Tropicalization in symplectic geometry and degeneration to real polarizations
or (Geometric degeneration to tropical varieties in symplectic geometry)

José Mourão
Técnico Lisboa, U Lisboa

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work in collaboration with Thomas Baier (U Porto), Carlos Florentino (Técnico Lisboa), Will Kirwin (U Köln) and João P. Nunes (Técnico Lisboa)
Index

1. Summary ................................................................. 2
2. Motivation ................................................................. 4
3. $(\mathbb{C}^*)^n$ – case, flat family ..................................... 6
4. Toric Varieties – non-flat families ........................................ 12
5. Wick Rotation, i.e. imaginary time hamiltonian evolution?? ........ 16
6. References ................................................................. 24
1. Summary

New relation:  
[for completely integrable systems \( (X^{2n}, \omega, \mu : X \rightarrow \mathbb{R}^n) ) \]

\[
\text{Quantum Physics & Symplectic Geometry} \quad \overset{\text{dequantization}}{\longrightarrow} \quad \text{Tropical Geometry} \quad \overset{\text{decomplexification}}{\longrightarrow}
\]

Dequantization  \(-\)  \( \hbar \rightarrow 0 \)

Decomplexifications  \(-\) (Kähler) Geometric degenerations to tropical varieties:

\[ \begin{align*}
\text{:=} & \quad \text{Follow (to infinite time) a geodesic ray,} \\
& \text{in the space of Kähler metrics,} \\
& \text{generated by } H = ||\mu||^2 = f_1^2 + \cdots + f_n^2, \\
\text{= Do a Wick rotation, time } \sim is, \text{ followed by } s \rightarrow \infty \text{ for } H = ||\mu||^2
\end{align*} \]
Imaginary time $s \to \infty$ limit
2. Motivation

Start with a hypersurface \( \hat{Y} \subset (\mathbb{C}^*)^n \) and let \( t = e^s \),

\[
(\mathbb{C}^*)^n \supset \hat{Y} = \{ w \in (\mathbb{C}^*)^n : \sum_{m \in \mathcal{P}} c_m w^m = 0 \}
\]

\[
T^*T^n \supset Y_t = \{ (\theta, x) \in T^n \times \mathbb{R}^n : \sum_{m \in \mathcal{P}} c_m e^{sm \cdot x + im \cdot \theta} = 0 \}
\]

**Q1** Can we see \( w_t = e^{sx_j + i\theta_j} \) as defining a one parameter family of complex structures in \( T^*T^n \)?

**A1** Yes

**Q2** Can this be extended to toric varieties = (partial equivariant) compactifications of \( (\mathbb{C}^*)^n \)?

**A2** Not directly. For the complex structure to be defined in the added divisors we need to add a constant term to the holomorphic coordinates which doesn’t change with \( s \! \).

\[
w_t = e^{yQ + sx + i\theta}
\]

or, more generally, \( w_t = e^{yQ + s \partial H / \partial x + i\theta} \)

where \( Q \) is the polytope, image of the moment map, \( Q = \mu(X) \).
Q3 In this framework, what is the symplectic geometry interpretation of the (generalization of the) $\text{Log}_t$ map?

A3 Is just the moment map (or its $H$-Legendre transform): $L_H \circ \mu$

Q4 Can A2 and A3 be extended to more general integrable systems?

A4 Yes proved in some cases – conjectured always – work in progress
3. $(\mathbb{C}^*)^n$—case, flat family

3.1 - Complex and Symplectic Pictures

Log$_t$ map as a moment map

$T^*\mathbb{T}^n$ - symplectic: $\omega = dx \wedge d\theta$

$(\mathbb{C}^*)^n$ - complex
\[
(T^*\mathbb{T}^n, \omega, J_t) \xrightarrow{\tilde{\psi}_t} (T\mathbb{T}^n, \tilde{\omega}_t, \tilde{J}) \xrightarrow{\varphi} ((\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J})
\]

\[
\tilde{\psi}_t(\theta, x) = (\theta, sx), \quad s = \log(t)
\]

\[
\varphi(\theta, y) = e^{y+i\theta} \quad \text{inverse of the polar decomposition}
\]

\[
\psi_t(\theta, x) = \varphi \circ \tilde{\psi}_t(\theta, sx) = e^{sx+i\theta},
\]

\[
\mu(\theta, x) = x \quad \text{moment map in the symplectic picture}
\]

\[
\hat{\mu}_t(w) = \text{Log}_t(w) - \hat{\omega}_t\text{-moment map in the complex picture}
\]
\( \hat{Y} = \left\{ \sum_{m \in \mathcal{P}} c_m w^m = 0 \right\} - \text{complex picture} \)

\( Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{s m \cdot x + im \cdot \theta} = 0 \right\} - \text{symplectic picture} \)

\[
\begin{align*}
Y_t &\xrightarrow{\psi_t} \hat{Y} \\
\mu_t &\xrightarrow{\psi_t} (\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J}) \\
\mathbb{R}^n &\xrightarrow{\mu_t}
\end{align*}
\]

The amoebas coincide

\[
A_t = \hat{\mu}_t \circ \hat{\iota}(\hat{Y}) = t\text{-dependent } \hat{\omega}_t\text{-moment image of a fixed subvariety}
\]

\[
= \mu \circ \iota_t(Y_t) = \text{image of a subvariety moving, due to change in } J_t
\]
3.2 - $J_t$ degenerates to what?

“$J_\infty$” as a (GQ) real polarization

$J_t$-polarized functions are solutions of the CR equations

$$\frac{\partial}{\partial \bar{z}_t} f = 0 \iff \frac{1}{\log t} \frac{\partial}{\partial x} f + i \frac{\partial}{\partial \theta} f = 0$$

“$J_\infty$” can be defined through its polarized functions which are solutions of

$$\frac{\partial}{\partial \theta} f = 0 \iff f(\theta, x) = F(x)$$

i.e. $\mathbb{T}^n$-invariant functions.
3.3 - GH collapse, Tropicalization of (the amoebas of) Divisors & Decomplexification

- **GH collapse**: Metrically, as $t \to \infty$, the Kähler manifold $(T^*T^n, \omega, J_t)$ collapses to $\mathbb{R}^n$

\[ \gamma_t = \log(t)dx^2 + \frac{1}{\log(t)}d\theta^2 \]
• **Decomplexification:** The complex (Kähler) structure degenerates to a real (GQ) polarization

• **Tropicalization:** In the same limit the amoebas of divisors become tropical

\[ A_\infty = \lim_{t \to \infty} \mu \circ \iota_t(Y_t) = A_{\text{trop}} \]
4. Toric Varieties – non-flat families

4.1 - Adding Divisors to $(\mathbb{C}^*)^n$

**Question:** Can this (the degeneration of the complex structure to the toric real polarization – decomplexification) be extended to a (partial) compactification of $(\mathbb{C}^*)^n$?

**Answer:** Yes, but we need an important modification. Recalling that the moment map is no longer surjective to $\mathbb{R}^n$ and from Guillemin (JDG, 1994), Abreu (IJM, 1999) and Baier-Florentino-M-Nunes (JDG, 2011) the previous one-parameter family has to change as follows.
\( (X_P, \omega, J_t) \cong (\mathbb{T}^n \times \bar{P}, \omega, J_t) \xrightarrow{\psi_t} ((\mathbb{C}^*)^n, \hat{\omega}_t, \hat{J}) \)

\( (\bar{X}_P, \omega, J_t) \)

\( (X_P, \omega, J_t) \xrightarrow{\psi_t} (\bar{X}_P, \hat{\omega}_t, \hat{J}) \)

\[ \psi_t(\theta, x) = e^{yt+i\theta} = e^{\partial g_t/\partial x+i\theta} \]

\[ g_t(x) = g_P(x) + sH(x) = \sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) + sH(x) \]

\[ \mu(\theta, x) = x - \text{moment map in the symplectic picture} \]

\[ H - \text{a convex function of } x, \text{ i.e. } H = x^2 = \|\mu\|^2 \]
4.2 - GH collapse, Tropicalization & Decomplexification for Toric varieties

Baier-Florentino-M-Nunes, JDG (2011)

• GH collapse: Metrically, as $t \to \infty$, the Kähler manifold $(X_P, \omega, J_t)$ collapses to $P$ with metric $HessH$ on $\bar{P}$

\[
\frac{1}{\log(t)}\gamma_t = \frac{1}{\log(t)}(Hess(g_t)dx^2 + Hess(g_t)^{-1}d\theta^2) \xrightarrow{t \to \infty} Hess(H)dx^2
\]
• **Decomplexification:** The complex (Kähler) structure degenerates to the real toric polarization

• **Tropicalization:** In the same limit (part of) the amoebas of divisors tropicalize

\[ Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial x}} + s m \cdot x + i m \cdot \theta = 0 \right\} \xrightarrow{\mu} P \]

**Theorem**[Baier-Florentino-M-Nunes, 2011]

\[ \lim_{t \to \infty} \mu(Y_t) = \pi(A_{trop}) \]
5. Wick Rotation, i.e. imaginary time hamiltonian evolution??

5.1 - Introduction to $Ham_C(X)$

**Question** - What decomplexification (to a given real polarization) has to do with the Wick rotation?

Hamiltonian evolution in complex time has been around in quantum physics for a long time.

Our approach is a development of an approach coming from 2 very different sides.
• **Kähler Geometry** Semmes (1992) and Donaldson (1999) considered the space of fixed cohomology class Kähler metrics on a manifold $M$ as an infinite dimensional symmetric space corresponding to $\text{Ham}(M)_\mathbb{C}/\text{Ham}(M)$, with geodesics given by imaginary time one-parameter subgroups of $\text{Ham}(M)_\mathbb{C}$.

• **Quantum Gravity** Thiemann (1995) considered imaginary time one-parameter subgroups of $\text{Ham}(M)_\mathbb{C}$ to map real observables to complex observables in an attempt to link the usual spin-connection representation with the complex Ash-tekar connection representation.
There are different versions on how to define the action of $Ham_C(M)$.

- Complexify $M$, act on $M_C$ and then project back to $M$ – see Burns-Lupercio-Uribe (2013)

- We will follow an approach closer to the one proposed initially by Thiemann and further adapted to the context of geometric quantization by Hall-Kirwin (2011) and Kirwin-M-Nunes (2013).
5.2 - Flat family on \((\mathbb{C}^*)^n\) and \(\text{Ham}_\mathbb{C}(\mathbb{C}^*)^n\)

Recall the trivial flat family \((T^*\mathbb{T}^n, dx \wedge d\theta, J_s)\)

\[ J_s : w_s = e^{sx+i\theta} \text{ are } J_s\text{-holomorphic coordinates} \]

\textbf{Metric} - \(\gamma_s = sdx^2 + \frac{1}{s}d\theta^2\)

Includes two GH collapses to \(\mathbb{R}^n\) (as the complex structure degenerates to the real horizontal or toric polarization at \(s = \infty\)) and to \(\mathbb{T}^n\) (as the complex structure degenerates to the real vertical or toric polarization at \(s = 0\))
Let $2H = ||\mu||^2 = x^2$. The whole family (including the pseudo-Kähler metrics corresponding to $s < 0$)) is the closure of the orbit of the imaginary time one-parameter subgroup $\mathcal{C} \subset Ham_\mathbb{C}(T^*\mathbb{T}^n)$ through $J_1$ generated by this hamiltonian, $\mathcal{C} = \{ e^{i\tilde{s}X_H}, \tilde{s} \in \mathbb{R} \}$

$$e^{i\tilde{s}X_H} w_1 = e^{i\tilde{s}x \partial / \partial \theta} e^{x+\theta} = w_{\tilde{s}+1} = w_s$$

For $\tilde{s} \neq -1 \iff s \neq 0$ these define diffeomorphisms of $T^*\mathbb{T}^n$, $\varphi_s$

$$\varphi_s(w) = |w|^s \frac{w}{|w|}$$

So that we get

$$\xymatrix{ Ham_\mathbb{C}(T^*\mathbb{T}^n) & \mathcal{C} \setminus \{ e^{-iX_H} \} & Diff(T^*\mathbb{T}^n) }$$
5.3 - Non flat family on a toric variety $X_P$ and $\text{Ham}_\mathbb{C}(X_P)$

Kirwin-M-Nunes, 2013-14

The calculations here are surprisingly similar to the flat case.
Consider again $2H = ||\mu||^2 = x^2$. Then $X_H = x\frac{\partial}{\partial \theta}$

\[
\begin{align*}
    w_1 & = e^{y_P + i\theta} \\
    y_P & = \frac{\partial}{\partial x} g_P = \frac{\partial}{\partial x} \left( \sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) \right) \\
    w_t & = e^{isX_H w_1} = e^{sx} w_1
\end{align*}
\]
To illustrate consider the sphere $S^2 = \mathbb{CP}^1$, with polytope $P = \{0 \leq x \leq 1\}$ so that

\[
\begin{align*}
g_P &= \frac{1}{2} x \log x + \frac{1}{2} (1 - x) \log(1 - x) \\
\gamma_t &= G_t dx^2 + \frac{1}{G_t} d\theta^2 \\
G_t &= \frac{1}{2} \frac{1}{x(1-x)} + s
\end{align*}
\]

and the metric picture is represented in the previous toric figure (p. 14) and (for $s < -2$) by
5.4 - Summary - Decomplexification $\equiv$ (Wick Rotation) $+$ ($s \to \infty$)

Thus, to decomplexify a Kähler manifold $M$ in the direction of a (local) integrable system with moment map $\mu$:

- **i)** Choose $H = ||\mu||^2$
- **ii)** Wick rotate: take the one parameter subgroup of $Ham_\mathbb{C}(M)$ with imaginary time $is$ generated by $H$.
- **iii)** Take the limit $s \to \infty$

**Theorem [M-Nunes, 2013-14]** In algebraically completely integrable systems, in equivariant neighborhoods and in some other interesting examples the above method gives the GH collapse. The one-parameter Kähler metrics correspond to geodesic rays.

**Theorem [Kirwin-M-Nunes, 2013-14]** In toric varieties the above method is well defined and gives GH collapse and tropicalization of divisors.

**Compact Riemann surfaces** Kirwin-M-Nunes, work in progress.
6. References

Journal References


Work in Progress

- J. Mourão and J. P. Nunes, *A note on complexified Hamiltonian flows and geodesics on the space of Kähler metrics*

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Thank you!