Non–uniqueness of quantization, (no–)reality conditions, complex time evolution and generalized coherent state transforms

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Summary

$$(M,\omega) \rightsquigarrow \begin{cases} F &= (F_1,\ldots,F_n) \rightsquigarrow (\omega, J_F, \gamma_F) \\ \mathcal{H}_F^{\mathsf{Q}} &= \left\{\Psi = \psi(F)\, e^{-k_F}, ||\Psi|| < \infty\right\} \subset \mathcal{H}^{\mathsf{prQ}} \\ F &\mapsto \widehat{F}^{\mathsf{prQ}}|_{\mathcal{H}_F^{\mathsf{Q}}} = F \end{cases}$$

I. Ambiguity of quantization and preferred observables

I.1 Introduction

With $>$ 100 years of General Relativity and $>$ 90 years of Quantum Mechanics it is becoming increasingly embarassing the fact that there is not a fully consistent theory of Quantum Gravity.

The candidates to succeed as e.g. String Theory, Loop Quantum Gravity, Causal Dynamical Triangulations, Group Field Theory continue facing conceptual and technical problems.

One of the problems one is faced with and the one we will address today is that of nonuniqueness of quantization of a classical system.

The dream of the founders of quantum mechanics was to have quantization as a well defined process assigning a quantum system to every classical system and satisfying the correspondence principle

$$\text{Quantization Functor (?)} : (M, \omega) \mapsto Q_\hbar(M, \omega) \overset{\hbar \to 0}{\mapsto} (M, \omega)$$

It was soon realized that this can never be the case even for the simplest systems.

Particle in the line (1 dof)

Classical

$$(M,\omega) = (\mathbb{R}^2, dp\wedge dq, H = \frac{1}{2}p^2 + V(q)),$$

$$f \rightsquigarrow X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p} \qquad X_H = p\frac{\partial}{\partial q} - V'(q)\frac{\partial}{\partial p}$$

Quantum

$$Q_\hbar^{Sch}(\mathbb{R}^2, dp\wedge dq, H):$$

$$\mathcal{H}^{\mathsf{Q}}_{Sch} = L^2(\mathbb{R}, dq)$$

$$q \mapsto Q_\hbar^{\mathsf{Sch}}(q) = \widehat{q} = q$$

$$p \mapsto Q_\hbar^{\mathsf{Sch}}(p) = \widehat{p} = i\hbar\frac{\partial}{\partial q}$$

$$f(q,p) \mapsto ??$$

$$H = \frac{1}{2}p^2 + V(q) \mapsto Q_\hbar^{\mathsf{Sch}}(H) = \widehat{H} = -\frac{\hbar^2}{2}\frac{\partial}{\partial q^2} + V(q)$$

$$\mathcal{H}^{\mathsf{Q}}_{\mathsf{Sch}} = \mathcal{H}^{\mathsf{Q}}_q$$

Groenewold (1946) – van Hove (1951) no go Thm:

It is impossible, even for systems with one degree of freedom, to quantize all observables exactly as Dirac hoped

$$\begin{aligned} Q_\hbar(f) &= \widehat{f} \\ [Q_\hbar(f), Q_\hbar(h)] &= i\hbar\, Q_\hbar(\{f, g\}) \end{aligned}$$

and satisfy natural additional requirements like irreducibility of the quantization.

In order to quantize one needs to add additional data to the classical system. *eg* choose a (sufficiently big but not too big ...) (Lie) subalgebra of the algebra of all observables

$$\mathcal{A} = \mathrm{Span}_{\mathbb{C}}\{1, q, p\}$$

Then we have to study the dependence of the quantum theory on the additional data.

I.2 Geometric Quantization

Geometric quantization is mathematically perhaps the best defined quantization

$$(M, \omega), \quad \frac{1}{2\pi\hbar}[\omega] \in H^2(M, \mathbb{Z})$$

Prequantum data: (L, ∇, h), $L \to M, F_\nabla = \frac{\omega}{\hbar}$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^\infty(M, L) : ||s||^2 = \int_M h(s, s)\, \frac{\omega^n}{n!} < \infty \right\}}$$

Pre-quantum observables: $\widehat{f} = Q^{\mathrm{prQ}}_\hbar(f) = \widehat{f}^{\mathrm{prQ}} = i\hbar\nabla_{X_f} + f$

This almost works! But the Hilbert space is too large, the representation is reducible.
We need a smaller Hilbert space: Prequantization $\Rightarrow$ Quantization

Additional Data in Geometric Quantization

Generalizing what is done in the Schrödinger representation, for systems with one degree of freedom, to fix a quantization one chooses (locally) a preferred observable – $F(q,p)^*$ – and then works with wave functions of the form

$$\mathcal{H}^{\mathrm{prQ}} \rightsquigarrow \mathcal{H}_F^Q = \left\{\Psi \in \mathcal{H}^{\mathrm{prQ}} : \nabla_{X_F}\Psi = 0, ||\Psi|| < \infty\right\} = \\ = \left\{\Psi(q,p) = \psi(F)\, e^{-k(q,p)}, \, ||\Psi|| < \infty\right\} \subset \mathcal{H}^{\mathrm{prQ}}$$

on which the preferred observable F and functions of it $u(F)$ act diagonally

$$Q_\hbar^F(u(F)) = \widehat{u(F)}^{\mathrm{prQ}}|_{\mathcal{H}_F^Q} = u(F).$$

*for systems with n degrees of freedom one chooses (locally) n independent observables in involution $F_1, \ldots, F_n$, $\{F_j, F_k\} = 0$. The distribution $\mathcal{P} = < X_{F_j}, j = 1, \ldots n >$ is called polarization associated with this choice.

(Non–)Equivalence of different Quantizations

Are all these quantizations (for different choices of F) physically equivalent?

NO!

Consider the observable: $H_\lambda = \frac{p^2}{2} + \frac{q^2}{2} + \lambda \frac{q^4}{4},\ \lambda \geq 0$
and let $Sp^{\mathsf{Sch}}(H_\lambda)$ denote the (discrete) spectrum of H_λ in the Schrödinger quantization, i.e. the spectrum of the operator

$$Q_\hbar^{\mathsf{Sch}}(H_\lambda) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{q^2}{2} + \lambda \frac{q^4}{4}$$

acting on $\mathcal{H}^Q_{\mathsf{Sch}} = L^2(\mathbb{R}, dq)$.

Now consider the 1−parameter family of quantizations with Hilbert spaces $\mathcal{H}^{\mathrm{Q}}_{H_\lambda}$ for which the role of preferred observable is played by H_λ. Then, one finds that

$$\begin{aligned}\mathcal{H}^{Q}_{H_\lambda} &= \left\{\Psi(q,p) : \nabla_{X_{H_\lambda}}\Psi = 0\right\} = \\ &= \left\{\Psi(q,p) = \psi(H_\lambda)\, e^{i\,G_\lambda(q,p)}\right\} = \\ &= \left\{\sum_{n=0}^{\infty}\psi_n\,\delta(H_\lambda - E^\lambda_n)\, e^{iG_\lambda(q,p)}\right\}, \qquad (1)\end{aligned}$$

where E^λ_n are defined by the Bohr-Sommerfeld conditions

$$\oint_{H_\lambda = E^\lambda_n} pdq = \hbar(n + \frac{1}{2}). \qquad (2)$$

Since H_λ acts diagonally on this quantization we conclude from (1) that its spectrum in this quantization is given by (2) $Sp^{H_\lambda}(H_\lambda) = \{E^\lambda_n, n \in \mathbb{N}_0\}$.

It is known that on one hand $Sp^{\mathrm{Sch}}(H_0) = Sp^{H_0}(H_0)$ but on the other hand $Sp^{\mathrm{Sch}}(H_\lambda) \neq Sp^{H_\lambda}(H_\lambda)$ for all $\lambda > 0$ so that the two quantizations $Q^{\mathrm{Sch}}_\hbar$ and $Q^{X_{H_\lambda}}_\hbar$ are physically inequivalent if $\lambda > 0$! Wins $Q^{\mathrm{Sch}}_\hbar$!

I.3 Ambiguity of quantization and reality conditions

LQG is facing a similar problem with the Ashtekar–Barbero connection as preferred observable

$$A_\beta = \Gamma(E) + \beta\, K \Rightarrow \Psi_\beta(E,K) = \psi(A_\beta)\, e^{iG_\beta(E,K)}.$$

Are the quantizations based on the choice of connections with different (Immirzi) parameters equivalent? No, because they lead to different spectra of the area operator.

Here it is less obvious which one is the "correct" one. Studies of the black hole entropy formula seemed to indicate the value

$$\beta = \ln(3)/\sqrt{8}\pi \;??$$

Other, recent studies (e.g. Pranzetti, Sahlmann, Phys Lett 2015, Ben Achour, Livine, arXiv:1705.03772) however seem to point back to $\beta = \sqrt{-1}$. This corresponds to the Ashtekar connection

$$A_{\sqrt{-1}} = \Gamma + \sqrt{-1}K$$

The study of quantizations based on compex valued observables like this has been the focus of most of our recent work.

It turns out that for some choices of complex observables quantization is in fact mathematically better defined then quantization based on real observables and this may help addressing some of the technical issues faced by LQG.

Complex observables and reality conditions: rescued by the power of complex analysis

Let us illustrate the general situation with a one degree of freedom system.

Consider the quantum observable

$$z_f = q + if(p)\,, \qquad dz_f \wedge \overline{dz_f} = -2if'(p)\, dq \wedge dp\,.$$

It turns out that if $f'(p) > 0$ then several remarkable simplifying facts occur:

$$F_f = z_f = q + i f(p)$$

1. Complex Structure: There is a unique complex structure J_f on $\mathbb{R}^2$ for which z_f is a global holomorphic coordinate.

2. Kähler Metric: The symplectic form together with the complex structure J_f define on $\mathbb{R}^2$ a Kähler metric

$$\begin{aligned} \gamma_f &= \frac{1}{f'(p)} \, dq^2 + f'(p) \, dp^2 \\ R(\gamma_f) &= -\left(\frac{1}{f'(p)}\right)'' . \end{aligned}$$

3. Quantum Hilbert space much better defined than in the case of quantizations based on real observables:

$$\mathcal{H}^{Q}_{X_{z_f}} = \left\{ \Psi(q,p) = \psi(z_f)\, e^{-k_f(p)},\ ||\Psi|| < \infty \right\}$$

where ψ is a J_f–holomorphic function and
$k_f(p) = pf(p) - \int f(p)dp$ is a Kähler potential.

4. The inner product is not ambiguous and it fixes the reality conditions:

$$< \Psi_1, \Psi_2 >= \int_{\mathbb{R}^2} \overline{\psi_1(z_f)} \psi_2(z_f)\, e^{-2k_f(p)}\, dqdp\,.$$

II. **Geometry on the (infinite dimensional) space of Kähler structures $\mathcal{H}$ ($\subset$ space of quantizations)**

II.1 **Kähler manifolds and space of Kähler metrics**

Kähler manifolds (M, ω, J) are symplectic manifolds (M, ω) with a compatible complex structure J, *ie* such that the bilinear form $\gamma(X, Y) := \omega(X, JY)$ is a Riemannian metric.
We get then 3 compatible geometric structures, (M, ω, J, γ).

A symplectic manifold may not have compatible complex structures but if it has one it has an infinite dimensional space of them.

The symplectic form is automatically of type $(1,1)$ for any compatible complex structure and has a locally defined J-dependent Kähler potential k_J,

$$\omega = \frac{i}{2}\partial_J \overline{\partial_J}\, k_J$$

Example - $\mathbb{CP}^n$

The Fubini-Study Kähler form reads

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\, k_{FS} = \frac{i}{2}\partial\bar{\partial} \log(1 + |z_1|^2 + \cdots + |z_n|^2)$$

On the other hand, fixing J, on a compact manifold M, two J–compatible closed 2–forms ω and ω' are in the same cohomology class iff their Kähler potentials k, k' can be chosen to differ by a global function

$$k' = k + \phi, \quad \phi \in C^\infty(M)$$

Then, the space of Kähler metrics compatible with J, in the given cohomology class, is naturally given by

$$\mathcal{H}(\omega, J) \cong \left\{ \phi \in C^\infty(M) \; : \; \omega_\phi = \omega + \frac{i}{2} \partial_J \bar{\partial}_J \phi > 0 \right\} / \mathbb{R}$$

This (infinite dimensional) manifold has a natural metric introduced by Mabuchi,

$$G_\phi(h_1, h_2) = \int_M h_1 \, h_2 \, \frac{\omega_\phi^n}{n!}, \qquad \text{where} \, \omega_\phi = \omega + \frac{i}{2} \partial_J \bar{\partial}_J \phi \tag{3}$$

Example - $\mathbb{CP}^n$

The space of Kähler metrics on $\mathbb{CP}^n$ is then given by the following open subset of $C^\infty(\mathbb{CP}^n)$:

$$\mathcal{H}(\omega_{FS}, J) = \left\{ \phi \in C^\infty(\mathbb{CP}^n) \;:\; \omega_\phi = \frac{i}{2}\partial\bar{\partial}\left[\log(1 + |z_1|^2 + \cdots + |z_n|^2) + \phi\right] > 0 \right\} / \mathbb{R}$$

Remark. Notice that this metric does not coincide with the pullback of the standard metric in the space of metrics. The pullback of that metric to the space of Kähler metrics is called Calabi metric.

II.2. Geometry on the space of Kähler metrics on M and HCMA

Theorem 1 (Donaldson) *The geodesics for the metric (3) are the stationary points of the energy functional*

$$E(\phi) = \int_0^1 \int_M \dot{\phi}_t^2 \, dt \, \frac{\left(\omega + \frac{i}{2}\partial\bar{\partial}\phi_t\right)^n}{n!} \, .$$

Donaldson further shows that $\mathcal{H}$ with the Mabuchi metric corresponds to the infinite dimensional symmetric space

$$\text{“}\mathrm{Ham}_{\mathbb{C}}(M,\omega)\text{''} / \mathrm{Ham}(M,\omega) \cong \mathcal{H}$$

being the ananalogue of the symmetric spaces of non–compact type of the form

$$PSL(N,\mathbb{C})/PSU(N) \, ,$$

with $PSL(N,\mathbb{C})$–invariant metric.

(I) First argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:

$\mathcal{H}$ as a quotient

Let

$$Ham_{\mathbb{C}}(M,\omega) \;:=\; \left\{\psi \in Diff(M) \,:\, \left(\psi^{-1}\right)^{*}(\omega) \in \mathcal{H}\right\} \quad (4)$$

$$\underset{\subset}{\text{not a subgr}} \; \text{Diff}(M)$$

we obtain, from Moser theorem,

$$\begin{aligned} Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega) &\cong \mathcal{H} \\ \psi &\mapsto \left(\psi^{-1}\right)^{*}(\omega) \,. \end{aligned}$$

(II) Second argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:
Tangent space at a Kähler potential

We have $T_{\omega_\phi}\mathcal{H} \cong C^\infty(M)/\mathbb{R}$
and

$$\mathcal{L}_{JX_H^{\omega_\phi}}\left(\omega_\phi\right) = -\frac{i}{2}\partial\bar{\partial}H,$$

(III) Third argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:
Curvature formulas

Theorem 2 (Donaldson) *The curvature of the Mabuchi metric (3) and the sectional curvature read*

$$R_\phi(f_1,f_2)f_3 = -\frac{1}{4}\{\{f_1,f_2\}_\phi, f_3\}_\phi, \; K_\phi(f_1,f_2) = -\frac{1}{4}||\{f_1,f_2\}_\phi||^2_\phi .$$

for all $f_1, f_2, f_3 \in T_\phi\mathcal{H}$*, where*

$$T_\phi\mathcal{H} = \left\{ f \in C^\infty(M) \,:\, \int_M f \,\frac{\omega_\phi}{n!} = 0 \right\} \cong Lie(Ham(M,\omega_\phi)) .$$

Remark

The above expressions are in full agreement with the formulas for the curvature of the finite dimensional symmetric spaces $K_{\mathbb{C}}/K$,

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$

and

$$K(X,Y) = -\frac{1}{4}||[X,Y]||^2 .$$

for all $X, Y, Z \in T_0 K_{\mathbb{C}}/K \cong iLie(K) \cong Lie(K)$ and the Lie brackets are calculated in $Lie(K)$.

(IV) Fourth argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$:
Limit of spaces of Bergman metrics

$$\mathcal{H} = \lim_{N\to\infty} PSL(N,\mathbb{C})/PSU(N)$$

Let $L \to M$ be a very ample holomorphic line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$ and $\dim H^0(M,L^p) = d_p+1$. Every ordered basis $\underline{s} = (s_0,\dots,s_{d_p+1})$ defines an embedding $i_{\underline{s}} : M \to \mathbb{CP}^{d_p}$ and the p–th root of the pullback of the Fubini-Study hermitian structure defines an hermitian structure on $\mathcal{H}$,

$$\begin{aligned}
FS_p(\underline{s}) &= \left(i_{\underline{s}}^* h_{FS}\right)^{1/p} = \frac{1}{(\sum_{j=0}^{d_p} |s_j(z)|^2)^{1/p}} \\
\mathcal{B}_p &= \left\{k = -\log(FS_p(\underline{s})) : \underline{s} \text{ a basis of } H^0(M,L^p)\right\} \cong GL(d_p+1)/U(d_p+1) \\
\mathcal{B}_p &\subset \mathcal{H}
\end{aligned}$$

Every $k \in \mathcal{H}$ defines an inner product on $H^0(M, L^p)$ via the Hermitean structure $h_p(k) = e^{-pk}$

$$\langle s, \tilde{s} \rangle_k = \int_M (s, \tilde{s})_{h_p(k)} \frac{\omega^m_{h^k_p}}{m!}$$

Let $\underline{s}_p(k)$ be an orthonormal basis for $\langle \cdot, \cdot \rangle_k$ and let

$$\begin{array}{rcl} \mathcal{H}(\omega, J) & \longrightarrow & \mathcal{B}_p \cong GL(d_p + 1)/U(d_p + 1) \\ k & \mapsto k_p & = -\log\left(FS_p(\underline{s}_p(k))\right) \end{array}$$

Theorem 3 *(Tian, 1990)*

$$k = \lim_{p \to \infty} k_p \,.$$

(V) Fifth argument supporting $\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega)$: Geodesic equations on $\mathcal{H}$ and imaginary time Hamiltonian flows

The Homogeneuous Complex Monge–Ampère (HCMA) equation is the following nonlinear equation on a complex $(n+1)$–dimensional manifold N

$$MA(K) := \det\left(\frac{\partial^2 K}{\partial z_j \partial \bar{z}_l}\right) = 0,$$

or, equivalently,

$$\left(\partial\bar{\partial}K\right)^{n+1} = 0. \tag{5}$$

It is a very difficult equation with very few (genuinly complex) rank n solutions known.

Even for $n = 1$ the CHMA equation is very nontrivial.

Relation with geodesics on $\mathcal{H}$

Let us for simplicity consider the case $n = 1$.

Functions K on (open subsets of) $N = [0, T] \times S^1 \times M \ni (t, \theta, z)$, which are

(a) S^1–invariant and

(b) such that $g_{1\bar{1}} = \frac{\partial^2 K}{\partial z \partial \bar{z}}(t, z, \bar{z}) > 0$

so that $k_t = K(t, \cdot)$ is a path of Kähler potentials on M).

The CHMA equation for these functions coincides with the geodesic equations for k_t.

$$\left(\partial_N \bar{\partial}_N K\right)^2 = 0 \quad \Leftrightarrow$$

$$\frac{\partial^2 K}{\partial t^2} \frac{\partial^2 K}{\partial z \partial \bar{z}} - \left|\frac{\partial^2 K}{\partial t \partial \bar{z}}\right|^2 = 0 \Leftrightarrow$$

$$\frac{\partial^2 K}{\partial t^2} = g^{1\bar{1}} \left|\frac{\partial^2 K}{\partial t \partial \bar{z}}\right|^2 \quad \Leftrightarrow$$

$$k_t'' = ||\nabla k_t'||^2_{\phi_t} \tag{6}$$

Ellaborating on an idea of Semmes and Donaldson we will show how to reduce the Cauchy problem for (6), with $k_t = k + \phi_t$.

$$\begin{cases} k_t'' &= ||\nabla k_t'||^2_{k_t} \\ k_0 &= k, \\ k_0' &= -H, \end{cases} \qquad k_t \in C^\infty(U), H \in C^\infty(M). \tag{7}$$

to the problem of finding the integral curves of the Hamiltonian vector field X_H^ω, where $\omega = \frac{i}{2}\partial\bar{\partial}k$, followed by "rotating" t to the imaginary axis (in the complex t–plane)

$$\exp(sX_H^\omega) \rightsquigarrow \exp(\sqrt{-1}tX_H^\omega) \in Ham_{\mathbb{C}}(M,\omega) \overset{??}{\subset} Diff(M), \tag{8}$$

in a certain way.

To make sense of (8) we must be working on the symplectic picture (see section 3 below) in which ω is fixed and the complex structure J_t changes.

Burns–Lupercio–Uribe [BLU]:
The imaginary time integral curves in (8) are solutions of the following coupled system

$$\begin{cases} \dot{x}_t &= J_t X_H^{\omega} = \nabla^{\gamma_t} H \\ J_t &= \left(\exp(\sqrt{-1}tX_H^{\omega})\right)^* (J). \end{cases} \tag{9}$$

Then Donaldson shows that a solution of (7) is given by the Kähler potential ϕ_t of ω_t in

$$\omega_t = \left(\left(\exp(\sqrt{-1}tX_H^{\omega})\right)^{-1}\right)^* (\omega) \, . \tag{10}$$

This is the so called Donaldson formal solution of the CHMA.

The problem is that to find the imaginary time flow $\exp(\sqrt{-1}\, tX_H^\omega)$ with (9) is equivalent to solving a complicated system of PDE (see [BLU]). So it is not clear what have we gaigned in going from the original HCMA (7) to the coupled system (9).

NO PDE needed!

In the next Section II.3 we will describe a method, proposed in [MN], to integrate (9) by only finding the real time flow $\exp(tX_H^\omega)$ and then rotating to imaginary time and finding

$$\exp(\sqrt{-1}\, tX_H^\omega) \in \mathrm{Diff}(M)$$

without the need of solving any PDE.

This method will be used in Section 4 to find infinite dimensional spaces of new solutions of HCMA equations in different manifolds.

II.3 Explicit "rotation" of hamiltonian flows to imaginary time

The missing step to transform Donaldson formal solution of the Cauchy problem (7) for the HCMA given by (10) into an actual solution is the rotation (8)

$$\exp(sX_H^\omega) \rightsquigarrow \exp(\sqrt{-1}tX_H^\omega).$$

In the present section we will describe our solution to this problem obtained in [M-Nunes, IMNR2015].

- One key technical tool to rotate the flow is the Gröbner theory of Lie series of vector fields (which is still very popular in numerical methods in astronomy – satelite motion, exoplanets, etc).

- Another was to notice that the complicated and incomplete (in general) evolution equation $f_t' = J_t\,X_H\,f_t$ becomes complete if we restrict ourselves to J_t–holomorphic functions as for those,

$$f_t' = J_t\,X_H\,f_t = i\,X_H\,f_t \qquad \Rightarrow \qquad f_t = e^{itX_H}\,f_0$$

Theorem 4 (M-Nunes) *Let (M, J) be a compact complex manifold and $X \in \mathcal{X}(M)$ an analytic vector field. There exist local charts $((z_j), U)$ in neighourhoods of every point and $T > 0$ such that for all $\tau \in \mathbb{D}_T$ the functions*

$$z_j^\tau = e^{\tau X} z_j = u_j^\tau(x, y) + \sqrt{-1} v_j^\tau(x, y), \qquad (11)$$

where $x_j = \Re(z_j), y_j = \Im(z_j), u_j^\tau(x, y) = \Re(z_j^\tau), v_j^\tau(x, y) = \Im(z_j^\tau)$, define on $V \subset U$ local J_τ–holomorphic charts for a unique complex structure J_τ and there exists a unique diffeomorphism $\varphi_\tau^{X,J}$ such that

$$\left(\varphi_\tau^{X,J}\right)^* J = J_\tau.$$

The complex time flow is then given explicitly locally by

$$\varphi_\tau^{X,J}(x, y) = (u^\tau(x, y), v^\tau(x, y)), \qquad (12)$$

We see that, as expected, if $\tau = t \in \mathbb{R}$ the complex time flow is J–independent and coincides with the real time flow

$$\varphi_t^{X,J} = \varphi_t^X \,.$$

Theorem 5 (M-Nunes) *Consider the Cauchy problem for the HCMA (7) on $I \times M$ (where we are already supressing the angular coordinate of the first factor in $A \times M$). Then by replacing* exp$(\sqrt{-1} t X_H^\omega)$*, in the formal solution (10), by $\varphi_{it}^{X_H,J}$ obtained as in (12) one obtains a solution of (7).*

II.4 New solutions to the geodesic equation on the space of Kähler structures on an elliptic curve

Let us now obtain an infinite dimensional family of nonsymmetric solutions of the HCMA on an elliptic curve $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with J^ϵ defined by the holomorphic coordinate $z = x + \epsilon \sin(x) + iy$, where $|\epsilon| < 1$ and (x, y) are the standard periodic coordinates on $\mathbb{T}^2$. We choose $\omega = dx \wedge dy$, which corresponds to choosing an initial Kähler potential $k_0 = k$. Let $\dot{k}_0(x, y) = -H(y)$, a (periodic) function of y only.

Remark 6 The calculations remain simple if we consider the more general initial Kähler structure

$$z = u(x, y) + iv(x, y)$$

but we keep H as a function of y (or x) alone. $\Diamond$

To solve the HCMA with the given initial conditions let us first find the real time hamiltonian flow of H. Since

$$X_H^\omega = H'(y)\,\frac{\partial}{\partial x},$$

we obtain

$$\varphi_t^{X_H^\omega}(x,y) = \big(x + tH'(y), y\big)\,.$$

To rotate the flow to the imaginary axis we find

$$z^{it} = \exp(itX_H^\omega)(z) = x + \epsilon\sin(x)\cosh(tH'(y)) + i\left(y + tH'(y)\epsilon\cos(x)\sinh(tH'(y)\right) \tag{13}$$

We see that, as expected, though the evolution is linear in the geodesic (= imaginary hamiltonian) time t only in the symmetric (with respect to translations in x) case $\epsilon = 0$, the explicit expressions can be found also for $\epsilon \neq 0$ and for any function $H(y)$. From (12) and (13) we see that

$$\varphi_{it}^{X_H,J^\epsilon}(x,y) = \big(x + \epsilon\sin(x)\cosh(tH'(y), y + tH'(y) + \epsilon\cos(x)\sinh(tH'(y)\big)\,.$$

III. Generalized CST = KSH transforms as liftings of geodesics from $\mathcal{H}$ to the quantum bundle

III.1 Applications of the geometry of $\mathcal{H}$

The main applications so far:

1. Donaldson–Tian theory of stability of Kähler manifolds
   Extend Kempf–Ness to the "action" of $Ham_{\mathbb{C}}(M, \omega)$ on $\mathcal{H}$.

2. Quantization

3. Representation theory

4. Hele–Shaw flow on Riemann surfaces

We will concentrate on the applications 2 and 3. In fact they are intimately linked via geometric quantization.

III.2 Quantum Bundle, geodesics and generalized CST

Let $\mathcal{T}$ be the space of polarizations. In $\mathcal{T}$ we have $\mathcal{H}$ and in its boundary real and mixed polarizations.

Geometric quantization gives us the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T} \supset \mathcal{H}$$

and the tools to study the dependence of quantization on the choice of the complex structure or, more generaly, on the choice of polarization.

Integral transforms relating different quantizations

Step 1 Given two polarizations $\mathcal{P}_1$ and $\mathcal{P}_2$ we can hope to link them with a geodesic on $\mathcal{T}$, i.e. that there exists an Hamiltonian $H \in C^\omega(M)$ such that

$$\begin{aligned} \mathcal{P}_2 &= e^{it\,\mathcal{L}_{X_H}}|_{t=1}\,\mathcal{P}_1 = e^{it\,\mathcal{L}_{X_H}}|_{t=1} < X_{F_1}, \ldots, X_{F_n} >= \\ &= < X_{e^{iX_H}(F_1)}, \ldots, X_{e^{iX_H}(F_n)} > \end{aligned} \tag{14}$$

Step 2 Then geometric quantization gives us a way of lifting the geodesics to the quantum bundle and thus construct construct an integral transform

$$C^{iH}_{\mathcal{P}_1\mathcal{P}_2} : \mathcal{H}^{\mathrm{Q}}_{\mathcal{P}_1} \longrightarrow \mathcal{H}^{\mathrm{Q}}_{\mathcal{P}_2}$$

Interpretation

Case 1 If the transform in Step 2 is unitary, as for $M = T^*K$, K a compact Lie group, $\mathcal{P}_1$ the vertical or Schrödinger (real) polarization and $\mathcal{P}_2$ the standard Kähler polarization (called adapted) for the bi-invariant metric on K and H is the norm square of the $K-$moment map, then we have established the equivalence of the two quantizations.

Case 2 If not then we may still use the transform to study the difference of the two quantizations. In cases in which we have "preferred polarizations" (i.e. preferred quantizations) we may use the transforms in step 2 to "correct" other, nonpreferred, quantizations.

Some terminology – CST versus KSH

If the starting polarization $\mathcal{P}_1$ in step 2 above is real and $\mathcal{P}_2$ is Kähler then the integral transform is called a Coherent State Transform (CST) and H is called a Thiemann complexifier. The name CST comes from the fact that they generalize the Segal–Bargmann CST for $M = \mathbb{R}^{2n}$

$$C_{\mathcal{P}_{\mathsf{Sch}}\mathcal{P}_{Fock}} : L^2(\mathbb{R}^n, dx) \longrightarrow \mathcal{H}L^2(\mathbb{C}^n, e^{-|z|^2} dxdy)\,.$$

In general the transforms $C_{\mathcal{P}_1\mathcal{P}_2}$ are called Kostant–Souriau–Heisenberg (KSH) transforms or generalized coherent state transforms.

III.3 Generalized Coherent State Transforms (CST)

III.3.1 Hall transform

In 1994 Brian Hall constructed an unitary transform for Lie groups of compact type G

$$\begin{aligned} U \,:\, L^2(G, dx) &\longrightarrow \mathcal{H}L^2(G_{\mathbb{C}}, d\nu(g)) \\ U &= \mathcal{C} \circ e^{\frac{\Delta}{2}} \end{aligned} \tag{15}$$

where $G_{\mathbb{C}}$ is the unique complexification of G, $\mathcal{H}L^2$ means holomorphic L^2 functions and ν is the averaged heat kernel measure on $G_{\mathbb{C}}$.

III.3.2 Case $G = \mathbb{R}, M = T^*\mathbb{R} \cong \mathbb{R}^2$

Let us show how geometric quantization reveals the intimate relation of the two factors in the rhs of (15).

Then (15) reads

$$\begin{array}{rcl} U \,:\, L^2(\mathbb{R}, dq) & \longrightarrow & \mathcal{H}L^2(\mathbb{C}, e^{-p^2} dpdq) \\ U & = & \mathcal{C} \circ e^{\frac{\Delta}{2}} \\ \psi(q) & \mapsto & (e^{\frac{\Delta}{2}}\psi)(q) \mapsto (e^{\frac{\Delta}{2}}\psi)(q + \sqrt{-1}p)\,. \end{array}$$

Notice that, for $H = \frac{p^2}{2}$, $X_H = p\frac{\partial}{\partial q}$ and therefore

$$e^{\tau X_H}(q)|_{\tau=i} = (q + \tau p)|_{\tau=i} = q + ip = z$$

We see therefore that, for $H = \frac{p^2}{2}$,

$$\mathcal{C} = e^{iX_H}$$

and since $\widehat{H}^{\mathsf{prQ}} = iX_H - \frac{p^2}{2}$, we conclude that

$$e^{-i\tau \widehat{H}^{\mathsf{prQ}}}|_{\tau=i} = e^{\widehat{H}^{\mathsf{prQ}}} = \mathcal{C} \circ e^{-\frac{p^2}{2}}\,.$$

On the other hand, since, $\widehat{p}^{\mathsf{Sch}} = -i\frac{\partial}{\partial q}$, we have also

$$e^{\frac{\Delta}{2}} = e^{-\widehat{H}^{\mathsf{Sch}}} = e^{-i\tau \widehat{H}^{\mathsf{Sch}}}|_{\tau=-i},$$

We see therefore that the Hall CST transform in (15) is equivalent to the follwing transform lifting the complex canonical transformation, $e^{\tau X_H}|_{\tau=i} = e^{ip\frac{\partial}{\partial q}}$:

$$\begin{aligned} \mathcal{H}^{Q}_{\mathsf{Sch}} = \mathcal{H}^{\mathsf{Q}}_{q} &\overset{C^{iH}}{\longrightarrow} \mathcal{H}^{Q}_{z} = \mathcal{H}^{\mathsf{Q}}_{\mathsf{Fock}} \\ C^{iH} &= e^{-i\tau \widehat{H}^{\mathsf{prQ}}}|_{\tau=i} \circ e^{-i\tau \widehat{H}^{\mathsf{Sch}}}|_{\tau=-i} = \\ &= e^{+\widehat{H}^{\mathsf{prQ}}} \circ e^{-\widehat{H}^{\mathsf{Sch}}} \end{aligned} \tag{16}$$

with the (extra bonus of the) averaged heat kernel measure being absorbed into the prequantization of the complexified canonical transformation.

III.3.3 Representation Theoretic meaning of the factors in the (abelian) CST

Notice that the prequantization of the observables q, p preserve both Hilbert spaces $\mathcal{H}^Q_{\text{Sch}}$ and $\mathcal{H}^Q_{\text{Fock}}$ so that there is a $*-$representation of the complexified Heisenberg algebra on both.

One can check that the first factor to act in (16) maps the self-adjoint $\widehat{q}^{\text{Sch}}$ to the non self-adjoint $\widehat{q - ip}^{\text{Sch}}$ and the second factor to act maps $\widehat{q}^{\text{Sch}}$ to $\widehat{q + ip}^{\text{Fock}}$ and therefore C^{iH} maps $\widehat{q}^{\text{Sch}}$ to $\widehat{q}^{\text{Fock}}$.

Then C^{iH} intertwines $\widehat{q}^{\text{Sch}}$ and $\widehat{p}^{\text{Sch}}$ with $\widehat{q}^{\text{Fock}}$ and $\widehat{p}^{\text{Fock}}$, respectively, which makes its projective unitarity a consequence of Schur's lemma.

III.3.4 Case $G = \mathbb{T}^n, M = T^*\mathbb{T}^n$

The two best known (real) polarizations on $T^*\mathbb{T}^n \cong \mathbb{T} \times \mathbb{R}^n$ are the **vertical (or Schrödinger)**

$$\begin{aligned}
\mathcal{P}_{\mathsf{Sch}} &= \langle X_{x_j} = -\frac{\partial}{\partial p_j} \quad j = 1, \ldots, n \rangle \\
\mathcal{H}_{\mathcal{P}_{\mathsf{Sch}}} &= \{\psi(x_1, \ldots, x_n)\} = \\
&= L^2\left(\mathbb{T}^n, \frac{dx}{(2\pi)^n}\right)
\end{aligned}$$

and the **momentum polarizations**

$$\begin{aligned}
\mathcal{P}_{\mathsf{mom}} &= \langle X_{p_j} = \frac{\partial}{\partial x_j} \quad j = 1, \ldots, n \rangle \\
\mathcal{H}_{\mathcal{P}_{\mathsf{Sch}}} &= \langle \delta(p - m)\, e^{im\theta},\, m \in \widehat{\mathbb{T}}^n = \mathbb{Z}^n \rangle = \\
&= L^2(\widehat{\mathbb{T}}^n)
\end{aligned}$$

Remark The leaves of the two polarizations $\mathcal{P}_{\mathsf{Sch}}$ and $\mathcal{P}_{\mathsf{mom}}$ on $M = T^*\mathbb{T}^n$ have different topologies and therefore there can not be a diffeomorphism taking one to the other.

However, as we will see below, the one parameter family of "imaginary time symplectomorphisms" (which are not even diffeomorphisms) generated by the analogous function to the one used in the previous example ($M = T^*\mathbb{R}$), $H(x,p) = \frac{1}{2}||p||^2$,

$$(\varphi_{it}^{X_H})^* = e^{it\mathcal{L}_{X_H}}$$

maps $\mathcal{P}_{\mathsf{Sch}}$ to $\mathcal{P}_{\mathsf{mom}}$ in the limit $t \to \infty$

$$\mathcal{P}_{it} = e^{it\mathcal{L}_{X_H}} \mathcal{P}_{\mathsf{Sch}} \xrightarrow{t\to\infty} \mathcal{P}_{\mathsf{mom}}$$

$t = \pi/2$

If instead of $M = T^*\mathbb{T}^n$ we had $T^*\mathbb{R}^n$ there would be ofcourse many different ways of getting from $\mathcal{P}_{\mathsf{Sch}}$ to $\mathcal{P}_{\mathsf{mom}}$ and we would not need to go through Kähler polarizations as we could use e.g. the simple (real) canonical transformation generated by $H = \frac{1}{2}(||x||^2 + ||p||^2)$ (at time $t = \pi/2$) to achieve that (and open the way to define fractional Fourier Transforms).

$$e^{\frac{\pi}{2}\mathcal{L}_{X_H}}(\mathcal{P}_{\mathsf{Sch}}) = \mathcal{P}_{\mathsf{mom}}$$

$t = is$ can change the type of a polarization

On the other hand, as we saw before, using one–parameter "groups" of imaginary time canonical transformations we can change the type of the polarization (from real or mixed to Kähler and back) and even connect real polarizations with different topologies if we let the imaginary (or geodesic) time go to infinity. This can happen because imaginary time canonical transformations may be very well defined even when they don't correspond to diffeomorphisms!

The (finite) imaginary time (say $\tau = i$) symplectomorphisms generated by the function H transforming a real polarization to a Kähler correspond algebraically to the analytic continuation of functions from a totally real submanifold $R \subset M$ to M with a given complex structure I. Geometrically they are the inverse of the collapse of (M, I) to R. This is literally so as the imaginary time symplectomorphisms generated by the same function H, at time $\tau = -i$, correspond to the collapse of (M, I) to R.

$t = i\infty$ can change the topology of a polarization

Let us show that indeed imaginary time canonical transformations generated by $H = ||\mu||^2/2 = ||p||^2/2$ take us from $\mathcal{P}_{\mathsf{Sch}}$ to $\mathcal{P}_{\mathsf{mom}}$ in infinite imaginary time $t = i\infty$. For $n = 1$

$$e^{it\,\mathcal{L}_{X_H}}\langle X_x\rangle = \langle X_{x+itp}\rangle = \langle \frac{1}{it}X_x + X_p\rangle \stackrel{\mathsf{t}\to\infty}{\longrightarrow} \langle X_p = \frac{\partial}{\partial x}\rangle$$

It turns out that this limit can be extended to a very wide context.

$\mathbf{A}_1$ For ****all**** (infinite dimensional (!) space of) starting real, mixed or Kähler polarizations $\mathcal{P}$ for which the limit

$$\lim_{t\to\infty} e^{it\,\mathcal{L}_{X_H}}\,\mathcal{P}$$

exists, it is equal to the momentum polarization, $\mathcal{P}_{\mathrm{mom}}$. This includes cases for which the Schrödinger polarization does not exist (as is the case of toric manifolds).

$\mathbf{A}_2$ Taking $H = ||\mu||^2$ and $t = i\infty$ can also be extended to (compact) nonabelian groups G and $M = T^*G$ leading to a natural (mixed) polarization which we call Peter–Weyl/Kirwin–Wu, $\mathcal{P}_{\mathrm{PW-KW}}$, due to its relation with the Peter–Weyl theorem and its discovery by Kirwin and Wu,

$$\mathcal{P}_{\mathrm{PW-KW}} := \lim_{t\to\infty} e^{it\,\mathcal{L}_{X_H}}\,(\mathcal{P}_0) = \lim_{t\to\infty} e^{it\,\mathcal{L}_{X_H}}\,(\mathcal{P}) \qquad \forall\mathcal{P}$$

III.3.5 Case G a compact semisimple group and $M = T^*G$

In particular we get [Kirwin–Wu (unpublished)] and [Baier–Hilgert-Kaya-M-Nunes (reproved and extended to symmetric spaces and soon to all $K_{\mathbb{C}}$–manifolds with invariant Kähler structure)] that $\mathcal{P}_{\mathsf{PW-KW}}$ is a mixed polarization generated by Casimir functions of μ and complex valued functions on $K \times \mathcal{O}_\xi$ that are pullbacks of meromorphic functions on $\mathcal{O}_{-\xi} \times \mathcal{O}_\xi$. For $\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}_{\mathsf{PW-KW}}}$ we get

$$\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}_{\mathsf{PW-KW}}} = \sum_{\lambda \in \Lambda^+_{\mathbb{Z}}} \delta(\mu^{\mathsf{Kir}}(g) - \lambda - \rho)\, H^0(L_{-\lambda-\rho} \boxtimes L_{\lambda+\rho})$$

where $\mu^{\mathsf{Kir}}(g)$ means the image of $g \in K_{\mathbb{C}}$ under the Kirwan moment map.

Earlier, very different, attempts by eg Guedes–Oriti–Raasakka.

Some of our papers on this subject:

- T. Baier, J. Hilgert, O. Kaya, J.Mourão and J.P. Nunes, *Geodesics in the space of Kähler metrics on cotangent bundles of compact symmetric spaces and PW-KW quantization*, work in progress.
- W. Kirwin, J.Mourão, J.P. Nunes and T. Thiemann, *Hyperbolic complexifiers and geometric quantization*, work in progress.
- W. Kirwin, J.Mourão and J.P. Nunes, *Complex symplectomorphisms and pseudo-Kahler islands in the quantization of toric manifolds*, Math Analen **364** (2016) 1–28.
- J.Mourão and J.P. Nunes, *On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics*, Int Math Research Notices (2015) 10624–10656

- W. Kirwin, J.Mourão and J.P. Nunes, *Coherent state transforms and the Mackey-Stone-Von Neumann theorem*, Journ. Math. Phys. **55** (2014) 102101.

- W. Kirwin, J.Mourão and J.P. Nunes, *Complex time evolution in geometric quantization and generalized coherent state transforms*, J. Funct. Anal. **265** (2013) 1460–1493.

- W. Kirwin, J.Mourão and J.P. Nunes, *Degeneration of Kaehler structures and half-form quantization of toric varieties*, Journ. Sympl. Geom. **11** (2013) 603–643.

- T. Baier, J.Mourão and J.P. Nunes, *Toric Kahler Metrics Seen from Infinity, Quantization and Compact Tropical Amoebas*, Journ. Differ. Geometry **89** (2011) 411–454 .

Thank you!