Imaginary time flow in geometric quantization and in Kahler geometry, degeneration to real polarizations and tropicalization

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Index

1. Summary

Three main topics of the seminar:

1. Hamiltonian complex time evolution in Kähler geometry and in quantum physics

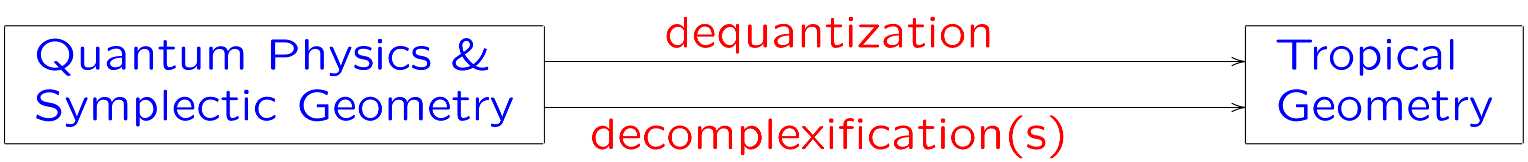
$$e^{tX_h} \rightsquigarrow e^{itX_h}$$

2. Problem that motivated us initially: For a completely integrable system $(M, \omega, \mu = (H_1, \ldots, H_n) \; : \; M \to \mathbb{R}^n)$ define its quantization in the (usually singular) real polarization $\mathcal{P}_\mu$ defined by μ,

$$\mathcal{H}^Q_{\mathcal{P}_\mu} = ?? \qquad \begin{cases} \mathcal{P}_\mu & = \; \lim_{t\to\infty} e^{itX_h} \mathcal{P}_0 \\ \mathcal{H}^Q_{\mathcal{P}_\mu} & := \; \text{"} \lim_{t\to\infty} \mathcal{H}^Q_{\mathcal{P}_{it}} \text{"} \end{cases} \qquad h = ||\mu||^2$$

3. New relation:

[for (some?) completely integrable systems $(M, \omega, \mu \ : \ M \to \mathbb{R}^n)$]



Dequantization	$-$	$\hbar \to 0$
Decomplexifications	$-$	(Kähler) Geometric degenerations to tropical varieties
	$=$	geometric tropicalization:
	$=:$	Follow (to infinite time) a geodesic ray, in the space of Kähler metrics, generated by $h = \|\|\mu\|\|^2 = H_1^2 + \cdots + H_n^2$,
	$=$	Do a Wick rotation, time $\rightsquigarrow is$, followed by $s \to \infty$ for $h = \|\|\mu\|\|^2$

2. Complex time evolution

2.1 Imaginary time: introduction

Geometric quantization

$$(M, \omega), \qquad \frac{1}{\hbar}[\omega] \in H^2(M, \mathbb{Z})$$

Prequantum data: (L, ∇, h), $L \to M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^{\infty}(M, L) : ||s||^2 = \int_M h(s, s) \, \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f} = Q_{\hbar}(f) = -i\hbar \nabla_{X_f} + f$

Kähler quantization: fix a complex structure I on M such that (M, ω, I, γ) is a Kähler manifold.

$$\mathcal{H}_I^Q = \left\{ s \in \mathcal{H}^{\mathrm{prQ}} \ : \ \nabla_{\bar{\partial}_I} s = 0 \right\} = H^0(M, L_I)$$

Get the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

Need to study the dependence of quantization on the choice of the complex structure.

It is precisely to study the dependence of $Q_{\hbar,\mathcal{P}}$ on the choice of the complex structure (or more generally on the polarization) that evolution in imaginary time enters the scene.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in order to transform the spin connection to the Ashtekar connections $\Gamma_\mu \mapsto A_\mu^{\mathbb{C}}$. [Thiemann, Class. Quant. Grav., 1996]

In non-Hermitian Quantum Mechanics one considers time evolution associated with complex valued observables [Moiseyev, Cambridge University Press, 2011]

In $\mathcal{P}T$–symmetric quantum mechanics one deals with complex valued observables which may sometimes have meaningful real spectrum – Bessis and Zinn-Justin example: $H = p^2 + ix^3$ [Dorey-Dunning-Tateo, J.Phys.A, 2007, Section 2]

What we are studying is a new way of looking at imaginary time evolution in (some situations in) quantum mechanics giving it a precise geometric meaning.

In Kähler geometry imaginary time evolution leads to geodesics and is used to study the stability of varieties [Semmes '92 and Donaldson '99].

2.2 Complex time evolution in Kähler geometry

$G = \mathrm{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with $\mathrm{Lie}(\mathrm{Ham}(M, \omega)) = C^\infty(M)$

$G_{\mathbb{C}} = \mathrm{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits. There are natural orbits of two types passing through a Kähler pair (ω, J_0).

Complex picture

Let us start with the smaller one

$$\mathcal{H}(\omega, J_0) = \left\{ f \in C^\infty(M) \ : \ \omega + i\bar{\partial}_0 \partial_0 f > 0 \right\} / \mathbb{R} =: \\ =: \ \{(\varphi^* \omega, J_0), \varphi \in G_{\mathbb{C}}\} = (G_{\mathbb{C}} \cdot \omega, J_0)$$

So we get a orbit-dependent "definition" of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of Diff(M)

$\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space: $\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ and Donaldson shows that the Mabuchi metric has constant negative curvature mimiking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \mathrm{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that $\mathrm{Aut}_0(\mathrm{M}, \mathrm{J}_0)$ is trivial the second natural orbit is the total space of a G principal bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0),\ \varphi \in G_{\mathbb{C}}\}$$

with projection

$$\begin{array}{rcl} \pi\ :\ \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) & \longrightarrow & \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \\ \pi(\omega, \varphi_* J_0) & = & \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{array}$$

Gröbner Lie series

Let (M, ω) be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\text{an}}(M)$

Theorem [Gröbner]

Let $f \in C^{\text{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} \; : \; f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{1}$$

converges absolutely to a (complex) real analytic function.

So $\text{Ham}_{\mathbb{C}}(M, \omega)$ acts locally on $C^{\text{an}}(M)$. To descend to an "action" on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local J_0-holomorphic coordinates. Then we use (1) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \tag{2}$$

Theorem [Burns-Lupercio-Uribe, 2013; M-Nunes, 2013]
Under natural conditions on J_0 there exists a $T_1 > 0$ such that (2) defines, for every $|\tau| < T_1$, a unique map $\varphi_\tau^{X_h} : M \to M$ for which

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

There exists a $T_2 \leq T_1, T_2 > 0$ such that, for every $\tau : |\tau| < T_2$, $\varphi_\tau^{X_h} : (M, J_\tau) \to (M, J_0)$ is a biholomorphism.

Remarks

1. The map $\tau \mapsto \varphi_\tau^{X_h}$ is not (not even a local) homomorphism from (the additive group of complex numbers) $\mathbb{C}$ to Diff(M) [though its restriction to $\mathbb{R}$ of course is]. It satisfies a grupoid property though, which follows from Burns-Lupercio-Uribe. $\mathbb{C}$ acts on the space of polarizations via (2), leading to the action grupoid $\{(\tau, \mathcal{P})\}$ with composition

$$(\tau_2, \mathcal{P}_2) \circ (\tau_1, \mathcal{P}_1) = (\tau_1 + \tau_2, \mathcal{P}_1)$$

defined only if $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$. Then notice that the map $\tau \mapsto \varphi_\tau^{X_h}$ depends on the polarization one starts with. By extending the domain to the action grupoid

$$(\tau, \mathcal{P}) \mapsto \varphi_\tau^{X_h, \mathcal{P}} \tag{3}$$

one can show that the following composition law is verified. If $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$ then

$$\varphi_{\tau_2}^{X_h, \mathcal{P}_2} \circ \varphi_{\tau_1}^{X_h, \mathcal{P}_1} = \varphi_{\tau_1 + \tau_2}^{X_h, \mathcal{P}_1}$$

or, more precisely, the map (3) is a functor from a subcategory of the action grupoid to (the one-object category) Map(M).

2. If the polarization $\mathcal{P}_1$ is complex and the polarization $\mathcal{P}_2 = \tau \cdot \mathcal{P}_1$ is real or mixed then $\varphi_{\tau}^{X_h,\mathcal{P}_1}$ exists but $\varphi_{-\tau}^{X_h,\mathcal{P}_2}$ does not exist eventhough, $\mathcal{P}_1 = (-\tau) \cdot \mathcal{P}_2$.

Example 1

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_{\mathrm{st}})$ and $h = y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then

$\varphi_t^{X_h}(x, y) = (x + ty, y)$

and

$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$

Indeed: $e^{ity\frac{\partial}{\partial y}}(x + iy) = x + ity + iy = x + i(t + 1)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow J_{it}X_h = \nabla^{\gamma_{it}} h = \frac{1}{1+t}\, y\frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t}dx^2 + (1 + t)dy^2$.

Example 2a

Let $(M, \omega, J_0, \mu) = (\mathbb{R}^2, dx \wedge dy, J_{\text{st}}, \mu_{ho})$, where $\mu_{ho} = H = \frac{1}{2}(x^2 + y^2)$.

Then $h = \frac{1}{2}||\mu_{ho}||^2 = \frac{1}{2}H^2/2 \Rightarrow X_h = -H\frac{\partial}{\partial\theta}$

Then

$$\begin{aligned} z_{it} &= e^{itX_h}(x+iy) = e^{-itH\frac{\partial}{\partial\theta}}\sqrt{2H}\,e^{i\theta} = \sqrt{2H}\,e^{tH}e^{i\theta} = \\ &= e^{\frac{1}{2}\log(H)+tH}e^{i\theta} = e^{\frac{d}{dH}\left(\frac{1}{2}H\log(H)-\frac{1}{2}H+t\,\frac{1}{2}H^2\right)}e^{i\theta} \end{aligned}$$

and

$$\gamma_{it} = \left(\frac{1}{2H}+t\right)\,dH^2 + \left(\frac{1}{2H}+t\right)^{-1} d\theta^2$$

Example 2b

Let $(M, \omega, \mathcal{P}_0, \mu) = (\mathbb{R}^2, dx \wedge dy, \mathcal{P}_{Sch} =< X_x >, \mu_{ho})$.

Then

$$\begin{aligned} z_{it} &= e^{itX_h}(x) = e^{-itH\frac{\partial}{\partial\theta}}\sqrt{2H}\,\cos(\theta) = \sqrt{2H}\,\cos(\theta - itH) = \\ &= \frac{1}{2}e^{\frac{1}{2}\log(H)+tH}e^{i\theta}\left(1+e^{-2tH-2i\theta}\right) \end{aligned}$$

Example 3 - Toric Varieties

P − Delzant polytope

$P = \{H = (H_1, \ldots, H_n) \in \mathbb{R}^n \ : \ \ell_F(H) = \nu_F \cdot H + \lambda_F \geq 0, F \text{ facet of } P\}.$

Let X_P be the associated toric variety,

$\breve{X}_P \cong \mathbb{T}^n \times \breve{P} \subset T^*\mathbb{T}^n, \qquad \omega = dH \wedge d\theta$

Toric complex structures are are in one-to-one correspondence with the space of symplectic potentials (Guillemin-Abreu theory)

$$g(H) = g_P(H) + \phi(H) = \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H)$$

via the Legendre transform

$$\left(\bar{X}_P, \omega, J_\phi\right) \cong \left(\mathbb{T}^n \times \bar{P}, \omega, J_\phi\right) \xrightarrow{\psi_\phi} ((\mathbb{C}^*)^n, \widehat{\omega}_\phi, \widehat{J})$$

where

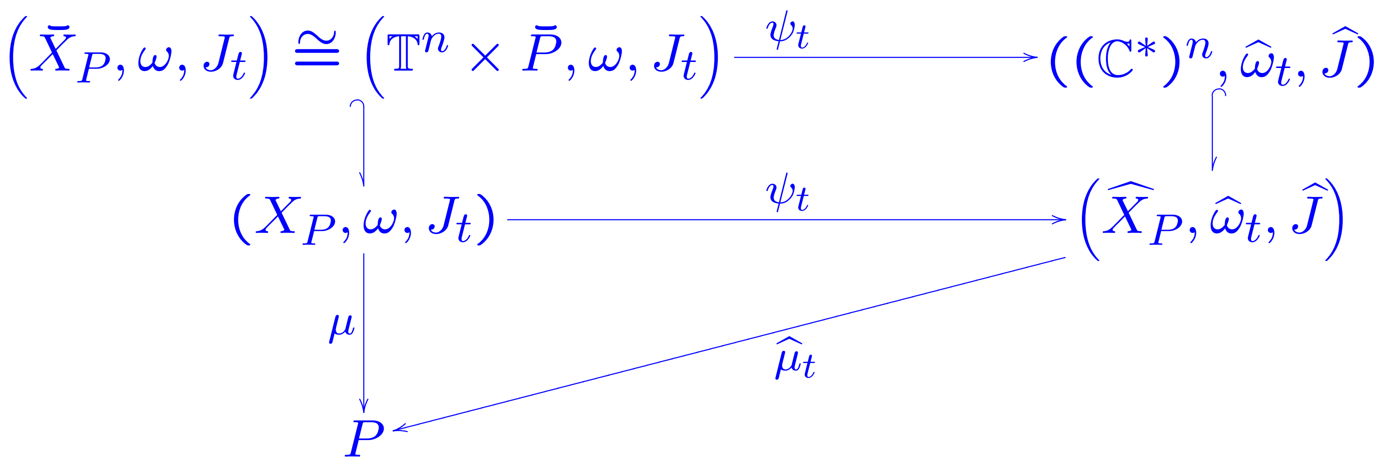
$$\psi_\phi(H, \theta) = e^{z_\phi} = e^{\frac{\partial g}{\partial H} + i\theta}$$

For a toric $h(H)$ we have $X_h = -\sum_j \frac{\partial h}{\partial H_j} \frac{\partial}{\partial \theta_j}$ and

$$e^{itX_h} e^{\frac{\partial g}{\partial H} + i\theta} = e^{\frac{\partial g}{\partial H} + i(\theta - it\frac{\partial h}{\partial H})} = e^{\frac{\partial (g+th)}{\partial H} + i\theta}$$

as in the example 2a above.

Thus we get



$$
\begin{array}{lcl}
\psi_t(\theta, H) & = & e^{y_t + i\theta} = e^{\partial g/\partial H + t\,\partial h/\partial H + i\theta} \\
g_t(H) & = & g_P(H) + \phi(H) + th(H) = \\
 & = & \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H) + t\,h(H) \\
\mu(\theta, H) & = & H \ - \text{moment map in the } \textit{symplectic} \text{ picture} \\
 & & \varphi_{it}^{X_h} = \psi_0^{-1} \circ \psi_t
\end{array}
$$

Example 4 - Complex reductive groups $K_{\mathbb{C}}$

The "adapted" Kähler structure is [Guillemin-Stenzel, Lempert-Szöke, 1991]

$$\begin{array}{rcl} (T^*K, \omega, J_0) & \xrightarrow{\psi_0} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_0) \\ (x, Y) & \mapsto & xe^{iY}, \end{array}$$

where we used $T^*K \cong TK \cong K \times \mathrm{Lie}K$

General bi-K-invariant Kähler structures on T^*K are in one-to-one correspondence with strictly convex Ad–invariant "symplectic potentials" $\breve{g}$ on Lie$K$ [Kirwin-M-Nunes, 2013].

They define and are defined by their (Weyl–invariant) restriction g to the Cartan subalgebra. One has

$$\begin{array}{ccc} (T^*K, \omega, J_{\breve{g}}) & \xrightarrow{\psi_{\breve{g}}} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_{\breve{g}}) \\ (x, Y) & \mapsto & xe^{iu(Y)}, \quad u = \dfrac{\partial \breve{g}}{\partial Y} \end{array}$$

The adapted Kähler structure corresponds to $\breve{g} = ||Y||^2/2$.

Choosing an h bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial y_j} X_j$, where $Y = \sum_j y_j T_j$. Then

$$e^{itX_h} \cdot xe^{iY} = xe^{i(Y + t\frac{\partial h}{\partial Y})} = xe^{i\frac{\partial \breve{g}_t}{\partial Y}},$$

where $\breve{g}_t = \frac{||Y||^2}{2} + th$.

2.3 Complex time evolution in quantum mechanics

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{prQ} \overset{J_0}{\rightsquigarrow} \mathcal{H}^Q_{J_0} = H^0_{J_0}(M, L) = \left\{ s \in \mathcal{H}^{prQ} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned} \mathcal{H}^Q_{J_0} &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow < \frac{\partial}{\partial \bar{z}^j_{J_0}} > = \mathcal{P}_{J_0} \\ &= \text{space of sections depending only on } z_{J_0} \\ &\quad (= \text{i.e. } J_0\text{-holomorphic}) \end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$
\begin{aligned}
\mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} < \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} > \\
&= e^{\tau \mathcal{L}_{X_h}} < X_{z^1}, \dots, X_{z^n} > = < X_{e^{\tau X_h}(z^1)}, \dots, X_{e^{\tau X_h}(z^n)} > = \\
&= < X_{z^1_\tau}, \dots, X_{z^n_\tau} > = (\varphi_\tau^{X_h})^* (\mathcal{P}_0)
\end{aligned}
$$

For some values of $\tau \neq 0$ the polarization may be real or mixed.

In our flat example, $h = y^2/2, \tau = it$,

$$\begin{aligned} \mathcal{P}_{it} &= < \frac{\partial}{\partial \bar{z}_{it}} >=< X_{z_{it}} >=< X_{x+i(1+t)y} >= \\ &= < -\frac{\partial}{\partial y} + i(1+t)\frac{\partial}{\partial x} > \stackrel{t \to -1}{\longrightarrow} < \frac{\partial}{\partial y} >= \mathcal{P}_{-i} \end{aligned}$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}^Q_{it}$ – holomorphic functions of z_{it} – Fock-like rep.

$t = -1 \Rightarrow \mathcal{H}^Q_{-i}$ – L^2-functions of x – Schrödinger representation

If two quantum quantum theories – with Hilbert spaces $\mathcal{H}_0^Q$ and $\mathcal{H}_\tau^Q$ – are equivalent there must be a unitary operator

$$U_\tau \ : \ \mathcal{H}_0^Q \longrightarrow \mathcal{H}_\tau^Q$$

intertwining the representations of relevant observable algebras.

3. Decomplexification and defintion of $\mathcal{H}^Q_{\mathcal{P}_\mu}$

One problem that motivated us initially was the fact that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic oscillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \dots, H_n) : M \to \mathbb{R}^n$ is the moment map of a $\mathbb{R}^n$ action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu =< X_{H_1}, \ldots, X_{H_n} >$$

So that $\mathcal{H}^Q_{\mathcal{P}_\mu} =??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations $\mathcal{P}_t$ degenerating to $\mathcal{P}_\mu$ as $t \to \infty$.

Theorem [M-Nunes, in preparation]
For a class of completely integrable systems (M, ω, μ), $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

1. (proved for big class)
$\lim_{t\to\infty} \mathcal{P}_t = \lim_{t\to\infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0 = \mathcal{P}_\mu.$

2. (proved for big class)
The evolution of sections under $e^{-it\widehat{h}_{GQ}}$ concentrates exponentially fast around Bohr-Sommerfeld leaves.

3. (so far proved for much smaller class)
The family $\mathcal{P}_t$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M, L)$, with a L^2-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty} \sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^Q_{\mathcal{P}_\mu} = \text{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

Proof sketch of **1.**:

$$\mathcal{P}_t = (\varphi_{it}^{X_h})^* \mathcal{P}_0 = \langle X_{z_1^{(it)}}, \ldots, X_{z_n^{(it)}} \rangle,$$

where $z_j^{(it)} = e^{-it \sum_j H_j \frac{\partial}{\partial \alpha_j}} z_j = z_j(H, \alpha + itH)$ Then

$$\frac{1}{t} X_{z_j^{(it)}} = \sum_j \left[\left(i \frac{\partial z_j}{\partial \alpha_k}(H, \alpha + itH) + \frac{1}{t} \frac{\partial z_j}{\partial H_k}(H, \alpha + itH) \right) X_{H_k} + \right.$$
$$\left. + \frac{1}{t} \frac{\partial z_j}{\partial \alpha_k}(H, \alpha + itH)\, X_{\alpha_k} \right]$$

4. Geometric tropicalization of varieties and divisors

4.1. Tropicalization: flat case

Recall the origin of tropical geometry

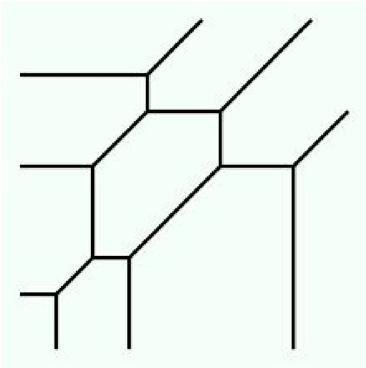
$$\hat{Y} = \left\{ w \in (\mathbb{C}^*)^n \ : \ \sum_{m \in P} c_m \, w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

$$\begin{array}{rcl} \hat{Y} & \stackrel{\mathrm{Log}_T}{\longrightarrow} & \mathbb{R}^n \\ \mathrm{Log}_T(w) & = & \dfrac{1}{t}(\log|w_1|, \ldots, \log|w_n|) \end{array}$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T \to \infty} \mathrm{Log}_T(\hat{Y})$

An example for a generic degree 3 plane curve is:



Some geometric and enumerative properties become very simple in tropical geometry.

Simple examples for plane curves are the following. Let $C, \widetilde{C}$ be nonsingular plane curves.

Genus–degree formula

$$g(C) = \frac{1}{2}(d(C) - 1)(d(C) - 2)$$

Bezout Theorem

$$|C \cap \widetilde{C}| = d(C)\ d(\widetilde{C})$$

Let us make a geodesic interpretation of the $T \to \infty$ limit.

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J) & \xrightarrow{\psi} & ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}) \\ (\theta, H) & \mapsto & e^{H+i\theta} \end{array}$$

We see that the initial moment map $\widehat{\mu}_0 \; : \; (\mathbb{C}^*)^n \to \mathbb{R}^n$ of the torus action is

$$\widehat{\mu}_0 = \mathsf{Log}_1(w) = H$$

Now let us consider a geodesic starting at $\widehat{\omega}_0$ in the direction of $h = (H_1^2 + \cdots + H_n^2)/2$ so that $X_h = -H \cdot \frac{\partial}{\partial\theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h}\, e^{H+i\theta} = e^{(1+\tilde{t})H+i\theta} = e^{tH+i\theta}$$

with $t = \tilde{t} + 1$. We have

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}_0) \\ (\theta, H) & \mapsto & e^{tH+i\theta} \end{array}$$

As $\widehat{\omega}_t = \frac{1}{t} du \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t}\, u$ so that

$$\widehat{\mu}_t(w) = \mathsf{Log}_T(w) = H$$

We see that, in this interpretation, the T of amoeba theory is exponential of the geodesic time, $T = e^t$, and the Log_T map is the moment map of the complex picture.

4.2. Definition of Geometric Tropicalization

Definition

Tropicalization (Liouville integrable) Kähler manifolds

Let $(\widehat{M}, \widehat{\omega}_t, I, \widehat{\gamma}_t)$ be a $||\mu||^2$-geodesic ray. The geometric tropicalization of $(\widehat{M}, \widehat{\omega}_0, I, \widehat{\gamma}_0)$, in the direction of $||\mu||^2$, is the Gromov-Hausdorff limit

$$\begin{aligned}(M_{\text{gtrop}}, d_{\text{gtrop}}) &= \lim_{t\to\infty} (\widehat{M}, \widehat{\gamma}_t) \\ M_{\text{gtrop}} &= \mu(M) \text{ if the fibers are connected}\end{aligned}$$

Tropicalization of divisors (for connected fibers)

The geometric tropicalization of an hypersurface $\widehat{Y} \subset \widehat{M}$, in the direction of $||\mu||^2$, is the Hausdorff limit of $\widehat{\mu}_t(\widehat{Y})$ in $\mu(M) = M_{\text{gtrop}}$,

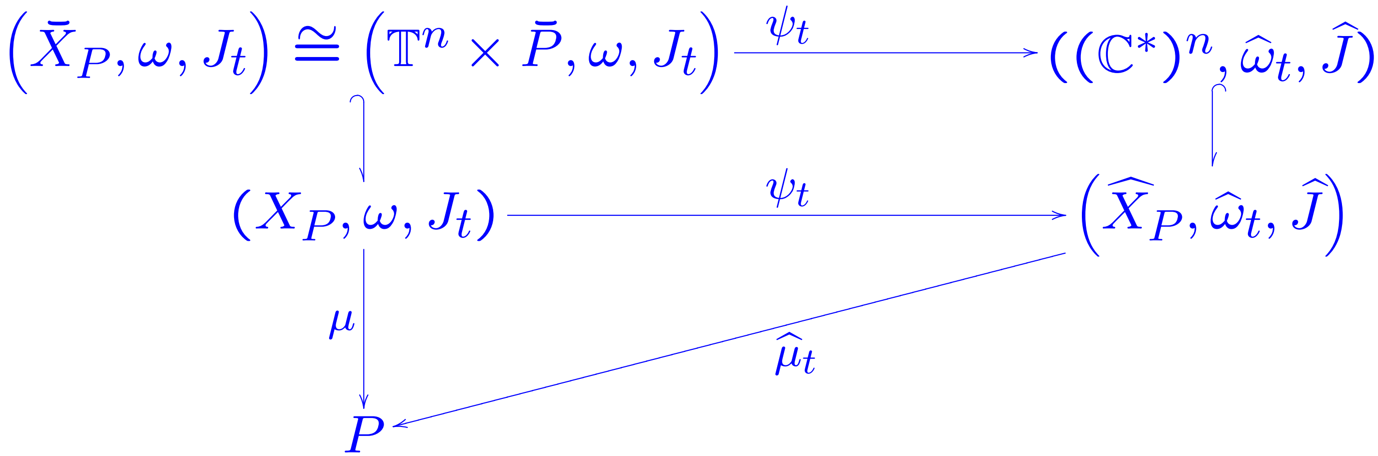
$$Y_{\text{gtrop}} = \lim_{t \to \infty} \widehat{\mu}_t(\widehat{Y}) \subset \mu(M)$$

4.3. Geometric tropicalization: toric case

What we did in [Baier-Florentino-M-Nunes, J. Diff. Geom., 2011] was (equivalent) to show that the flat picture extends to (nonflat) toric varieties

$$h = \frac{1}{2}||\mu||^2 = \frac{1}{2}(H_1^2 + \cdots + H_n^2) \Rightarrow w_t = e^{\frac{\partial}{\partial H}(g+th)+i\theta} = e^{\frac{\partial}{\partial H}(g)+tH+i\theta}$$

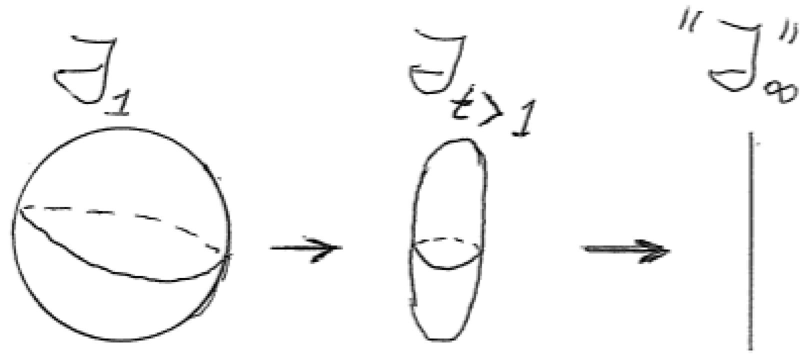
through the diagram we had before



$$
\begin{aligned}
\psi_t(\theta, H) &= e^{\partial g_t/\partial H + i\theta} = e^{\frac{\partial}{\partial H}(g) + tH + i\theta} \\
g_t(H) &= \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi + t\, h(H) \\
\mu(\theta, H) &= H \; - \text{moment map in the } \textit{symplectic} \text{ picture}
\end{aligned}
$$

[Baier-Florentino-M-Nunes, J. Diff. Geom., 2011]

- GH collapse or geometric tropicalization of toric manifolds: Metrically, as $t \to \infty$, the Kähler manifold (X_P, ω, J_t) collapses to P with metric Hess(h) on $\bar{P}$



$$\frac{1}{t}\gamma_t = \frac{1}{t}(\text{Hess}(g_t) dH^2 + \text{Hess}(g_t)^{-1} d\theta^2) \stackrel{t\to\infty}{\longrightarrow} \text{Hess}(h)\, dH^2$$

- Decomplexification: The complex (Kähler) structure degenerates to the real toric polarization

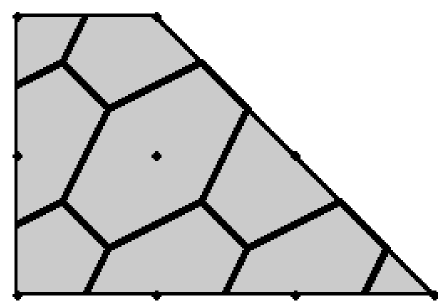
- Geometric Tropicalization of hypersurfaces: In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial H} + t m \cdot H + i m \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem (Baier-Florentino-M-Nunes, 2011)

$$\lim_{t \to \infty} \widehat{\mu}_t(Y) = \lim_{t \to \infty} \mu(Y_t) = \pi(A_{\text{trop}})$$

where π denotes the convex projection to P.



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Thank you!