

Imaginary time flow in geometric quantization and in Kahler geometry, degeneration to real polarizations and tropicalization

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1. Summary

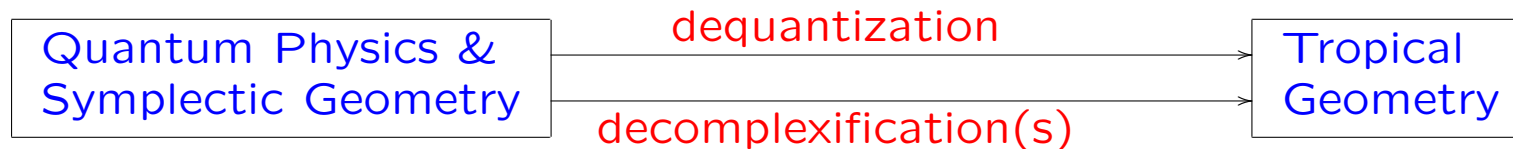
Three main topics of the seminar:

1. Hamiltonian complex time evolution in Kähler geometry and in quantum physics
2. Problem that motivated us initially: For a completely integrable system $(M, \omega, \mu = (H_1, \dots, H_n) : M \rightarrow \mathbb{R}^n)$ define its quantization in the (usually singular) real polarization \mathcal{P}_μ defined by μ ,

$$\mathcal{H}_{\mathcal{P}_\mu}^Q = ??$$

3. New relation:

[for (some?) completely integrable systems $(M, \omega, \mu : M \rightarrow \mathbb{R}^n)$]



Dequantization

– $\hbar \rightarrow 0$

Decomplexifications

– (Kähler) Geometric degenerations to tropical varieties
= geometric tropicalization:

=: Follow (to infinite time) a geodesic ray,
in the space of Kähler metrics,

generated by $h = \|\mu\|^2 = H_1^2 + \dots + H_n^2$,

= Do a Wick rotation, time $\rightsquigarrow is$, followed by $s \rightarrow \infty$
for $h = \|\mu\|^2$

2. Complex time evolution

2.1 Imaginary time: introduction

Geometric quantization

$$(M, \omega), \quad \frac{1}{\hbar}[\omega] \in H^2(M, \mathbb{Z})$$

Prequantum data: $(L, \nabla, h), L \rightarrow M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\text{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^\infty(M, L) : \|s\|^2 = \int_M h(s, s) \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\hat{f} = Q_{\hbar}(f) = -i\hbar\nabla_{X_f} + f$

Kähler quantization: fix a complex structure I on M such that (M, ω, I, γ) is a Kähler manifold.

$$\mathcal{H}_I^Q = \{s \in \mathcal{H}^{\text{prQ}} : \nabla_{\bar{\partial}} s = 0\} = H^0(M, L_I)$$

Get the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

Need to study the dependence of quantization on the choice of the complex structure.

It is precisely to study the dependence of $Q_{\hbar, \mathcal{P}}$ on the choice of the complex structure (or more generally on the polarization) that evolution in imaginary time enters the scene.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in order to transform the spin connection to the Ashtekar connections $\Gamma_\mu \mapsto A_\mu^{\mathbb{C}}$. [Thiemann, Class. Quant. Grav., 1996]

In non-Hermitian Quantum Mechanics one considers time evolution associated with complex valued observables [Moiseyev, Cambridge University Press, 2011]

In \mathcal{PT} -symmetric quantum mechanics one deals with complex valued observables which may sometimes have meaningful real spectrum – Bessis and Zinn-Justin example: $H = p^2 + ix^3$ [Dorey-Dunning-Tateo, J.Phys.A, 2007, Section 2]

What we are studying is a **new way of looking at imaginary time evolution** in (some situations in) quantum mechanics giving it a precise geometric meaning.

In Kähler geometry imaginary time evolution leads to geodesics and is used to study the stability of varieties [Semmes '92 and Donaldson '99].

2.2 Complex time evolution in Kähler geometry

$G = \text{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with $\text{Lie}(\text{Ham}(M, \omega)) = C^\infty(M)$

$G_{\mathbb{C}} = \text{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits. There are natural orbits of two types passing through a Kähler pair (ω, J_0) .

Complex picture

Let us start with the smaller one

$$\begin{aligned}\mathcal{H}(\omega, J_0) &= \{f \in C^\infty(M) : \omega + i\bar{\partial}_0\partial_0 f > 0\} / \mathbb{R} =: \\ &=: \{(\varphi^*\omega, J_0), \varphi \in G_{\mathbb{C}}\} = (G_{\mathbb{C}} \cdot \omega, J_0)\end{aligned}$$

So we get a orbit-dependent “definition” of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of $\text{Diff}(M)$

$\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space:

$\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ and Donaldson shows that the Mabuchi metric has constant negative curvature mimiking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \text{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that $\text{Aut}_0(M, J_0)$ is trivial the second natural orbit is the total space of a G principal bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0), \varphi \in G_{\mathbb{C}}\}$$

with projection

$$\begin{aligned} \pi : \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) &\longrightarrow \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \\ \pi(\omega, \varphi_* J_0) &= \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{aligned}$$

Gröbner Lie series

Let (M, ω) be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter subgroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\text{an}}(M)$

Theorem [Gröbner]

Let $f \in C^{\text{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} : f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \quad (1)$$

converges absolutely to a (complex) real analytic function.

So $\text{Ham}_{\mathbb{C}}(M, \omega)$ acts locally on $C^{\text{an}}(M)$. To descend to an “action” on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$ be local J_0 -holomorphic coordinates. Then we use (1) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \quad (2)$$

Theorem [Burns-Lupercio-Urbe, 2013; M-Nunes, 2013]

Under natural conditions on J_0 there exists a $T_1 > 0$ such that (2) defines, for every $|\tau| < T_1$, a unique map $\varphi_\tau^{X_h} : M \rightarrow M$ for which

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

There exists a $T_2 \leq T_1, T_2 > 0$ such that, for every $\tau : |\tau| < T_2$, $\varphi_\tau^{X_h} : (M, J_\tau) \rightarrow (M, J_0)$ is a biholomorphism.

Remarks

1. The map $\tau \mapsto \varphi_{\tau}^{X_h}$ is not (not even a local) homomorphism from (the additive group of complex numbers) \mathbb{C} to $\text{Diff}(M)$ [though its restriction to \mathbb{R} of course is]. It satisfies a grupoid property though, which follows from Burns-Lupercio-Urbe. \mathbb{C} acts on the space of polarizations via (2), leading to the action grupoid $\{(\tau, \mathcal{P})\}$ with composition

$$(\tau_2, \mathcal{P}_2) \circ (\tau_1, \mathcal{P}_1) = (\tau_1 + \tau_2, \mathcal{P}_1)$$

defined only if $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$. Then notice that the map $\tau \mapsto \varphi_{\tau}^{X_h}$ depends on the polarization one starts with. By extending the domain to the action grupoid

$$(\tau, \mathcal{P}) \mapsto \varphi_{\tau}^{X_h, \mathcal{P}} \tag{3}$$

one can show that the following composition law is verified. If $\mathcal{P}_2 = \tau_1 \cdot \mathcal{P}_1$ then

$$\varphi_{\tau_2}^{X_h, \mathcal{P}_2} \circ \varphi_{\tau_1}^{X_h, \mathcal{P}_1} = \varphi_{\tau_1 + \tau_2}^{X_h, \mathcal{P}_1}$$

or, more precisely, the map (3) is a functor from a subcategory of the action grupoid to (the one-object category) $\text{Map}(M)$.

2. If the polarization \mathcal{P}_1 is complex and the polarization $\mathcal{P}_2 = \tau \cdot \mathcal{P}_1$ is real or mixed then $\varphi_{\tau}^{X_h, \mathcal{P}_1}$ exists but $\varphi_{-\tau}^{X_h, \mathcal{P}_2}$ does not exist eventhough, $\mathcal{P}_1 = (-\tau) \cdot \mathcal{P}_2$.

Example 1 - $M = \mathbb{R}^2$

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_0)$ and $h = y^2/2 \Rightarrow X_h = y \frac{\partial}{\partial x}$

Then

$$\varphi_t^{X_h}(x, y) = (x + ty, y)$$

and

$$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$$

Indeed: $e^{ity \frac{\partial}{\partial y}}(x + iy) = x + i(1 + t)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow \nabla^{\gamma_{it}} h = J_{it} X_h = \frac{1}{1+t} y \frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t} dx^2 + (1 + t) dy^2$.

Example 2 - Toric Varieties

P – Delzant polytope

$$P = \{H = (H_1, \dots, H_n) \in \mathbb{R}^n : \ell_F(H) = \nu_F \cdot H + \lambda_F \geq 0, F \text{ facet of } P\}.$$

Let X_P be the associated toric variety,

$$\tilde{X}_P \cong \mathbb{T}^n \times \tilde{P} \subset T^*\mathbb{T}^n, \quad \omega = dH \wedge d\theta$$

Toric complex structures are in one-to-one correspondence with the space of symplectic potentials (Guillemin-Abreu theory)

$$g(H) = g_P(H) + \phi(H) = \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H)$$

via the Legendre transform

$$\left(\bar{X}_P, \omega, J_\phi \right) \cong \left(\mathbb{T}^n \times \bar{P}, \omega, J_\phi \right) \xrightarrow{\psi_\phi} \left((\mathbb{C}^*)^n, \hat{\omega}_\phi, \hat{J} \right)$$

where

$$\psi_\phi(H, \theta) = e^{z\phi} = e^{\frac{\partial g}{\partial H} + i\theta}$$

For a toric $h(H)$ we have $X_h = -\sum_j \frac{\partial h}{\partial H_j} \frac{\partial}{\partial \theta_j}$ and

$$e^{itX_h} e^{\frac{\partial g}{\partial H} + i\theta} = e^{\frac{\partial g}{\partial H} + i(\theta - it \frac{\partial h}{\partial H})} = e^{\frac{\partial(g+th)}{\partial H} + i\theta}$$

Thus we get

$$\begin{array}{ccc}
 (\bar{X}_P, \omega, J_t) \cong (\mathbb{T}^n \times \bar{P}, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J}) \\
 \downarrow & & \downarrow \\
 (X_P, \omega, J_t) & \xrightarrow{\psi_t} & (\widehat{X}_P, \hat{\omega}_t, \hat{J}) \\
 \downarrow \mu & & \nearrow \hat{\mu}_t \\
 P & &
 \end{array}$$

$$\begin{aligned}
 \psi_t(\theta, H) &= e^{yt+i\theta} = e^{\partial g_t / \partial H + i\theta} \\
 g_t(H) &= g_P(H) + \phi(H) + th(H) = \\
 &= \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H) + th(H) \\
 \mu(\theta, H) &= H - \text{moment map in the symplectic picture} \\
 \varphi_{it}^{X_h} &= \psi_0^{-1} \circ \psi_t
 \end{aligned}$$

Example 3 - Complex reductive groups $K_{\mathbb{C}}$

The “adapted” Kähler structure is [Guillemin-Stenzel, Lempert-Szöke, 1991]

$$\begin{aligned} (T^*K, \omega, J_0) &\xrightarrow{\psi_0} (K_{\mathbb{C}}, \hat{\omega}, \hat{J}_0) \\ (x, Y) &\mapsto xe^{iY}, \end{aligned}$$

where we used $T^*K \cong TK \cong K \times \text{Lie}K$

General bi- K -invariant Kähler structures on T^*K are in one-to-one correspondence with strictly convex Ad -invariant “symplectic potentials” \check{g} on $\text{Lie}K$ [Kirwin-M-Nunes, 2013].

They define and are defined by their (Weyl–invariant) restriction g to the Cartan subalgebra. One has

$$\begin{aligned} (T^*K, \omega, J_{\check{g}}) &\xrightarrow{\psi_{\check{g}}} (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_{\check{g}}) \\ (x, Y) &\mapsto xe^{iu(Y)}, \quad u = \frac{\partial \check{g}}{\partial Y} \end{aligned}$$

The adapted Kähler structure corresponds to $\check{g} = \|Y\|^2/2$.

Choosing an h bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial y_j} X_j$, where $Y = \sum_j y_j T_j$. Then

$$e^{itX_h} \cdot xe^{iY} = xe^{i(Y + t \frac{\partial h}{\partial Y})} = xe^{i \frac{\partial \check{g}_t}{\partial Y}},$$

where $\check{g}_t = \frac{\|Y\|^2}{2} + th$.

2.3 Complex time evolution in quantum mechanics

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{prQ} \xrightarrow{J_0} \mathcal{H}_{J_0}^Q = H_{J_0}^0(M, L) = \left\{ s \in \mathcal{H}^{prQ} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned} \mathcal{H}_{J_0}^Q &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow \left\langle \frac{\partial}{\partial \bar{z}_{J_0}^j} \right\rangle = \mathcal{P}_{J_0} \\ &= \text{space of sections depending only on } z_{J_0} \\ &\quad (= \text{i.e. } J_0\text{-holomorphic}) \end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$\begin{aligned}
 \mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} \left\langle \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\rangle \\
 &= e^{\tau \mathcal{L}_{X_h}} \langle X_{z^1}, \dots, X_{z^n} \rangle = \langle X_{e^{\tau X_h}(z^1)}, \dots, X_{e^{\tau X_h}(z^n)} \rangle = \\
 &= \langle X_{z_\tau^1}, \dots, X_{z_\tau^n} \rangle = (\varphi_\tau^{X_h})^*(\mathcal{P}_0)
 \end{aligned}$$

For some values of $\tau \neq 0$ the polarization may be real or mixed.

In our flat example, $h = y^2/2, \tau = it,$

$$\begin{aligned} \mathcal{P}_{it} &= \left\langle \frac{\partial}{\partial \bar{z}_{it}} \right\rangle = \left\langle X_{z_{it}} \right\rangle = \left\langle X_{x+i(1+t)y} \right\rangle = \\ &= \left\langle -\frac{\partial}{\partial y} + i(1+t)\frac{\partial}{\partial x} \right\rangle \xrightarrow{t \rightarrow -1} \left\langle \frac{\partial}{\partial y} \right\rangle = \mathcal{P}_{-i} \end{aligned}$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}_{it}^Q$ – holomorphic functions of z_{it} – **Fock-like rep.**

$t = -1 \Rightarrow \mathcal{H}_{-i}^Q$ – L^2 -functions of x – **Schrödinger representation**

If two quantum theories – with Hilbert spaces \mathcal{H}_0^Q and \mathcal{H}_T^Q – are equivalent there must be a unitary operator

$$U_T : \mathcal{H}_0^Q \longrightarrow \mathcal{H}_T^Q$$

intertwining the representations of relevant observable algebras.

3. Decomplexification and definition of $\mathcal{H}_{\mathcal{P}\mu}^Q$

One problem that motivated us initially was the fact that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic oscillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \dots, H_n) : M \rightarrow \mathbb{R}^n$ is the moment map of a \mathbb{R}^n action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu = \langle X_{H_1}, \dots, X_{H_n} \rangle$$

So that $\mathcal{H}_{\mathcal{P}_\mu}^Q = ??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations \mathcal{P}_t degenerating to \mathcal{P}_μ as $t \rightarrow \infty$.

Theorem [M-Nunes, 2013]

For a class of completely integrable systems (M, ω, μ) , $h = \|\mu\|^2$, and starting Kähler polarizations \mathcal{P}_0

1.(proved for big class)

$$\lim_{t \rightarrow \infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0 = \mathcal{P}_\mu.$$

2. (so far proved for much smaller class)

The family $\mathcal{P}_t = (\varphi_{it}^{X_h})^* \mathcal{P}_0$ selects a basis of holomorphic sections of $H_{\mathcal{P}_t}^0(M, L)$, with a L^2 -normalized holomorphic section $\sigma_{t, \mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization \mathcal{P}_μ . Then

$$\lim_{t \rightarrow \infty} \sigma_{t, \mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}_{\mathcal{P}_\mu}^Q = \text{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

4. Geometric tropicalization of varieties and divisors

4.1. Tropicalization: flat case

Recall the origin of tropical geometry

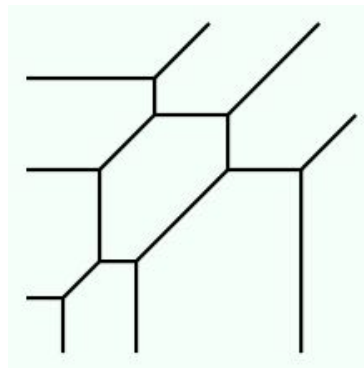
$$\hat{Y} = \left\{ w \in (\mathbb{C}^*)^n : \sum_{m \in P} c_m w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\text{Log}_T} & \mathbb{R}^n \\ \text{Log}_T(w) & = & \frac{1}{t} (\log |w_1|, \dots, \log |w_n|) \end{array}$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T \rightarrow \infty} \text{Log}_T(\hat{Y})$

An example for a generic degree 3 plane curve is:



Some geometric and enumerative properties become very simple in tropical geometry.

Simple examples for plane curves are the following. Let C, \tilde{C} be nonsingular plane curves.

Genus–degree formula

$$g(C) = \frac{1}{2}(d(C) - 1)(d(C) - 2)$$

Bezout Theorem

$$|C_1 \cap C_2| = d(C_1) d(C_2)$$

Let us make a **geodesic** interpretation of the $T \rightarrow \infty$ limit.

$$\begin{aligned} (T^*\mathbb{T}^n, \omega, J) &\xrightarrow{\psi} ((\mathbb{C}^*)^n, \hat{\omega}, \hat{J}) \\ (\theta, H) &\mapsto e^{H+i\theta} \end{aligned}$$

We see that the initial moment map $\hat{\mu}_0 : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$ of the torus action is

$$\hat{\mu}_0 = \text{Log}_1(w) = H$$

Now let us consider a geodesic starting at $\hat{\omega}_0$ in the direction of $h = (H_1^2 + \dots + H_n^2)/2$ so that $X_h = -H \cdot \frac{\partial}{\partial \theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h} e^{H+i\theta} = e^{(1+\tilde{t})H+i\theta} = e^{tH+i\theta}$$

with $t = \tilde{t} + 1$. We have

$$\begin{aligned} (T^*\mathbb{T}^n, \omega, J_t) &\xrightarrow{\psi_t} ((\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J}_0) \\ (\theta, H) &\mapsto e^{tH+i\theta} \end{aligned}$$

As $\widehat{\omega}_t = \frac{1}{t} du \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t} u$ so that

$$\widehat{\mu}_t(w) = \text{Log}_T(w) = H$$

We see that, in this interpretation, the T of amoeba theory is exponential of the geodesic time, $T = e^t$, and the Log_T map is the moment map of the complex picture.

4.2. Definition of Geometric Tropicalization

Definition

Tropicalization (Liouville integrable) Kähler manifolds

Let $(\widehat{M}, \widehat{\omega}_t, I, \widehat{\gamma}_t)$ be a $\|\mu\|^2$ -geodesic ray. The geometric tropicalization of $(\widehat{M}, \widehat{\omega}_0, I, \widehat{\gamma}_0)$, in the direction of $\|\mu\|^2$, is the Gromov-Hausdorff limit

$$\begin{aligned} (M_{\text{gtrop}}, d_{\text{gtrop}}) &= \lim_{t \rightarrow \infty} (\widehat{M}, \widehat{\gamma}_t) \\ M_{\text{gtrop}} &= \mu(M) \text{ if the fibers are connected} \end{aligned}$$

Tropicalization of divisors (for connected fibers)

The geometric tropicalization of an hypersurface $\hat{Y} \subset \hat{M}$, in the direction of $\|\mu\|^2$, is the Hausdorff limit of $\hat{\mu}_t(\hat{Y})$ in $\mu(M) = M_{\text{gtrop}}$,

$$Y_{\text{gtrop}} = \lim_{t \rightarrow \infty} \hat{\mu}_t(\hat{Y}) \subset \mu(M)$$

4.3. Geometric tropicalization: toric case

What we did in [Baier-Florentino-M-Nunes, J. Diff. Geom., 2011] was (equivalent) to show that the flat picture extends to (nonflat) toric varieties

$$h = \frac{1}{2} \|\mu\|^2 = \frac{1}{2} (H_1^2 + \dots + H_n^2) \Rightarrow w_t = e^{\frac{\partial}{\partial H}(g+th) + i\theta}$$

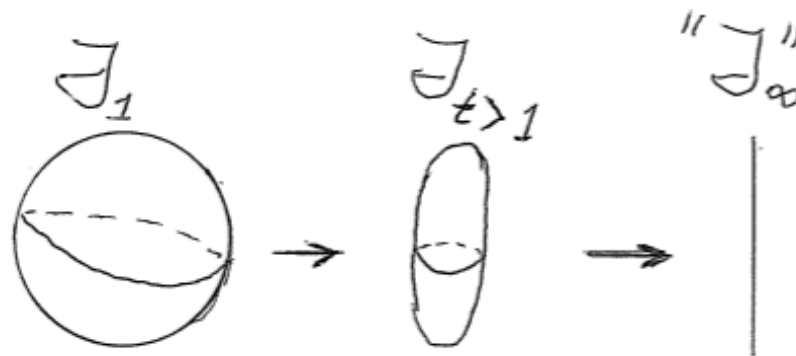
through the diagram we had before

$$\begin{array}{ccc}
(\bar{X}_P, \omega, J_t) \cong (\mathbb{T}^n \times \bar{P}, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J}) \\
\downarrow & & \downarrow \\
(X_P, \omega, J_t) & \xrightarrow{\psi_t} & (\hat{X}_P, \hat{\omega}_t, \hat{J}) \\
\downarrow \mu & & \downarrow \hat{\mu}_t \\
P & & P
\end{array}$$

$$\begin{aligned}
\psi_t(\theta, H) &= e^{u_t + i\theta} = e^{\partial g_t / \partial H + i\theta} \\
g_t(H) &= \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi + t h(H) \\
\mu(\theta, H) &= H - \text{moment map in the symplectic picture}
\end{aligned}$$

[Baier-Florentino-M-Nunes, J. Diff. Geom., 2011]

- **GH collapse or geometric tropicalization of toric manifolds:**
 Metrically, as $t \rightarrow \infty$, the Kähler manifold (X_P, ω, J_t) collapses to P with metric $\text{Hess}(h)$ on \bar{P}



$$\frac{1}{t}\gamma_t = \frac{1}{t}(\text{Hess}(g_t)dH^2 + \text{Hess}(g_t)^{-1}d\theta^2) \xrightarrow{t \rightarrow \infty} \text{Hess}(h) dH^2$$

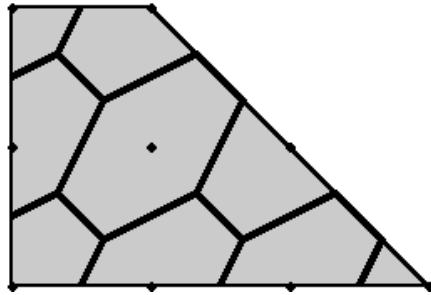
- **Decomplexification:** The complex (Kähler) structure degenerates to the real toric polarization
- **Geometric Tropicalization of hypersurfaces:** In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial H} + tm \cdot H + im \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem (Baier-Florentino-M-Nunes, 2011)

$$\lim_{t \rightarrow \infty} \hat{\mu}_t(Y) = \lim_{t \rightarrow \infty} \mu(Y_t) = \pi(A_{\text{trop}})$$

where π denotes the convex projection to P .



5. Some References (and work in progress)

Symplectic and Toric Geometry

- [Ab] M. Abreu, *Toric Kähler metrics*, Journal of Geometry and Symmetry in Physics, **17** (2010) 1–33.
- [AG] V. I. Arnold, A. B. Givental, *Symplectic Geometry*.
- [dS] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer LNM, 2008.
- [Gu] V. Guillemin, *Kähler structures on toric varieties*, J. Diff. Geom. (1994), 285–309.
- [Wu] S. Wu, *Lectures on Symplectic Geometry*, 2012.

Geometric quantization and related

- [AGL] J. Andersen, N. Gammelgaard, M. Lauridsen, *Hitchin's Connection in Half-Form Quantization* arXiv:0711.3995
- [CPS] E. Clader, N. Priddis, M. Shoemaker, *Geometric Quantization with Applications to Gromov-Witten Theory*, arXiv:1309.1150
- [Ga] N.L. Gammelgaard, *Kähler Quantization and Hitchin Connections*, PhD Thesis, 2010.

- [JW1] L. Jeffrey, J. Weitsman, *Half density quantization of the moduli space of flat connections and Witten's semiclassical manifold invariants*, *Topology*, **32** (1993) 509–529.
- [KW] W. Kirwin, S. Wu, *Momentum space for compact Lie groups and Peter-Weyl theorem*, in preparation.
- [ncl1] ncatlab, *Geometric Quantization*
- [Ta] T. Takakura, *Degeneration of Riemann surfaces and intermediate polarization of the moduli space of flat connections*, *Invent. Math.* **123** (1996) 431–452
- [Wi1] E. Witten, *Quantization and topological field theory, slides of a seminar*
- [Wi2] E. Witten, *A New Look At The Path Integral Of Quantum Mechanics*, arXiv:1009.6032
- [Wo] N. Woodhouse, *Geometric Quantization*, OUP, 1997

Integrable systems and related

- [AM] A. Alekseev, E. Meinrenken, *Ginzburg-Weinstein via Gelfand-Zeitlin*, J. Diff. Geom. **76** (2007) 1–34.
- [AT] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. Lond. Math. Soc. **14** (1982) 1–15.
- [BF] A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems, Geometry, Topology, Classification*, Chapman & Hall, 2004.
- [Bu] D. Burns, *Toric Degenerations and the Exact Bohr-Sommerfeld Correspondence*, slides, Steklov, 2010.
- [Ca] M. Carl, *Approaches to the Quantization of Integrable Systems with Singularities*, PhD Thesis, 2008.
- [CLL] K. Chan, S. Lau, N. C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Diff. Geom. **90** (2012) 177–250.
- [De] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bull. Soc. Math. France **116** (1988) 315–339.

- [GS1] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982) 491–513.
- [GS2] V. Guillemin and S. Sternberg, *The Gelfand–Cetlin system and quantization of flag manifolds*, Journ. Funct. Anal. **52** (1983) 106–128.
- [HaKo] M. Hamilton and H. Konno, *Convergence of Kähler to real polarizations on flag manifolds via toric degenerations*, arXiv:1105.0741.
- [HaKa] M. Harada, K. Kaveh, *Integrable systems, toric degenerations and Okounkov bodies*, arXiv:1205.5249.
- [JW2] L. Jeffrey, J. Weitsman, *Bohr-Sommerfeld Orbits in the Moduli Space of Flat connections and the Verlinde Dimension Formula*, Commun. Math. Phys. **150** (1992) 593–630.
- [KMNW] W. Kirwin, J. Mourão, J.P. Nunes, S. Wu, *Decomplexification of flag manifolds*, in preparation.
- [PV] A. Pelayo, S. Vũ Ngoc, *First steps in symplectic and spectral theory of integrable systems*, Discrete and Continuous Dynamical Systems, Series A, **32** (2012) 3325–3377.

Imaginary time in Kähler geometry and related

- [BFMN2] T. Baier, C. Florentino, J. M. Mourão, and J. P. Nunes, *Symplectic description of compactifications of reductive groups*, work in progress.
- [BLU] D. Burns, E. Lupercio and A. Uribe, *The exponential map of the complexification of Ham in the real-analytic case*, arXiv:1307.0493.
- [Do1] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Amer. Math. Soc. Transl. (2) **196** (1999), 13–33.
- [Do2] S. K. Donaldson, *Moment maps and diffeomorphisms*, Asian, J. Math., **3** (1999), 1–16.
- [Gr] W. Gröbner, *General theory of the Lie series*, in *Contributions to the method of Lie series*, W. Gröbner and H. Knapp Eds., Bibliographisches Institut Mannheim, 1967.
- [MN1] J. Mourão and J. P. Nunes, *On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics*, arXiv:1310.4025.

[RZ] Y. Rubinstein and S. Zelditch, *The Cauchy problem for the homogeneous Monge-Ampère equation, II. Legendre transform*, Adv. Math. **228** (2011), 2989-3025.

[Se] S. Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. **114** (1992), 495–550.

Imaginary time in geometric quantization

[BFMN1] T. Baier, C. Florentino, J. M. Mourão, and J. P. Nunes, *Toric Kähler metrics seen from infinity, quantization and compact tropical amoebas*, J. Diff. Geom. **89** (2011), 411-454.

[FMMN1] C. Florentino, P. Matias, J. M. Mourão, and J. P. Nunes, *Geometric quantization, complex structures and the coherent state transform*, J. Funct. Anal. **221** (2005), 303–322.

- [FMMN2] C. Florentino, P. Matias, J. M. Mourão, and J. P. Nunes, *On the BKS pairing for Kähler quantizations of the cotangent bundle of a Lie group*, J. Funct. Anal. **234** (2006)180–198.
- [GS3] V. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Diff. Geom. **34** (1991), no. 2, 561–570.
- [Ha] B. Hall, *Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type*, Comm. Math. Phys. **226** (2002) 233–268.
- [HK1] B. Hall and W. D. Kirwin, *Adapted complex structures and the geodesic flow*, Math. Ann. **350** (2011), 455–474.
- [HK2] B. Hall and W. D. Kirwin, *Complex structures adapted to magnetic flows*, arXiv:1201.2142.
- [KMN1] W. D. Kirwin, J. M. Mourão, and J. P. Nunes, *Degeneration of Kähler structures and half-form quantization of toric varieties*, Journ. Symplectic Geometry, **11** (2013) 603–643

- [KMN2] W. D. Kirwin, J. M. Mourão, and J. P. Nunes, *Complex time evolution in geometric quantization and generalized coherent state transforms*, Journ. Funct. Anal. **265** (2013) 1460–1493.
- [LS1] L. Lempert and R. Szöke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of riemannian manifolds*, Math. Annalen **319** (1991), no. 4, 689–712.
- [LS2] L. Lempert and R. Szöke, *A new look at adapted complex structures*, Bull. Lond. Math. Soc. **44** (2012), 367–374.
- [MN2] J. Mourão and J. P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*, in preparation.
- [Th1] T. Thiemann, *Reality conditions inducing transforms for quantum gauge field theory and quantum gravity*, Class.Quant.Grav. **13** (1996), 1383–1404.
- [Th2] T. Thiemann, *Gauge field theory coherent states (GCS). I. General properties*, Classical Quantum Gravity **18** (2001), no. 11, 2025–2064.

Tropical geometry, mirror symmetry and Gromov-Hausdorff limits

[BBI] D. Burago, Y. Burago, S. Ivanov. *A course in metric geometry*, GSM, American Math Soc., 2001.

[Gr] M. Gross, *Tropical geometry and mirror symmetry*, CBMS, American Math Soc., 2011.

[MS] D. Mclagan, B. Sturmfels, *Introduction to Tropical Geometry*, book in progress

[Mi] G. Mikhalkin, *Tropical geometry and its applications*, Proceedings ICM, Madrid 2006, pp. 827–852.

[ncl2] ncatlab, *Tropical geometry*

Decomplexification and Geometric Tropicalization

[BFMN1], [KMN1], [MN2]

Thank you!