Imaginary time flow in geometric quantization and in Kahler geometry, degeneration to real polarizations and tropicalization

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1. Summary

Three main topics of the seminar:

1. Hamiltonian complex time evolution in Kähler geometry and in quantum physics

$$e^{tX_h} \rightsquigarrow e^{itX_h}$$

2. Problem that motivated us initially: For a completely integrable system $(M, \omega, \mu = (H_1, \ldots, H_n) \; : \; M \to \mathbb{R}^n)$ define its quantization in the (usually singular) real polarization $\mathcal{P}_\mu$ defined by μ,

$$\mathcal{H}^Q_{\mathcal{P}_\mu} = ?? \qquad \begin{cases} \mathcal{P}_\mu & = \quad \lim_{t\to\infty} e^{itX_h}\mathcal{P}_0 \\ \mathcal{H}^Q_{\mathcal{P}_\mu} & := \quad \text{"} \lim_{t\to\infty} \mathcal{H}^Q_{\mathcal{P}_{it}} \text{"} \end{cases} \qquad h = ||\mu||^2$$

Possible Applications:

1. Relate crystal bases appearing in the representations of reductive Lie groups and cluster algebras with Bohr-Sommerfeld bases.

2. (long term ...) Study unitarizability of the (KZB–) Hitchin connection for non-abelian theta functions.

3. New relation:

[for (some?) completely integrable systems $(M, \omega, \mu \;:\; M \to \mathbb{R}^n)$]

Quantum Physics & Symplectic Geometry —— dequantization ——→ Tropical Geometry

Quantum Physics & Symplectic Geometry —— decomplexification(s) ——→ Tropical Geometry

Dequantization − $\hbar \to 0$

Decomplexifications − (Kähler) Geometric degenerations to tropical varieties
= geometric tropicalization:
=: Follow (to infinite time) a geodesic ray, in the space of Kähler metrics, generated by $h = ||\mu||^2 = H_1^2 + \cdots + H_n^2$,
= Do a Wick rotation, time $\rightsquigarrow is$, followed by $s \to \infty$ for $h = ||\mu||^2$

2. Geometric quantization

Geometric quantization

$$(M,\omega), \qquad \frac{1}{\hbar}[\omega] \in H^2(M,\mathbb{Z})$$

Prequantum data: (L,∇,h), $L \to M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathsf{prQ}} = \Gamma_{L^2}(M,L) = \overline{\left\{ s \in \Gamma^\infty(M,L) : ||s||^2 = \int_M h(s,s)\, \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f} = Q_\hbar(f) = -i\hbar\nabla_{X_f} + f$

(uncorrected) Quantization: $\mathcal{H}^{prQ}$ is too large. Choose a polarization $\mathcal{P}$, $\mathcal{P}_m \subset T_m M_{\mathbb{C}}$ - Lagrangian, integrable. The quantum Hilbert space consists of $\mathcal{P}$–polarized sections

$$\mathcal{H}^{Q}_{\mathcal{P}} = \{\psi \in \mathcal{H}^{prQ} \ : \ \nabla_X \psi = 0, \forall X \in \Gamma(\mathcal{P})\}$$

Real or Kähler polarizations divide adapted Darboux charts in "coordinates and momenta" or in holomorphic and anti-holomorphic coordinates. Polarized functions are those which do not depend on half of these coordinates.

Two extreme cases of polarizations

$\mathcal{P}$ is real $\mathcal{P} = \overline{\mathcal{P}}$ [we will allow polarizations with certain kinds of singularities]
So $\mathcal{P} = \langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \rangle = \langle X_{q_1}, \ldots, X_{q_n} \rangle$ defines a (possibly) singular fibration on M with the regular fibers being Lagrangian.

$\mathcal{H}^Q_{\mathcal{P}} = \{\psi \in \mathcal{H}^{prQ} \ : \ \nabla_{\frac{\partial}{\partial p_j}} \psi = 0, \quad j = 1, \ldots, n)\}$ consists of sections depending (essentially) only on half of the canonical variables corresponding to the directions transverse to the fibers (or leaves).

Furthermore, if the leaves have noncontractible 1-cycles then polarized sections will be supported only on those leaves with trivial ∇-holonomy, called Bohr-Sommerfeld (BS) leaves.

$\mathcal{P}$ is Kähler

Then $\mathcal{P} = \langle \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \rangle = \langle X_{z_1}, \ldots, X_{z_n} \rangle$ is equivalent to a compatible complex structure J. The pair (∇, J) defines on L the structure of an holomorphic line bundle $\mathcal{L}_J \to M$ and

$$\mathcal{H}^Q_{\mathcal{P}} = \mathcal{H}^Q_J \cong H^0(M, \mathcal{L}_J)$$

Mixed Polarizations

Some mixed polarizations are also very interesting as the Kirwin-Wu momentum polarization $\mathcal{P}_{\mathrm{mom}}$ on $K_{\mathbb{C}} \cong T^*K$ [KW] or the Takakura polarization $\mathcal{P}_T$ on moduli spaces of flat connections G on Riemann surfaces with $g > 1$ and compact simple groups with rank, $r \geq 2$, $\mathcal{M}(G, \Sigma)$.

Quantum Hilbert "bundle"

Thus we get a bundle, the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T},$$

where $\mathcal{T}$ denotes (a family in) the space of polarizations.

Degeneration of Kähler Polarizations

We will study the degeneration of Kähler polarizations to real polarizations

$$
\begin{array}{rcl}
\mathcal{P}_{J_t} & \stackrel{t\to\infty}{\longrightarrow} & \mathcal{P}^{\mathbb{R}}, \\
\text{e.g.} \quad z_t & = & x + ity \\
\dfrac{\partial}{\partial \bar{z}_t} & = & \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{i}{t}\dfrac{\partial}{\partial y}\right) \stackrel{t\to\infty}{\longrightarrow} \dfrac{1}{2}\dfrac{\partial}{\partial x}
\end{array}
$$

as a tool to do two things:

a) study the dependence of the quantum theory on the choice of polarization.

b) find a proper definition of real polarized sections in cases with singular leaves, i.e. a proper definition of $\mathcal{H}^{Q}_{\mathcal{P}^{\mathbb{R}}}$.

3. Complex Hamiltonian symplectomorphisms

3.1 General Theory

$G = \mathrm{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with Lie(Ham(M, ω)) $\cong C^\infty(M)/\mathbb{R}$

$G_{\mathbb{C}} = \mathrm{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits. There are natural orbits of two types passing through a Kähler pair (ω, J_0).

Complex picture

Let us start with the smaller one

$$\mathcal{H}(\omega, J_0) = \left\{ f \in C^\infty(M) \; : \; \omega + i\bar{\partial}_0\partial_0 f = \omega + \frac{1}{2}\mathcal{L}_{J_0 X_f}(\omega) > 0 \right\} / \mathbb{R} =:$$
$$=: \; \{(\varphi^*\omega, J_0), \varphi \in G_{\mathbb{C}}\} = (G_{\mathbb{C}} \cdot \omega, J_0) \cong G_{\mathbb{C}}/G$$

So we get a orbit-dependent “definition” of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of Diff(M).

$\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space: $\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ and Donaldson shows that the Mabuchi metric has constant negative curvature mimiking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \mathrm{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that $\mathrm{Aut}_0(\mathrm{M}, \mathrm{J}_0)$ is trivial the second natural orbit is the total space of a G principal bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0),\ \varphi \in G_{\mathbb{C}}\} \cong G_{\mathbb{C}}$$

with projection

$$\begin{aligned} \pi \ : \ G_{\mathbb{C}} \cong \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) \ &\longrightarrow \ \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \cong G_{\mathbb{C}}/G \\ \pi(\omega, \varphi_* J_0) \ &= \ \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{aligned}$$

3.2 Complex Hamiltonian symplectomorphisms: concrete realization of one-parameter subgroups

Gröbner Lie series

Let (M, ω) be compact, real analytic. We can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\mathrm{an}}(M)$.

Theorem [Gröbner]
Let $f \in C^{\mathsf{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} \; : \; f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{1}$$

converges, for $|\tau| < T_f$, absolutely to a (complex) real analytic function.

So $\mathrm{Ham}_{\mathbb{C}}(M,\omega)$ acts locally on $C^{\mathsf{an}}(M)$. To descend to an "action" on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local J_0-holomorphic coordinates. Then we use (1) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \tag{2}$$

Theorem [Burns-Lupercio-Uribe, 2013; M-Nunes, 2013]
Under natural conditions on J_0 there exists a $T_1 > 0$ such that (2) defines, for every $|\tau| < T_1$, a unique map $\varphi_\tau^{X_h} \; : \; M \to M$ for which

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

There exists a $T_2 \leq T_1, T_2 > 0$ such that, for every $\tau : |\tau| < T_2$, $\varphi_\tau^{X_h} \; : \; (M, J_\tau) \to (M, J_0)$ is a biholomorphism.

Remarks

1. The map $\tau \mapsto \varphi_\tau^{X_h}$ is not (not even a local) homomorphism from (the additive group of complex numbers) $\mathbb{C}$ to Diff(M) [though its restriction to $\mathbb{R}$ of course is]. It satisfies a grupoid property though, which follows from Burns-Lupercio-Uribe. $\mathbb{C}$ acts on the space of polarizations via (2), leading to the action grupoid $\{(\tau, \mathcal{P})\}$ with composition

$$(\tau_2, \mathcal{P}_2) \circ (\tau_1, \mathcal{P}_1) = (\tau_1 + \tau_2, \mathcal{P}_1)$$

defined only if $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$. Then notice that the map $\tau \mapsto \varphi_\tau^{X_h}$ depends on the polarization one starts with. By extending the domain to the action grupoid

$$(\tau, \mathcal{P}) \mapsto \varphi_\tau^{X_h, \mathcal{P}} \qquad (3)$$

one can show that the following composition law is verified. If $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$ then

$$\varphi_{\tau_2}^{X_h, \mathcal{P}_2} \circ \varphi_{\tau_1}^{X_h, \mathcal{P}_1} = \varphi_{\tau_1 + \tau_2}^{X_h, \mathcal{P}_1}$$

or, more precisely, the map (3) is a functor from the action grupoid of $\mathbb{C}$ on $\mathcal{T}$ to an extension of the action grupoid of Diff(M) on $\mathcal{T}$.

2. If the polarizations $\mathcal{P}_1$ and $\mathcal{P}_2 = \tau \cdot \mathcal{P}_1$ have different types then $\varphi_\tau^{X_h, \mathcal{P}_1}$ is not a diffeomorphism. May eg describe the collapse of the complex manifold M to a totally real submanifold $N \subset M$ or the embedding of N in M.

3. The simplification brought by geometric quantization is that the use of polarizations allows us to integrate time dependent vector fields as if they were time independent:

$$\varphi_\tau^{X_h, \mathcal{P}}(z_{\mathcal{P}}, \bar{z}_{\mathcal{P}}) = (e^{\tau X_h} z_{\mathcal{P}}, e^{\bar{\tau} X_h} \bar{z}_{\mathcal{P}}) \ .$$

3.3 Complex Hamiltonian symplectomorphisms: Examples

Example 1 – Constant velocity motion in imaginary time

Let $(M,\omega,J_0)=(\mathbb{R}^2, dx\wedge dy, J_{\mathrm{st}})$ and $h=y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then

$\varphi_t^{X_h}(x,y)=(x+ty,y)$

and

$\varphi_{it}^{X_h}(x,y)=(x,(1+t)y)$

Indeed:

$$\begin{aligned} e^{ity\frac{\partial}{\partial x}}(x+iy) &= \left(1+ity\frac{\partial}{\partial x}+\frac{1}{2}(ity\frac{\partial}{\partial x})^2+\dots\right)(x+iy) = \\ &= x+ity+iy = x+i(t+1)y = (\varphi_{it}^{X_h})^*(x+iy) \end{aligned}$$

Geometrically, the Moser "flow" for the imaginary time motion with constant velocity is

$$iX_h \leftrightarrow J_{it}X_h = \nabla^{\gamma_{it}}h = \frac{1}{1+t}\,y\frac{\partial}{\partial y}$$

where,

$$\gamma_{it} = \frac{1}{1+t}dx^2 + (1+t)dy^2.$$

Example 2a - harmonic oscillator:
S^1–invariant (or toric) degeneration

Let $(M, \omega, J_0, \mu) = (\mathbb{R}^2, dx \wedge dy, J_{\mathrm{st}}, \mu_{ho})$, where $\mu_{ho} = H = \frac{1}{2}(x^2 + y^2)$.

Then $h = \frac{1}{2}||\mu_{ho}||^2 = \frac{1}{2}H^2 \Rightarrow X_h = -H\frac{\partial}{\partial\theta,}$

$$\begin{aligned} z_{it} &= e^{itX_h}(x+iy) = e^{-itH\frac{\partial}{\partial\theta}}\sqrt{2H}\,e^{i\theta} = \sqrt{2H}\,e^{tH}e^{i\theta} = \\ &= e^{\frac{1}{2}\log(H)+tH}e^{i\theta} = e^{tH}\,z = e^{\frac{d}{dH}\left(\frac{1}{2}H\log(H)-\frac{1}{2}H+t\,\frac{1}{2}H^2\right)}e^{i\theta} \end{aligned}$$

and

$$\gamma_{it} = \left(\frac{1}{2H} + t\right) dH^2 + \left(\frac{1}{2H} + t\right)^{-1} d\theta^2$$

Also,

$$X_{z_{it}} = X_{e^{tH}z} = e^{tH}\,(tzX_H + X_z)$$

and therefore, for $z \neq 0$,

$$\lim_{t\to\infty}\langle X_{z_{it}}\rangle = \lim_{t\to\infty}\langle\frac{e^{-tH}}{t}\,X_{z_{it}}\rangle = \lim_{t\to\infty}\langle z\,X_H + \frac{1}{t}X_z\rangle = \langle X_H\rangle = \mathcal{P}_{ho}.$$

Example 2b - harmonic oscillator:
S^1–non invariant degeneration
Let $(M, \omega, \mathcal{P}_0, \mu) = (\mathbb{R}^2, dx \wedge dy, \mathcal{P}_{Sch} =< X_x >, \mu_{ho})$.

Then

$$\begin{aligned} z_{it} &= e^{itX_h}(x) = e^{-itH\frac{\partial}{\partial\theta}} \sqrt{2H} \cos(\theta) = \sqrt{2H} \cos(\theta - itH) = \\ &= \frac{1}{2} e^{\frac{1}{2}\log(H) + tH} e^{i\theta} \left(1 + e^{-2tH - 2i\theta}\right) = \\ &= \frac{1}{2} e^{tH} z \, \left(1 + e^{-2tH - 2i\theta}\right) \end{aligned}$$

We see that, under norm square of the moment map hamiltonian flow in imaginary time, the initial polarization $\mathcal{P}_0 = \langle X_x \rangle$ approaches a S^1–invariant Kähler polarization exponentially (e^{-2tH}) while degenerates to the real polarization $\mathcal{P}_{ho} = \langle X_H \rangle$ as $\frac{1}{t}$.

Example 3 - Toric Varieties

A toric Kähler variety is a T^n–equivariant (partial) compactification of $((\mathbb{C}^*)^n, \widehat{J})$. These can be described by studying $\mathbb{T}^n$–invariant Kähler potentials on $(\mathbb{C}^*)^n$, i.e. $\mathbb{T}^n$-invariant strictly plurisubharmonic (spsh) functions k.

Let $u = \mathrm{Log}(w) = (\log(|w_1|), \ldots, \log(|w_n|))$ so that, $k(w) = \widetilde{k}(u)$ and

$$\widehat{\omega} = \frac{i}{2}\partial\bar{\partial}\, k = \sum_{l,j} \mathrm{Hess}(\widetilde{k})_{lj}\; du_l \wedge d\theta_j$$

with $\mathrm{Hess}(\widetilde{k})$ positive definite.

We see that
$k(w)$is spsh $\Leftrightarrow \widetilde{k}(u)$is strictly convex. Then, defining,
$H = (H_1, \ldots, H_n) = \frac{\partial \widetilde{k}}{\partial u}$, we obtain

$$\widehat{\omega} = \sum_j dH_j \wedge d\theta_j$$

and therefore the map

$$\begin{array}{rcl} \psi^{-1} \; : \; ((\mathbb{C}^*)^n, \widehat{\omega}) & \longrightarrow & (T^*\mathbb{T}^n, \omega_{\mathrm{st}}) \\ w = e^{i\theta}e^u & \mapsto & (\theta, \dfrac{\partial \widetilde{k}}{\partial u}) \end{array} \tag{4}$$

is a symplectomorphism to its image and $\widehat{\mu}(w) = \frac{\partial \widetilde{k}}{\partial u} = H(u)$, is the moment map of the $\mathbb{T}^n$ action on $((\mathbb{C}^*)^n, \widehat{\omega})$.

Having invariant divisors D_j to which the Kähler structure $((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\text{st}})$ extends implies loss of surjectivity of the Legendre transform of the fibers in (4): we get $\ell_{F_j}(H) = \nu_{F_j} \cdot H + \lambda_{F_j} \geq 0$, where ν_{F_j} is the integral vector associated with D_j in the fan of the resulting toric variety $\widehat{X}$. If the corresponding Kähler manifold $(\widehat{X}, \widehat{\omega}, \widehat{J}_{\text{st}}) \supset ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\text{st}})$ is compact then the image of the fibers is a Delzant polytope P (Atiyah–Guillemin–Sternberg and Delzant theorems.)

$P = \{H = (H_1, \ldots, H_n) \in \mathbb{R}^n \ : \ \ell_F(H) = \nu_F \cdot H + \lambda_F \geq 0, F \text{ facet of } P\}$.

Let X_P be the associated toric variety.

The inverse map to

$$\psi^{-1} \;:\; ((\mathbb{C}^*)^n, \widehat{\omega}) \longrightarrow (\mathbb{T}^n \times \bar{P}, \omega_{\mathrm{st}}) \qquad (5)$$

is also a Legendre transform, associated with a function, with domain $\bar{P}$, called symplectic potential and given by,
$g(H) = \sum_j H_j u_j - \widetilde{k}(u)$.

This leads to Guillemin-Abreu theory:
Toric complex structures are are in one-to-one correspondence with the space of symplectic potentials

$$g(H) = g_P(H) + \phi(H) = \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H)$$

and are given by

$$\left(\bar{X}_P, \omega, J_\phi\right) \cong \left(\mathbb{T}^n \times \bar{P}, \omega, J_\phi\right) \xrightarrow{\psi_\phi} ((\mathbb{C}^*)^n, \widehat{\omega}_\phi, \widehat{J})$$

where

$$\psi_\phi(H,\theta) = e^{z_\phi} = e^{\frac{\partial g}{\partial H} + i\theta}$$

Imaginary time toric flows and degeneration

For a toric $h(H)$ we have $X_h = -\sum_j \frac{\partial h}{\partial H_j}\frac{\partial}{\partial \theta_j}$ and

$$w_j^{(it)} = e^{itX_h}(w_j) = e^{itX_h}\left(e^{\frac{\partial g}{\partial H_j}+i\theta_j}\right) = e^{\frac{\partial g}{\partial H_j}+i(\theta_j - it\frac{\partial h}{\partial H_j})} =$$

$$= e^{\frac{\partial (g+th)}{\partial H_j}+i\theta_j} = e^{tH_j}\, w_j$$

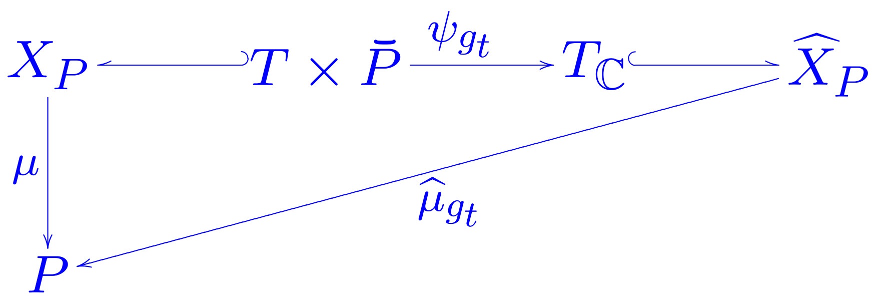
as in the example 2a above. As there we find

$$X_{w_j^{(it)}} = X_{e^{tH_j}w_j} = e^{tH_j}\left(tw_jX_{H_j} + X_{w_j}\right)$$

Therefore, for $H \in \breve{P}$, we have

$$\begin{aligned}
\lim_{t\to\infty} \langle X_{w_1^{(it)}}, \ldots, X_{w_n^{(it)}} \rangle &= \lim_{t\to\infty} \langle w_1 \frac{\partial}{\partial \theta_1} - \frac{1}{t} X_{w_1}, \ldots, w_n \frac{\partial}{\partial \theta_n} - \frac{1}{t} X_{w_n} \rangle = \\
&= \langle \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n} \rangle = \langle X_{H_1}, \ldots, X_{H_n} \rangle = \mathcal{P}_\mu .
\end{aligned}$$

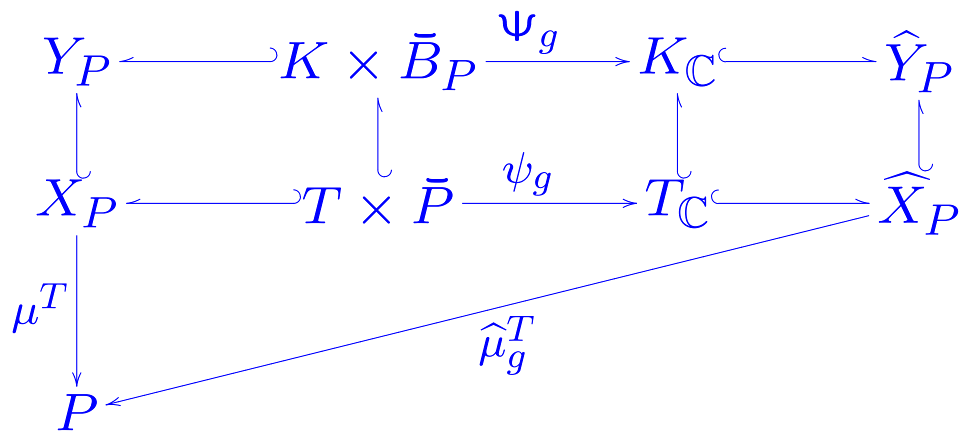
Summarizing



$$\begin{aligned}
\psi_{g_t}(\theta, H) &= e^{\partial g_t/\partial H + i\theta} = e^{\partial g/\partial H + t\,\partial h/\partial H + i\theta} = e^{t\,\frac{\partial h}{\partial H}} w \\
g_t(H) &= g_P(H) + \phi(H) + t h(H) = \\
&= \sum_{F \subset P} \tfrac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H) + t\, h(H) \\
\mu(\theta, H) &= H \ - \text{moment map in the } \textit{symplectic} \text{ picture} \\
& \quad \varphi_{it}^{X_h} = \psi_0^{-1} \circ \psi_t
\end{aligned}$$

Example 4 - Complex reductive groups $K_{\mathbb{C}}$

For $K \times K$ equivariant compactifications of a complex reductive group $K_{\mathbb{C}}$, the class of invariant Kähler structures is described by the Weyl–invariant T–Kähler structures on the corresponding toric variety compactification of the maximal torus $T_{\mathbb{C}}$ of $K_{\mathbb{C}}$. We get the following diagram



Where

$$\Psi_g(x,H) = xe^{i\,U}, \qquad\qquad U = \frac{\partial G}{\partial H}$$

and G is the unique $K \times K$ invariant extension to $B_P \subset Lie(K)^*$ of the Guillemin–Abreu symplectic potential g on P.

Imaginary time evolution

Choosing an h bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial H_j} \xi_j$, where $H = \sum_j H_j T_j$ and ξ_j are the left-invariant vector fields lifted to T^*K. Then

$$e^{itX_h}\left(xe^{iU}\right) = xe^{i(U + t\frac{\partial h}{\partial H})} = xe^{i\frac{\partial G_t}{\partial H}},$$

where $G_t = G + th$.

4. Decomplexification and the definition of $\mathcal{H}^Q_{\mathcal{P}_\mu}$

Theorem [M-Nunes, in preparation]
For a class of completely integrable systems (M, ω, μ), $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

1. $\lim_{t\to\infty} \mathcal{P}_t = \lim_{t\to\infty} (\varphi^{X_h}_{it})^* \mathcal{P}_0 = \mathcal{P}_\mu.$

2. The evolution of sections under $e^{-it\widehat{h}_{GQ}}$ concentrates exponentially fast around Bohr-Sommerfeld leaves.

3. (= **2.** + half-form correction) The family $\mathcal{P}_t$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M, L)$, with a L^2-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty} \sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^Q_{\mathcal{P}_\mu} = \mathrm{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

Proof sketch of **1.**:

$$\mathcal{P}_t = (\varphi_{it}^{X_h})^*\mathcal{P}_0 = \langle X_{z_1^{(it)}}, \ldots, X_{z_n^{(it)}} \rangle,$$

where $z_j^{(it)} = e^{-it\sum_j H_j \frac{\partial}{\partial \alpha_j}} z_j = z_j(H, \alpha + itH)$ Then

$$\frac{1}{t} X_{z_j^{(it)}} = \sum_j \left[\left(i \frac{\partial z_j}{\partial \alpha_k}(H, \alpha + itH) + \frac{1}{t}\frac{\partial z_j}{\partial H_k}(H, \alpha + itH) \right) X_{H_k} + \right.$$
$$\left. + \frac{1}{t}\frac{\partial z_j}{\partial \alpha_k}(H, \alpha + itH)\, X_{\alpha_k} \right]$$

Proof sketch of **2.**:

Let $F \in C^\infty(\mathbb{R}^n)$ be a function with the property that $h = F \circ \mu$ defines a strictly convex functions $\widetilde{h}$ of all local action variables $x = (x_1, \ldots, x_n)$ (standard candidate: $h = ||\mu||^2$), let $\mathcal{P}_{it} = e^{it\mathcal{L}_{H_h}}\mathcal{P}$ and

$$\widehat{h}_{GQ} = iX_h + \widetilde{h} - \sum_j x_j \frac{\partial \widetilde{h}}{\partial x_j}$$

on a invariant nbd $V \cong \mathbb{T}^n \times U \ni (\theta, x)$. Then one shows that

$$e^{-i\tau \widehat{h}_{GQ}}|_{\tau = it} = e^{t\widehat{h}_{GQ}} \ : \ \mathcal{H}_{\mathcal{P}} \longrightarrow \mathcal{H}_{\mathcal{P}_{it}}$$

Sections of L can be represented on V in the form

$$\sigma = \sum_{m\in\mathbb{Z}^n} a_m(x)\, e^{im\cdot\theta}$$

Consider the following global operator

$$\widehat{h}_E\,\sigma = \sum_{m\in\mathbb{Z}^n} \widetilde{h}(m) a_m(x)\, e^{im\cdot\theta}$$

Then

$$e^{t\widehat{h}_{GQ}} \circ e^{-t\widehat{h}_E}(\sigma) \;=\; \sum_{m\in\mathbb{Z}^n} e^{-t\left((x-m)\cdot\frac{\partial\widetilde{h}}{\partial x} - \widetilde{h} + \widetilde{h}(m)\right)} a_m(x)\, e^{im\cdot\theta}$$

$$\sim \sum_{m\,\in\,\mathbb{Z}^n\cap\, x(U)} e^{-\frac{t}{2}(x-m)\cdot \mathsf{Hess}(\widetilde{h})(m)\cdot(x-m)}\, a_m(x)\, e^{im\cdot\theta}$$

5. Geometric tropicalization of varieties and divisors

5.1. Tropicalization: flat case

Recall the origin of tropical geometry

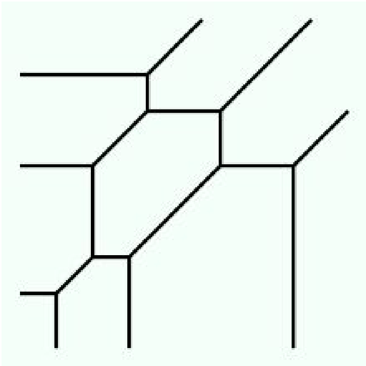
$$\widehat{Y} = \left\{ w \in (\mathbb{C}^*)^n \; : \; \sum_{m \in P} c_m \, w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

$$\begin{aligned} \widehat{Y} &\overset{\mathrm{Log}_T}{\longrightarrow} \mathbb{R}^n \\ \mathrm{Log}_T(w) &= \frac{1}{t}(\log|w_1|, \ldots, \log|w_n|) \end{aligned}$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T \to \infty} \mathrm{Log}_T(\widehat{Y})$

An example for a generic degree 3 plane curve is:



Some geometric and enumerative properties become very simple in tropical geometry.

Simple examples for plane curves are the following. Let $C, \tilde{C}$ be nonsingular plane curves.

Genus–degree formula

$$g(C) = \frac{1}{2}(d(C) - 1)(d(C) - 2)$$

Bezout Theorem

$$|C \cap \tilde{C}| = d(C)\ d(\tilde{C})$$

Let us make a geodesic interpretation of the $T \to \infty$ limit.

Consider a geodesic starting at $\widehat{\omega}_0$ in the direction of $h = (H_1^2 + \cdots + H_n^2)/2$ so that $X_h = -H \cdot \frac{\partial}{\partial \theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h}\, e^{H+i\theta} = e^{(1+\tilde{t})H+i\theta} = e^{tH+i\theta}$$

with $t = \tilde{t} + 1$. We have

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}_0) \\ (\theta, H) & \mapsto & e^{tH+i\theta} \end{array}$$

As $\widehat{\omega}_t = \frac{1}{t} du \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t} u$ so that

$$\widehat{\mu}_t(w) = \mathsf{Log}_T(w) = H$$

We see that, in this interpretation, the T of amoeba theory is exponential of the geodesic time, $T = e^t$, and the Log_T map is the moment map of the complex picture.

5.2. Definition of Geometric Tropicalization

Definition

Tropicalization (Liouville integrable) Kähler manifolds

Let $(\widehat{M}, \widehat{\omega}_t, I, \widehat{\gamma}_t)$ be a $||\mu||^2$-geodesic ray. The geometric tropicalization of $(\widehat{M}, \widehat{\omega}_0, I, \widehat{\gamma}_0)$, in the direction of $||\mu||^2$, is the Gromov-Hausdorff limit

$$\begin{aligned} (M_{\text{gtrop}}, d_{\text{gtrop}}) &= \lim_{t\to\infty} (\widehat{M}, \widehat{\gamma}_t) \\ M_{\text{gtrop}} &= \mu(M) \text{ if the fibers are connected} \end{aligned}$$

Tropicalization of divisors (for connected fibers)

The geometric tropicalization of an hypersurface $\widehat{Y} \subset \widehat{M}$, in the direction of $||\mu||^2$, is the Hausdorff limit of $\widehat{\mu}_t(\widehat{Y})$ in $\mu(M) = M_{\text{gtrop}}$,

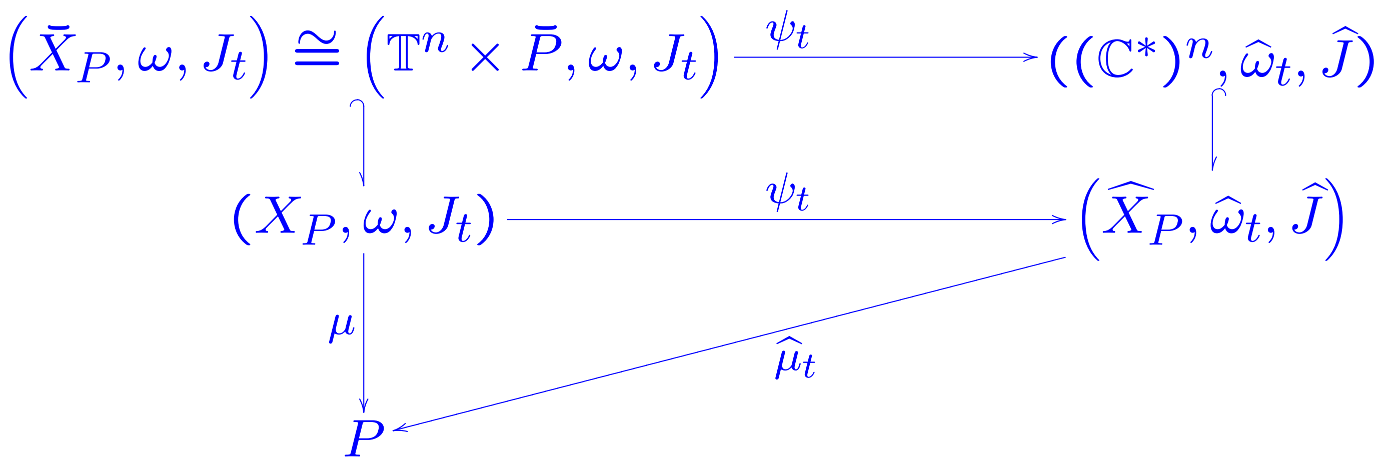
$$Y_{\text{gtrop}} = \lim_{t \to \infty} \widehat{\mu}_t(\widehat{Y}) \subset \mu(M)$$

5.3. Geometric tropicalization: toric case

What we did in [Baier-Florentino-M-Nunes, J. Diff. Geom., 2011] was (equivalent) to show that the flat picture extends to (nonflat) toric varieties

$$
\begin{aligned}
h = \frac{1}{2}||\mu||^2 &= \frac{1}{2}(H_1^2 + \cdots + H_n^2) \Rightarrow \\
\Rightarrow w_j^{(it)} &= e^{\frac{\partial}{\partial H_j}(g+th)+i\theta_j} \\
&= e^{\frac{\partial}{\partial H_j}(g)+tH_j+i\theta_j} = e^{tH_j}\, w_j
\end{aligned}
$$

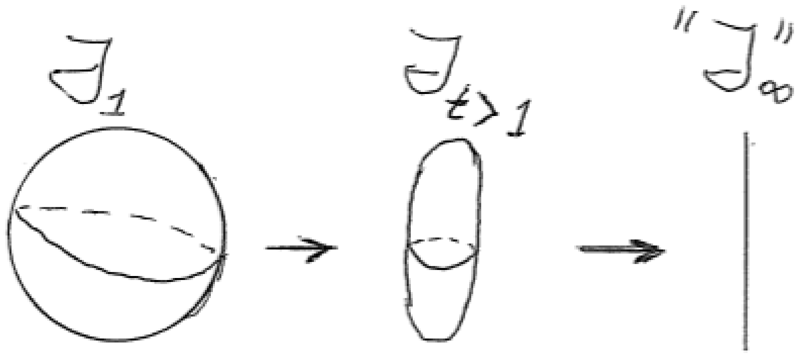
through the diagram we had before



$$
\begin{aligned}
\psi_t(\theta, H) &= e^{\partial g_t/\partial H + i\theta} = e^{\frac{\partial}{\partial H}(g) + tH + i\theta} = e^{tH}\, w \\
g_t(H) &= \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi + t\, h(H) \\
\mu(\theta, H) &= H \ - \text{ moment map in the } \textit{symplectic} \text{ picture}
\end{aligned}
$$

[Baier-Florentino-M-Nunes, J. Diff. Geom., 2011]

- GH collapse or geometric tropicalization of toric manifolds: Metrically, as $t \to \infty$, the Kähler manifold (X_P, ω, J_t) collapses to P with metric Hess(h) on $\bar{P}$



$$\frac{1}{t}\gamma_t = \frac{1}{t}(\text{Hess}(g_t) dH^2 + \text{Hess}(g_t)^{-1} d\theta^2) \stackrel{t\to\infty}{\longrightarrow} \text{Hess}(h)\, dH^2$$

- Decomplexification: The complex (Kähler) structure degenerates to the real toric polarization

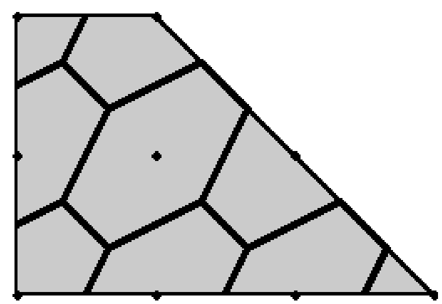
- Geometric Tropicalization of hypersurfaces: In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial H} + tm \cdot H + im \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem (Baier-Florentino-M-Nunes, 2011)

$$\lim_{t \to \infty} \widehat{\mu}_t(Y) = \lim_{t \to \infty} \mu(Y_t) = \pi(A_{\text{trop}})$$

where π denotes the convex projection to P.



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Thank you!