Tropicalization in symplectic geometry and degeneration to real polarizations

or (Geometric degeneration to tropical varieties in symplectic geometry)

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Conference: Tropical Aspects in Geometry and Topology

September, 2–6, Bonn MPIM

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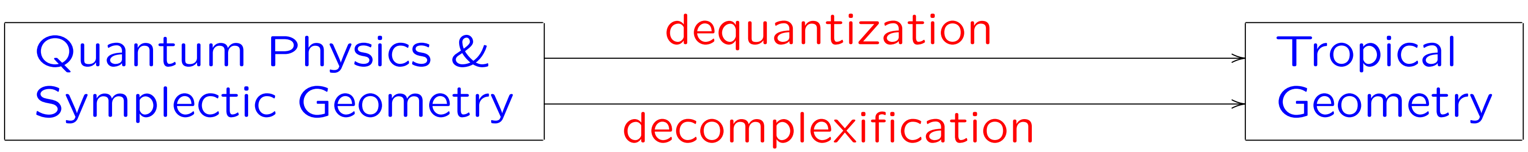
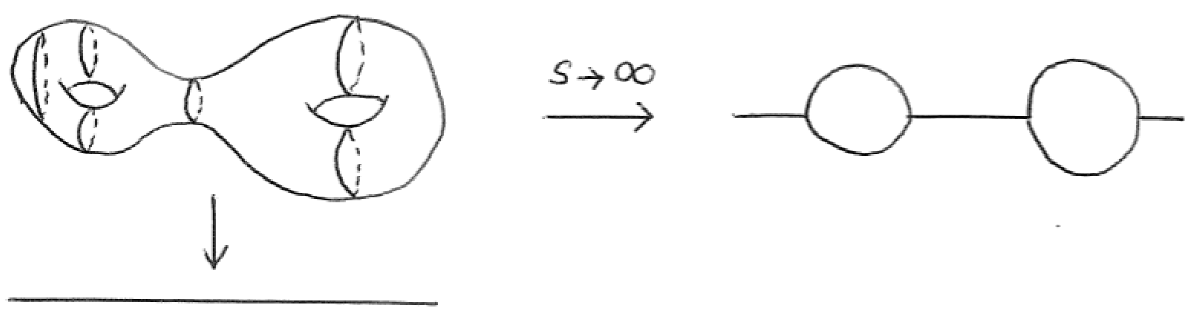
1. Summary

New relation:
[for completely integrable systems $(X^{2n}, \omega, \mu\ :\ X \to \mathbb{R}^n)$]



Dequantization	$-$	$\hbar \to 0$
Decomplexifications	$-$	(Kähler) Geometric degenerations to tropical varieties:
	$:=$	Follow (to infinite time) a geodesic ray, in the space of Kähler metrics, generated by $H = \|\|\mu\|\|^2 = f_1^2 + \cdots + f_n^2$,
	$=$	Do a Wick rotation, time $\rightsquigarrow is$, followed by $s \to \infty$ for $H = \|\|\mu\|\|^2$

Imaginary time $s \to \infty$ limit



2. Motivation

Start with a hypersurface $\widehat{Y} \subset (\mathbb{C}^*)^n$ and let $t = e^s$,

$$
\begin{aligned}
(\mathbb{C}^*)^n \supset \widehat{Y} &= \{w \in (\mathbb{C}^*)^n \ : \ \textstyle\sum_{m\in\mathcal{P}} c_m w^m = 0\} \\
T^*\mathbb{T}^n \supset Y_t &= \left\{(\theta, x) \in \mathbb{T}^n \times \mathbb{R}^n \ : \ \textstyle\sum_{m\in\mathcal{P}} c_m e^{sm\cdot x + im\cdot\theta} = 0\right\}
\end{aligned}
$$

Q1 Can we see $w_{t_j} = e^{sx_j + i\theta_j}$ as defining a one parameter family of complex structures in $T^*\mathbb{T}^n$?

A1 Yes

Q2 Can this be extended to toric varieties = (partial equivariant) compactifications of $(\mathbb{C}^*)^n$?

A2 Not directely. For the complex structure to be defined in the added divisors we need to add a constant term to the holomorphic coordinates which doesn't change with s!

$w_t = e^{y_Q + sx + i\theta}$ or, more generally, $w_t = e^{y_Q + s\partial H/\partial x + i\theta}$

where Q is the polytope, image of the moment map, $Q = \mu(X)$.

Q3 In this framework, what is the symplectic geometry interpretation of the (generalization of the) Log_t map?

A3 Is just the moment map (or its H-Legendre transform): $L_H \circ \mu$

Q4 Can A2 and A3 be extended to more general integrable systems?

A4 Yes proved in some cases – conjectured always – work in progress

3. $(\mathbb{C}^*)^n$– case, flat family

3.1 - Complex and Symplectic Pictures

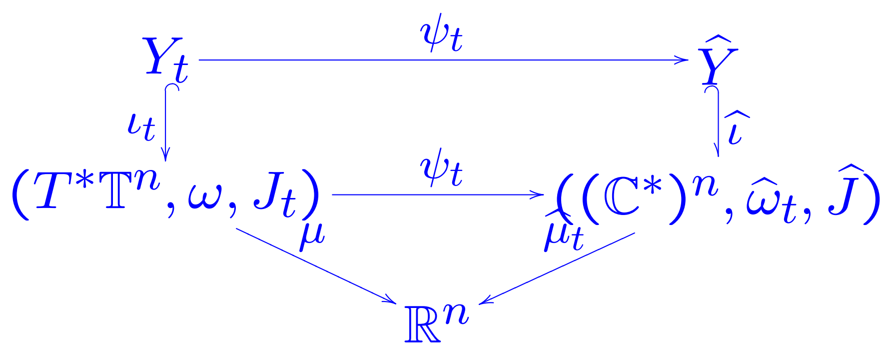
Log$_t$ map as a moment map

$T^*\mathbb{T}^n$ - symplectic: $\omega = dx \wedge d\theta$
$(\mathbb{C}^*)^n$ - complex

$$
\begin{array}{ccccc}
(T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\widetilde{\psi}_t} & (T\mathbb{T}^n, \widetilde{\omega}_t, \widetilde{J}) & \xrightarrow{\varphi} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}) \\
\downarrow \mu & & & \swarrow \widehat{\mu}_t & \\
\mathbb{R}^n & & & &
\end{array}
$$

$$
\begin{aligned}
\widetilde{\psi}_t(\theta, x) &= (\theta, sx),\ s = \log(t) \\
\varphi(\theta, y) &= e^{y+i\theta} \ \text{– inverse of the polar decomposition} \\
\psi_t(\theta, x) &= \varphi \circ \widetilde{\psi}_t(\theta, sx) = e^{sx+i\theta}, \\
\mu(\theta, x) &= x \ \text{– moment map in the } \textit{symplectic} \text{ picture} \\
\widehat{\mu}_t(w) &= \mathrm{Log}_t(w) \ \text{–}\ \widehat{\omega}_t\text{-moment map in the } \textit{complex} \text{ picture}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{Y} &= \left\{\textstyle\sum_{m\in\mathcal{P}} c_m w^m = 0\right\} \text{ -- complex picture} \\
Y_t &= \left\{\textstyle\sum_{m\in\mathcal{P}} c_m e^{sm\cdot x + im\cdot\theta} = 0\right\} \text{ -- symplectic picture}
\end{aligned}
$$



The amoebas coincide

$$
\begin{aligned}
A_t &= \widehat{\mu}_t \circ \widehat{\iota}(\widehat{Y}) = \text{t-dependent $\widehat{\omega}_t$-moment image of a fixed subvariety} \\
&= \mu \circ \iota_t(Y_t) = \text{image of a subvariety moving, due to change in J_t}
\end{aligned}
$$

3.2 - J_t degenerates to what?

"J_∞" as a (GQ) real polarization

J_t-polarized functions are solutions of the CR equations

$$\frac{\partial}{\partial \bar{z}_t} f = 0 \Leftrightarrow \frac{1}{\log t} \frac{\partial}{\partial x} f + i \frac{\partial}{\partial \theta} f = 0$$

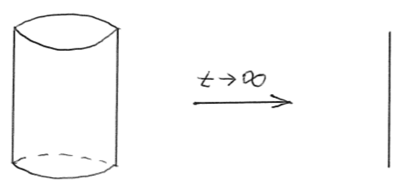
"J_∞" can be defined through its polarized functions which are solutions of

$$\frac{\partial}{\partial \theta} f = 0 \Leftrightarrow f(\theta, x) = F(x)$$

i.e. $\mathbb{T}^n$-invariant functions.

3.3 - GH collapse, Tropicalization of (the amoebas of) Divisors & Decomplexification

- GH collapse: Metrically, as $t \to \infty$, the Kähler manifold $(T^*\mathbb{T}^n, \omega, J_t)$ collapses to $\mathbb{R}^n$



$$\gamma_t = \log(t)dx^2 + \frac{1}{\log(t)}d\theta^2$$

- Decomplexification: The complex (Kähler) structure degenerates to a real (GQ) polarization

- Tropicalization: In the same limit the amoebas of divisors become tropical

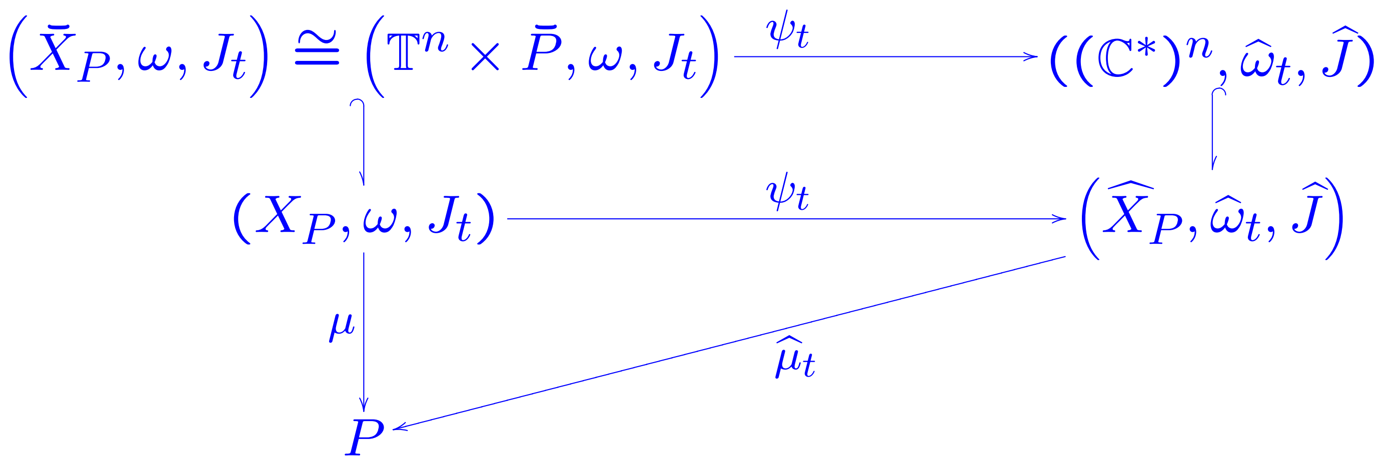
$$A_\infty = \lim_{t\to\infty} \mu \circ \iota_t(Y_t) = A_{\mathrm{trop}}$$

4. Toric Varieties – non-flat families

4.1 - Adding Divisors to $(\mathbb{C}^*)^n$

Question: Can this (the degeneration of the complex structure to the toric real polarization – decomplexification) be extended to a (partial) compactification of $(\mathbb{C}^*)^n$?

Answer: Yes, but we need an important modification. Recalling that the moment map is no longer surjective to $\mathbb{R}^n$ and from Guillemin (JDG, 1994), Abreu (IJM, 1999) and Baier-Florentino-M-Nunes (JDG, 2011) the previous one-parameter family has to change as follows.

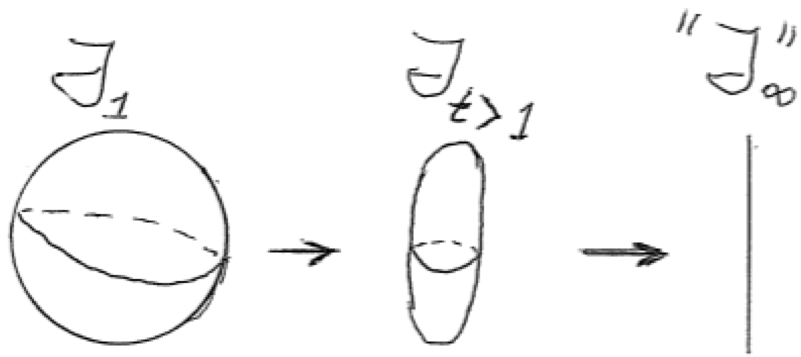


$$
\begin{aligned}
\psi_t(\theta, x) &= e^{y_t + i\theta} = e^{\partial g_t/\partial x + i\theta} \\
g_t(x) &= g_P(x) + sH(x) = \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) + sH(x) \\
\mu(\theta, x) &= x \; - \text{ moment map in the } \textit{symplectic} \text{ picture} \\
& \quad\; H - \text{a convex function of x, i.e. } H = x^2 = ||\mu||^2
\end{aligned}
$$

4.2 - GH collapse, Tropicalization & Decomplexification for Toric varieties

Baier-Florentino-M-Nunes, JDG (2011)

- GH collapse: Metrically, as $t \to \infty$, the Kähler manifold (X_P, ω, J_t) collapses to P with metric $HessH$ on $\bar{P}$



$$\frac{1}{\log(t)}\gamma_t = \frac{1}{\log(t)}(Hess(g_t)dx^2 + Hess(g_t)^{-1}d\theta^2) \overset{t\to\infty}{\longrightarrow} Hess(H)dx^2$$

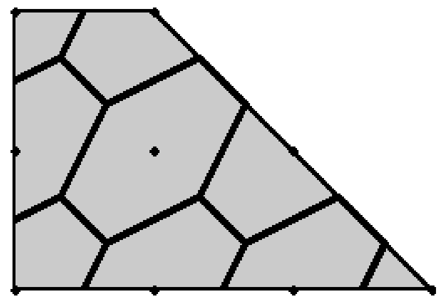
- Decomplexification: The complex (Kähler) structure degenerates to the real toric polarization

- Tropicalization: In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial x} + s m \cdot x + i m \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem[Baier-Florentino-M-Nunes, 2011]

$$\lim_{t \to \infty} \mu(Y_t) = \pi(A_{\mathrm{trop}})$$



5. Wick Rotation, i.e. imaginary time hamiltonian evolution??

5.1 - Introduction to $Ham_{\mathbb{C}}(X)$

Question - What decomplexification (to a given real polarization) has to do with the Wick rotation?

Hamiltonian evolution in complex time has been around in quantum physics for a long time.

Our approach is a development of an approach coming from 2 very different sides.

- Kähler Geometry Semmes (1992) and Donaldson (1999) considered the space of fixed cohomology class Kähler metrics on a manifold M as a infinite dimensional symmetric space corresponding to $Ham(M)_{\mathbb{C}}/Ham(M)$, with geodesics given by imaginary time one-parameter subgroups of $Ham(M)_{\mathbb{C}}$.

- Quantum Gravity Thiemann (1995) considered imaginary time one-parameter subgroups of $Ham(M)_{\mathbb{C}}$ to map real observables to complex observables in an attempt to link the usual spin-connection representation with the complex Ashtekar connection representation.

There are different versions on how to define the action of $Ham_{\mathbb{C}}(M)$.

- Complexify M, act on $M_{\mathbb{C}}$ and then project back to M – see Burns-Lupercio-Uribe (2013)

- We will follow an approach closer to the one proposed initially by Thiemann and further adapted to the context of geometric quantization by Hall-Kirwin (2011) and Kirwin-M-Nunes (2013).

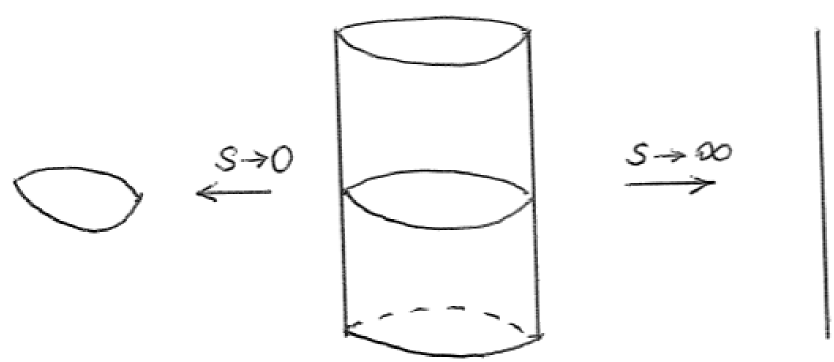
5.2 - Flat family on $(\mathbb{C}^*)^n$ and $Ham_{\mathbb{C}}((\mathbb{C}^*)^n)$

Recall the trivial flat family $(T^*\mathbb{T}^n, dx \wedge d\theta, J_s)$

J_s : $w_s = e^{sx+i\theta}$ are J_s-holomorphic coordinates

Metric - $\gamma_s = s dx^2 + \frac{1}{s} d\theta^2$

Includes two GH collapses to $\mathbb{R}^n$ (as the complex structure degenerates to the real horizontal or toric polarization at $s = \infty$) and to $\mathbb{T}^n$ (as the complex structure degenerates to the real vertical or toric polarization at $s = 0$)



Let $2H = ||\mu||^2 = x^2$. The whole family (including the pseudo-Kähler metrics corresponding to $s < 0$)) is the closure of the orbit of the imaginary time one-parameter subgroup $\mathcal{C} \subset Ham_{\mathbb{C}}(T^*\mathbb{T}^n)$ through J_1 generated by this hamiltonian, $\mathcal{C} = \left\{ e^{i\tilde{s}X_H}, \tilde{s} \in \mathbb{R} \right\}$

$$e^{i\tilde{s}X_H} w_1 = e^{i\tilde{s}x\partial/\partial\theta} e^{x+i\theta} = w_{\tilde{s}+1} = w_s$$

For $\tilde{s} \neq -1 \Leftrightarrow s \neq 0$ these define diffeomorphisms of $T^*\mathbb{T}^n$, φ_s

$$\varphi_s(w) = |w|^s \frac{w}{|w|}$$

So that we get

$$Ham_{\mathbb{C}}(T^*\mathbb{T}^n) \hookleftarrow \mathcal{C} \setminus \{e^{-iX_H}\} \longrightarrow Diff(T^*\mathbb{T}^n)$$

5.3 - Non flat family on a toric variety X_P and $Ham_{\mathbb{C}}(X_P)$

Kirwin-M-Nunes, 2013-14

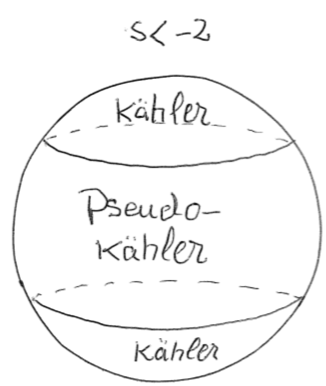
The calculations here are surprisingly similar to the flat case. Consider again $2H = ||\mu||^2 = x^2$. Then $X_H = x\frac{\partial}{\partial \theta}$

$$\begin{aligned} w_1 &= e^{y_P + i\theta} \\ y_P &= \frac{\partial}{\partial x} g_P = \frac{\partial}{\partial x}\left(\sum_{F\subset P} \frac{1}{2}\ell_F(x)\log \ell_F(x)\right) \\ w_t &= e^{isX_H} w_1 = e^{sx} w_1 \end{aligned}$$

To illustrate consider the sphere $S^2 = \mathbb{CP}^1$, with polytope $P = \{0 \leq x \leq 1\}$ so that

$$\begin{aligned} g_P &= \tfrac{1}{2} x \log x + \tfrac{1}{2}(1-x)\log(1-x) \\ \gamma_t &= G_t dx^2 + \tfrac{1}{G_t} d\theta^2 \\ G_t &= \tfrac{1}{2}\tfrac{1}{x(1-x)} + s \end{aligned}$$

and the metric picture is represented in the previous toric figure (p. 14) and (for $s < -2$) by



5.4 - Summary - Decomplexification = (Wick Rotation) + ($s \to \infty$)

Thus, to decomplexify a Kähler manifold M in the direction of a (local) integrable system with moment map μ:

i) Choose $H = ||\mu||^2$

ii) Wick rotate: take the one parameter subgroup of $Ham_{\mathbb{C}}(M)$ with imaginary time is generated by H.

iii) Take the limit $s \to \infty$

Theorem [M-Nunes, 2013-14] In algebraically completely integrable systems, in equivariant neighborhoods and in some other interesting examples the above method gives the GH collapse. The one-parameter Kähler metrics correspond to geodesic rays.

Theorem [Kirwin-M-Nunes, 2013-14] In toric varieties the above method is well defined and gives GH collapse and tropicalization of divisors.

Compact Riemann surfaces Kirwin-M-Nunes, work in progress.

6. References

Journal References

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- W. Kirwin, J.Mourão and J.P. Nunes, *Degeneration of Kaehler structures and half-form quantization of toric varieties*, To appear in Journ. Sympl. Geometry.
- T. Baier, C.Florentino, J.Mourão and J.P. Nunes, *Toric Kaehler metrics seen from infinity, quantization and compact tropical amoebas*, Journ. Diff. Geom. **89** (2011) 411–454.
- T. Baier, J.Mourão and J.P. Nunes, *Quantization of Abelian Varieties: distributional sections and the transition from Kaehler to real polarizations*, Journ. Funct. Anal. **258** (2010) 3388–3412.

Work in Progress

- J.Mourão and J.P. Nunes, *A note on complexified hamiltonian flows and geodesics on the space of Kähler metrics*

- J.Mourão and J.P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*

- W. Kirwin, J.Mourão and J.P. Nunes, *Decomplexification of toric varieties, geodesics in the space of toric Kähler metrics and quantization*

- W. Kirwin, J.Mourão and J.P. Nunes, *Decomplexification of Riemann surfaces and quantization*

Thank you!