathcal{G}\oplus i\mathcal{G} \quad \Rightarrow \quad T_{[e]}(G) \cong i\mathcal{G}$$

$G = \mathsf{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with Lie(Ham$(M, \omega)) = C^\infty(M)$

$G_{\mathbb{C}} = \mathsf{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits. There are natural orbits of two types passing through a Kähler pair $(\omega, J_0)$.

## Complex picture

Let us start with the smaller one

$$\begin{aligned}\mathcal{H}(\omega, J_0) &= \left\{ f \in C^\infty(M) \;:\; \omega + i\bar{\partial}_0 \partial_0 f > 0 \right\} / \mathbb{R} =: \\ &=: \{(\varphi^* \omega, J_0), \varphi \in G_{\mathbb{C}}\} = (G_{\mathbb{C}} \cdot \omega, J_0) \cong G_{\mathbb{C}}/G\end{aligned}$$

So we get a orbit-dependent "definition" of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of Diff$(M)$

$\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space: $\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ and Donaldson shows that the Mabuchi metric has constant negative curvature mimiking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \mathrm{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that $\mathrm{Aut}_0(M, J_0)$ is trivial the second natural orbit is the total space of a $G$ principal bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0),\ \varphi \in G_{\mathbb{C}}\}$$

with projection

$$\begin{aligned} \pi \ :\ \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) \ &\longrightarrow\ \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \\ \pi(\omega, \varphi_* J_0) \ &=\ \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{aligned}$$

## Gröbner Lie series

Let $(M, \omega)$ be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\text{an}}(M)$

**Theorem [Gröbner]**

Let $f \in C^{\text{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} \; : \; f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{6}$$

converges absolutely, for $|\tau| < T_f$, to a (complex valued) real analytic function.

So $\text{Ham}_{\mathbb{C}}(M, \omega)$ acts locally on $C^{\text{an}}(M)$. To descend to an "action" on $M$ we need to fix a complex structure.

Let $(M, \omega, J_0)$ be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local $J_0$-holomorphic coordinates. Then we use (6) to define the holomorphic coordinates of a new complex structure $J_\tau$

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \qquad (7)$$

Theorem [Burns-Lupercio-Uribe, 2013; M-Nunes, 2013]
Under natural conditions on $J_0$ the relations (7) define a unique map $\varphi_\tau^{X_h} : M \to M$ such that

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

For $\tau$ sufficiently small $\varphi_\tau^{X_h} : (M, J_\tau) \to (M, J_0)$ is a biholomorphism.

**Example 1** - $M = \mathbb{R}^2$

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_0)$ and $h = y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then

$\varphi_t^{X_h}(x, y) = (x + ty, y)$

and

$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$

Indeed: $e^{ity\frac{\partial}{\partial y}}(x + iy) = x + i(1 + t)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow \nabla^{\gamma_{it}} h = J_{it} X_h = \frac{1}{1+t}\, y\frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t} dx^2 + (1 + t) dy^2$.

## Example 2 - Toric Varieties

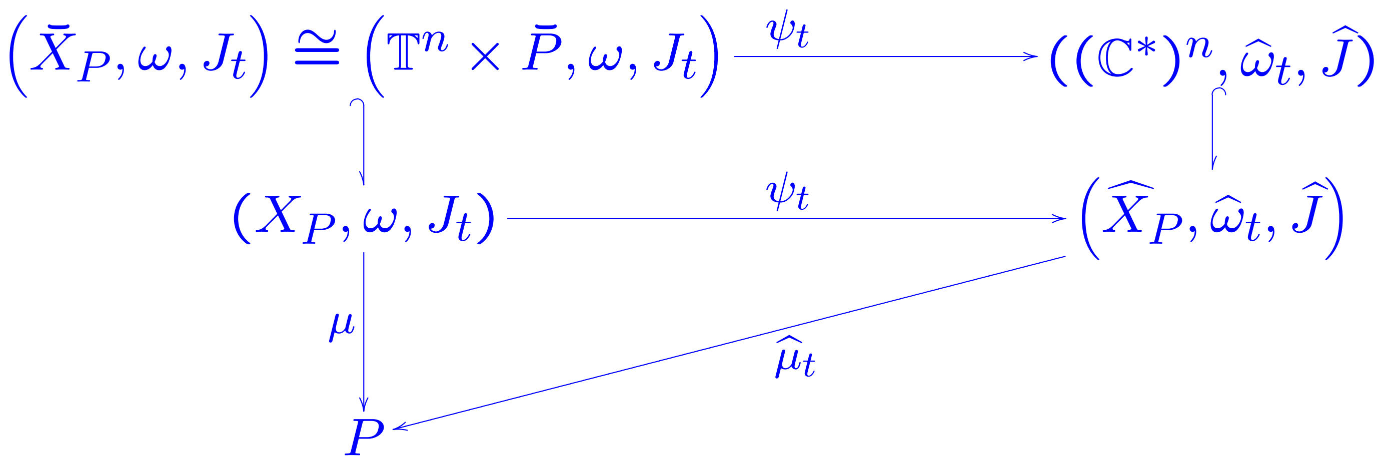
For a toric manifold let $h$ be $\mathbb{T}^n$–invariant, $h(H)$. Then, $X_h = -\sum_j \frac{\partial h}{\partial H_j}\frac{\partial}{\partial \theta_j}$ and

$$w_{it} = e^{itX_h}\, w = e^{itX_h}\, e^{\frac{\partial g}{\partial H}+i\theta} = e^{\frac{\partial g}{\partial H}+i(\theta - it\frac{\partial h}{\partial H})} = e^{\frac{\partial (g+th)}{\partial H}+i\theta} = e^{t\frac{\partial h}{\partial H}}\, w,$$

so that the change of the symplectic potential under imaginary time toric flow is that of a geodesic in flat space

$$g \rightsquigarrow g_t = g + th$$

Thus we get




$$\begin{aligned}
\psi_t(\theta, x) &= e^{u_t + i\theta} = e^{\partial g_t/\partial H + i\theta} \\
g_t(H) &= \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H) + t\, h(H) \\
\mu(\theta, H) &= H \;-\; \text{moment map in the } \textit{symplectic} \text{ picture} \\
& \quad \varphi_{it}^{X_h} = \psi_0^{-1} \circ \psi_t
\end{aligned}$$

## Example 3 - Complex reductive groups $K_{\mathbb{C}}$

The "adapted" Kähler structure is [Guillemin-Stenzel, Lempert-Szöke, 1991]

$$\begin{array}{ccc} (T^*K, \omega, J_0) & \stackrel{\psi_0}{\longrightarrow} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_0) \\ (x, Y) & \mapsto & xe^{iY}, \end{array} \qquad (8)$$

where we used $T^*K \cong TK \cong K \times \text{Lie}K$

General bi-$K$-invariant Kähler structures on $T^*K$ are in one-to-one correspondence with strictly convex $Ad$–invariant "symplectic potentials" $\breve{g}$ on Lie$K$ [Kirwin-M-Nunes, 2013].

They define and are defined by their (Weyl–invariant) restriction $g$ to the Cartan subalgebra. One has

$$\begin{array}{rcl}(T^*K, \omega, J_{\breve{g}}) & \xrightarrow{\psi_{\breve{g}}} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_{\breve{g}}) \\ (x, Y) & \mapsto & xe^{iu(Y)}, \quad u = \dfrac{\partial \breve{g}}{\partial Y}\end{array}$$

The adapted Kähler structure corresponds to $\breve{g} = ||Y||^2/2$.

Choosing an $h$ bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial y_j} X_j$, where $Y = \sum_j y_j T_j$. Then

$$e^{itX_h} \cdot xe^{iY} = xe^{i(Y + t\frac{\partial h}{\partial Y})} = xe^{i\frac{\partial \breve{g}_t}{\partial Y}},$$

where $\breve{g}_t = \frac{||Y||^2}{2} + th$.

Notice that if instead of $g(Y) = ||Y||^2/2$ on the dual of the Cartan subalgebra $\mathcal{H}^*$ we choose a Guillemin–Abreu symplectic potential on a Weyl–invariant Delzant polytope $P \subset \mathcal{H}^*$, then the Kähler structure defined by (8) on $(K_{\mathbb{C}}, \hat{\omega}, \hat{J}_{\check{g}})$ extends to an equivariant compactification $\overline{K_{\mathbb{C}}}$ of $K_{\mathbb{C}}$, having $\widehat{X_P}$ as a toric subvariety, corresponding to the compactification of the Cartan torus. In particular, if we take $K_{\mathbb{C}}$ of adjoint type and $P$ has vertices given by the Weyl orbit of a dominant weight then $\overline{K_{\mathbb{C}}}$ is a wondeful compactification. The imaginary time flow of a bi–$K$–invariant function acts in a similar way

$$e^{itX_h} \cdot xe^{i\frac{\partial \check{g}}{\partial Y}} = xe^{i\frac{\partial \check{g}_t}{\partial Y}},$$

where $\check{g}_t = \check{g} + th$.

## 2.3 Complex time evolution in quantization

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{prQ} \overset{J_0}{\rightsquigarrow} \mathcal{H}^{Q}_{J_0} = H^0_{J_0}(M,L) = \left\{ s \in \mathcal{H}^{prQ} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned}\mathcal{H}^{Q}_{J_0} &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow < \frac{\partial}{\partial \bar{z}^j_{J_0}} > = \mathcal{P}_{J_0} \\ &= \text{space of sections depending only on } z_{J_0} \\ &\quad (= \text{i.e. } J_0\text{-holomorphic})\end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$
\begin{aligned}
\mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} < \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} > \\
&= e^{\tau \mathcal{L}_{X_h}} < X_{z^1}, \ldots, X_{z^n} > = < X_{e^{\tau X_h}(z^1)}, \ldots, X_{e^{\tau X_h}(z^n)} > = \\
&= < X_{z_\tau^1}, \ldots, X_{z_\tau^n} > = (\varphi_\tau^{X_h})^*(\mathcal{P}_0)
\end{aligned}
$$

For some values of $\tau \neq 0$ the polarization may be real or mixed. Recall that mixed polarizations are spanned by real vector fields at every point.

In our flat example, $h = y^2/2, \tau = it$,

$$
\begin{aligned}
\mathcal{P}_{it} &= < \frac{\partial}{\partial \bar{z}_{it}} >=< X_{z_{it}} >=< X_{x+i(1+t)y} >= \\
&= < -\frac{\partial}{\partial y} + i(1+t)\frac{\partial}{\partial x} > \stackrel{t\to -1}{\longrightarrow} < \frac{\partial}{\partial y} >= \mathcal{P}_{-i}
\end{aligned}
$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}^Q_{it}$ − holomorphic functions of $z_{it}$ − Fock-like rep.

$t = -1 \Rightarrow \mathcal{H}^Q_{-i}$ − $L^2$-functions of $x$ − Schrödinger representation

If two quantum quantum theories – with Hilbert spaces $\mathcal{H}_0^Q$ and $\mathcal{H}_\mathcal{T}^Q$ – are equivalent there must be a unitary operator

$$U_\mathcal{T} \; : \; \mathcal{H}_0^Q \longrightarrow \mathcal{H}_\mathcal{T}^Q$$

intertwining the representations of relevant observable algebras.

## 3. Infinite imaginary (=geodesic) time & defintion of $\mathcal{H}^Q_{\mathcal{P}_\mu}$

One problem that motivated us initially was the problem that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic orcillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ is the moment map of a $\mathbb{R}^n$ action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu =< X_{H_1}, \ldots, X_{H_n} >$$

So that $\mathcal{H}^Q_{\mathcal{P}_\mu} =??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations $\mathcal{P}_t$ degenerating to $\mathcal{P}_\mu$ as $t \to \infty$.

Theorem [M-Nunes, 2013]
For a class of completely integrable systems $(M,\omega,\mu)$, $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

**1.**(proved for big class)
$\lim_{t\to\infty}(\varphi_{it}^{X_h})^*\mathcal{P}_0 = \mathcal{P}_\mu$.
**2.** (so far proved for much smaller class)
The family $\mathcal{P}_t = (\varphi_{it}^{X_h})^*\mathcal{P}_0$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M,L)$, with a $L^2$-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty}\sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^Q_{\mathcal{P}_\mu} = \text{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

# 4. Geometric tropicalization of varieties and divisors

## 4.1. Tropicalization: flat case

Recall the origin of tropical geometry

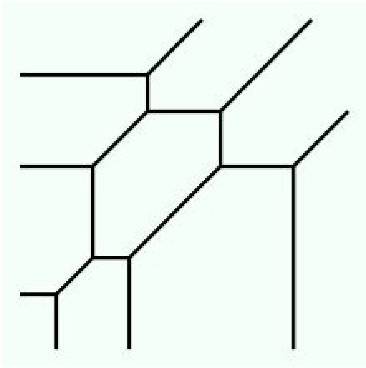
$$\widehat{Y} = \left\{ w \in (\mathbb{C}^*)^n \ : \ \sum_{m \in P} c_m \, w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\mathrm{Log}_T} & \mathbb{R}^n \\ \mathrm{Log}_T(w) & = & \dfrac{1}{t}(\log|w_1|, \ldots, \log|w_n|) \end{array}$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T \to \infty} \mathrm{Log}_T(\widehat{Y})$

An example for a generic degree 3 plane curve is:



Some geometric and enumerative properties become very simple in tropical geometry.

Simple examples for plane curves are the following. Let $C, \widetilde{C}$ be nonsingular plane curves.

Genus–degree formula

$$g(C) = \frac{1}{2}(d(C) - 1)(d(C) - 2)$$

Bezout Theorem

$$|C_1 \cap C_2| = d(C_1)\ d(C_2)$$

Let us make a geodesic interpretation of the $T \to \infty$ limit.

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J) & \xrightarrow{\psi} & ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}) \\ (\theta, H) & \mapsto & e^{H+i\theta} \end{array}$$

We see that the initial moment map $\widehat{\mu}_0 \ : \ (\mathbb{C}^*)^n \to \mathbb{R}^n$ of the torus action is

$$\widehat{\mu}_0 = \mathrm{Log}_1(w) = H$$

Now let us consider a geodesic starting at $\widehat{\omega}_0$ in the direction of $h = (H_1^2 + \cdots + H_n^2)/2$ so that $X_h = -H \cdot \frac{\partial}{\partial\theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h}\, e^{H+i\theta} = e^{(1+\tilde{t})H+i\theta} = e^{tH+i\theta}$$

with $t = \tilde{t} + 1$. We have

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}_0) \\ (\theta, H) & \mapsto & e^{tH+i\theta} \end{array}$$

As $\widehat{\omega}_t = \frac{1}{t} du \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t}\, u$ so that

$$\widehat{\mu}_t(w) = \mathsf{Log}_T(w) = H$$

We see that, in this interpretation, the $T$ of amoeba theory is exponential of the geodesic time, $T = e^t$, and the $\mathsf{Log}_T$ map is the moment map of the complex picture.

## 4.2. Definition of Geometric Tropicalization

**Definition**

Tropicalization (Liouville integrable) Kähler manifolds

Let $(\widehat{M}, \widehat{\omega}_t, I, \widehat{\gamma}_t)$ be a $||\mu||^2$-geodesic ray. The geometric tropicalization of $(\widehat{M}, \widehat{\omega}_0, \widehat{I}_0, \widehat{\gamma}_0)$, in the direction of $||\mu||^2$, is the Gromov-Hausdorff limit

$$(M_{\text{gtrop}}, d_{\text{gtrop}}) = \lim_{t\to\infty} (\widehat{M}, \widehat{\gamma}_t)$$

$$M_{\text{gtrop}} = \mu(M) \text{ if the fibers are connected}$$

## Tropicalization of divisors (for connected fibers)

The geometric tropicalization of an hypersurface $\widehat{Y} \subset \widehat{M}$, in the direction of $||\mu||^2$, is the Hausdorff limit of $\widehat{\mu}_t(\widehat{Y})$ in $\mu(M) = M_{\text{gtrop}}$,

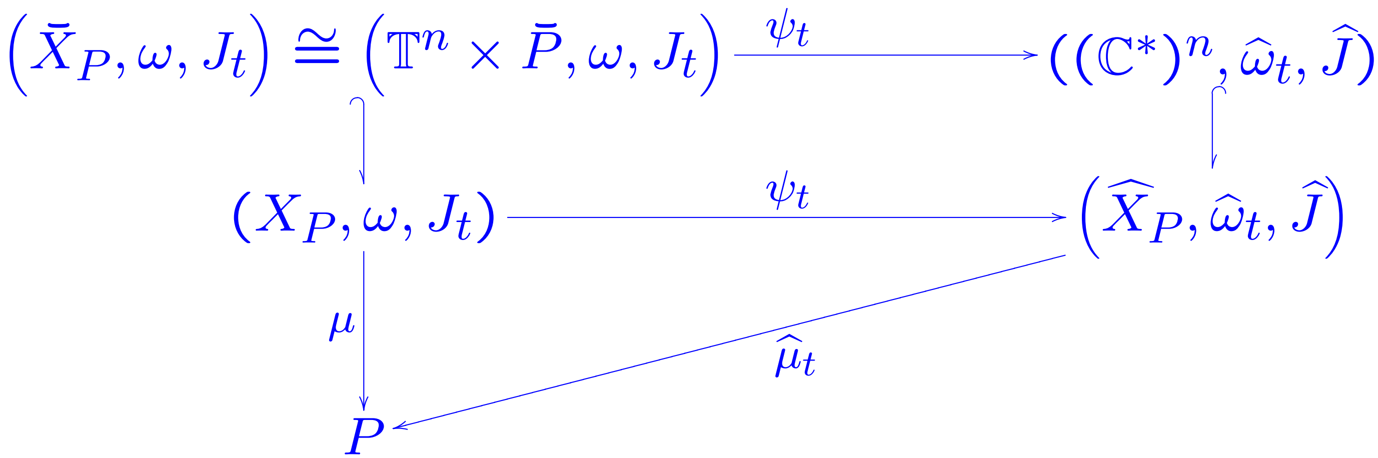
$$Y_{\text{gtrop}} = \lim_{t \to \infty} \widehat{\mu}_t(\widehat{Y}) \subset \mu(M)$$

## 4.3. Geometric tropicalization: toric case

What we did in [Baier-Florentino-M-Nunes, J. Diff. Geom., 2011] was (equivalent) to show that the flat picture extends to (nonflat) toric varieties

$$h = \frac{1}{2}||\mu||^2 = \frac{1}{2}(H_1^2 + \cdots + H_n^2) \Rightarrow w_t = e^{\frac{\partial}{\partial H}(g+th)+i\theta}$$

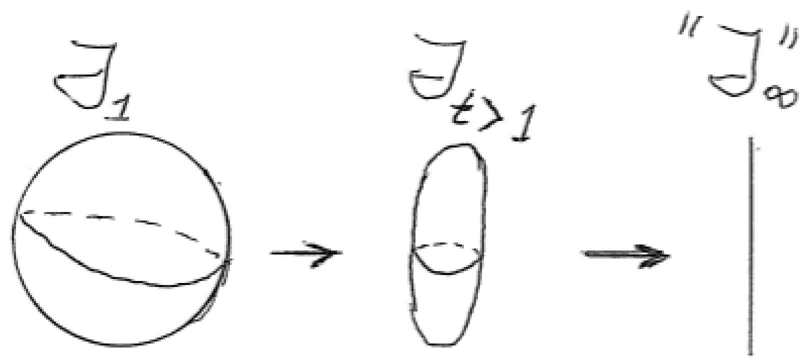
through the diagram we had before




$$
\begin{aligned}
\psi_t(\theta, H) &= e^{u_t + i\theta} = e^{\partial g_t / \partial H + i\theta} \\
g_t(H) &= \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi + t\, h(H) \\
\mu(\theta, H) &= H \text{ -- moment map in the \textit{symplectic} picture}
\end{aligned}
$$

[Baier-Florentino-M-Nunes, J. Diff. Geom., 2011]

- GH collapse or geometric tropicalization of toric manifolds: Metrically, as $t \to \infty$, the Kähler manifold $(X_P, \omega, J_t)$ collapses to $P$ with metric Hess($h$) on $\bar{P}$




$$\frac{1}{t}\gamma_t = \frac{1}{t}(\text{Hess}(g_t)dH^2 + \text{Hess}(g_t)^{-1}d\theta^2) \stackrel{t\to\infty}{\longrightarrow} \text{Hess}(h)\, dH^2$$

- Decomplexification: The complex (Kähler) structure degenerates to the real toric polarization

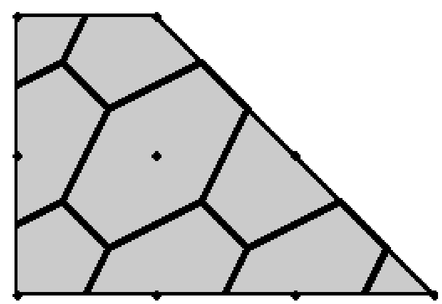
- Geometric Tropicalization of hypersurfaces: In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial H} + t m \cdot H + i m \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem (Baier-Florentino-M-Nunes, 2011)

$$\lim_{t \to \infty} \widehat{\mu}_t(Y) = \lim_{t \to \infty} \mu(Y_t) = \pi(A_{\mathrm{trop}})$$

where $\pi$ denotes the convex projection to $P$.



# 6. References

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## Work in Progress

- J.Mourão and J.P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*

- W. Kirwin, J.Mourão, J.P. Nunes and S. Wu, *Decomplexification of flag manifolds*

Thank you!