Decomplexification of integrable systems, quantization and Kähler geometry

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First of all let us explain the term decomplexification in the title of this set of lectures.

For a Liouville integrable system $(M, \omega, \mu = (H_1, \dots, H_n))$ [which we assume has Kähler polarizations] decomplexification in the direction of μ or μ–decomplexification, means that there is a geodesic flow (in the space of polarizations) converging, at infinite geodesic time, to the real polarization (i.e. singular Lagrangian foliation, $\mathcal{P}_\mu \subset TM$, with nonsingular leaves given by connected components of level sets of μ) defined by μ. This takes place for a wide class of starting Kähler and mixed polarizations.

The μ–decomplexifying geodesic flow is the flow generated by imaginary time $X_{||\mu||^2}$. This ray leads also to geometric tropicalization of M and of its complex hypersurfaces.

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Lecture 1

Quantization, integrable systems and toric geometry

1.1. Summary of Lecture 1

Aspects of Kähler geometry and of geometric quantization.

Liouville integrability (or *nice effective hamiltonian* $\mathbb{R}^k \times \mathbb{T}^{n-k}$ *action) of an hamiltonian system.*

Aspects of toric geometry (or effective hamiltonian $\mathbb{T}^n$ *action) needed for the course.*

Among the many available quantization schemes one scheme has been very successfull due to the fact that it gives a very well defined set of rules: Geometric Quantization (GQ).

We will concentrate in the ambiguity in the choice of the quantum Hilbert space, which is characteristic of GQ and also in the problem of defining geometric quantization for singular real polarizations, and in particular those defined by the moment map of a Liouville completely integrable system.

1.2. Kähler manifolds

Kähler manifolds (M, ω, J) are symplectic manifolds (M, ω) with a compatible complex structure J, *ie* such that the bilinear form $\gamma(X, Y) := \omega(X, JY)$ is a Riemannian metric.

A symplectic manifold may not have compatible complex structures but if it has one it has an infinite dimensional space of them.

The symplectic form is automatically of type $(1, 1)$ for any compatible complex structure and has a locally defined J-dependent Kähler potential k_J, $\omega = \frac{i}{2}\partial_J\overline{\partial_J}\,k_J$

On the other hand, fixing J, two J–compatible closed 2–forms ω and ω' are in the same cohomology class iff their Kähler potentials k, k' differ by a global function

$$k' = k + f,\ f \in C^\infty(M)$$

Then, the space of Kähler metrics compatible with J, is naturally given by

$$\mathcal{H}(\omega, J) = \left\{ f \in C^\infty(M)\ :\ \omega + \frac{i}{2}\partial_J \bar{\partial}_J f > 0 \right\} / \mathbb{R}$$

This (infinite dimensional) manifold has a natural metric introduced by Mabuchi,

$$G_f(h_1, h_2) = \int_M h_1\, h_2\, \frac{\omega_f^n}{n!}, \qquad \text{where}\, \omega_f = \omega + \frac{i}{2}\partial_J \bar{\partial}_J f$$

Remark. Notice that this metric does not coincide with the pullback of the standard metric in the space of metrics. The pullback of that metric to the space of Kähler metrics is called Calabi metric.

As showed by Donaldson, the Mabuchi metric is the metric of constant negative curvature associated with the realization of $\mathcal{H}(\omega, J)$ as the symmetric space $\text{Ham}_{\mathbb{C}}(M, \omega)/\text{Ham}(M, \omega)$. We will return to this in the second lecture.

1.3. Geometric Quantization

- Classical System: (M, ω) - symplectic manifold

- Classical Hamiltonian System: (M, ω, H), $H \in C^\infty(M)$ - symplectic manifold and chosen Hamiltonian $H \in C^\infty(M)$. The classical dynamics corresponds to the flow of the Hamiltonian vector field X_H associated with H.

$$X_H \ : \ dH(Y) = \omega(X_H, Y), \qquad \forall Y \in \mathcal{X}(M)$$

- Prequantization: Let $[\frac{\omega}{2\pi}] \in H^2(M,\mathbb{Z})$ and choose an Hermitian line bundle $L \to M$ with $c_1(L) = -i[\frac{\omega}{2\pi}]$, a compatible connection ∇, with curvature $F_\nabla = -i\omega$. The pre-quantum Hilbert space is

$$\begin{aligned} \mathcal{H}^{prQ} &= \Gamma_{L^2}(M,L) \\ f &\mapsto Q^{GQ}(f) = \hat{f} = -i\nabla_{X_f} + f \end{aligned}$$

- (uncorrected) Quantization: $\mathcal{H}^{prQ}$ is too large. Choose a polarization $\mathcal{P}$, $\mathcal{P}_m \subset T_m M_{\mathbb{C}}$ - Lagrangian, integrable. The quantum Hilbert space is

$$\mathcal{H}^Q_{\mathcal{P}} = \{\psi \in \mathcal{H}^{prQ} \ : \ \nabla_X \psi = 0, \forall X \in \Gamma(\mathcal{P})\}$$

$$\hat{f} \text{ acts on } \mathcal{H}^Q_{\mathcal{P}} \Leftrightarrow [X_f, \Gamma(\mathcal{P})] \subset \Gamma(\mathcal{P}) \Leftrightarrow f \in \mathcal{O}_{\mathcal{P}}$$

$\mathcal{O}_{\mathcal{P}}$–Poisson subalgebra of $\mathcal{P}$-quantizable observables.

- Half-form corrected quantization: Concrete examples show that one has to correct this quantization scheme. In a Kähler polarization the correction reads

$$\begin{aligned} L \quad &\rightsquigarrow \quad \widetilde{L} = L \otimes \sqrt{K_I} \\ \nabla \quad &\rightsquigarrow \quad \widetilde{\nabla} = \nabla \otimes 1 + 1 \otimes \nabla_I^C \\ F_\nabla = -i\omega \quad &\rightsquigarrow \quad F_{\widetilde{\nabla}} = -i(\omega - \frac{1}{2}\,\rho_I), \end{aligned}$$

where K_I denotes the canonical bundle of I-holomorphic top forms and ρ_I denotes the Ricci form. This is meaningful even if the line bundles L and $\sqrt{K_I}$ don't exist separately but $\widetilde{L}$ does with $c_1(\widetilde{L}) = [\frac{\omega}{2\pi}] - \frac{1}{2}c_1(M)$ (as for some toric varieties like the even dimensional complex projective spaces $\mathbb{CP}^{2n}$)

Real or Kähler polarizations divide adapted Darboux charts in coordinates and momenta or in holomorphic and anti-holomorphic coordinates. Polarized functions are those which do not depend on half of these coordinates.

Two extreme cases of polarizations

$\mathcal{P}$ is real $\mathcal{P} = \overline{\mathcal{P}}$ [we will allow polarizations with certain kinds of singularities]
So $\mathcal{P} = \langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \rangle = \langle X_{q_1}, \ldots, X_{q_n} \rangle$ defines a (possibly) singular fibration on M with the regular fibers being Lagrangian.

$\mathcal{H}^Q_{\mathcal{P}}$ consists of sections depending (essentially) only on half of the canonical variables corresponding to the directions transverse to the fibers (or leaves).

If the leaves have noncontractible 1-cycles then polarized sections will be supported only on those leaves with trivial ∇-holonomy, called Bohr-Sommerfeld (BS) leaves.

$\mathcal{P}$ is Kähler Then $\mathcal{P} = \langle \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \rangle = \langle X_{z_1}, \ldots, X_{z_n} \rangle$ is equivalent to a compatible complex structure J. The pair (∇, J) defines on L the structure of an holomorphic line bundle $\mathcal{L}_J \to M$ and

$$\mathcal{H}^Q_{\mathcal{P}} = \mathcal{H}^Q_J \cong H^0(M, \mathcal{L}_J)$$

Quantum Hilbert bundle

Thus we get a bundle, the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T},$$

where $\mathcal{T}$ denotes (a family in) the space of polarizations.

Need to study the dependence of quantization on the choice of the complex structure.

Mixed Polarizations

Some mixed polarizations are also very interesting as the Kirwin-Wu momentum polarization $\mathcal{P}_{\mathsf{mom}}$ on $K_{\mathbb{C}} \cong T^*K$ [KW] or the Takakura polarization $\mathcal{P}_T$ on moduli spaces of flat connections G on Riemann surfaces with $g > 1$ and compact simple groups with rank, $r \geq 2$, $\mathcal{M}(G, \Sigma)$ [Ta].

BKS pairing map:

Given two nonnegative polarizations $\mathcal{P}_1, \mathcal{P}_2$, geometric quantization gives us a map

$$B_{\mathcal{P}_1\mathcal{P}_2} \ : \ \mathcal{H}^Q_{\mathcal{P}_1} \longrightarrow \mathcal{H}^Q_{\mathcal{P}_2}$$

If $M = T^*S^1$ and

$\mathcal{P}_1 = \mathcal{P}_{\mathsf{Sch}} =< \frac{\partial}{\partial y} >$ and $\mathcal{P}_2 = \mathcal{P}_{\mathsf{hor}} =< \frac{\partial}{\partial \theta} >=< X_y >$

then $B_{\mathcal{P}_{\mathsf{Sch}}\mathcal{P}_{\mathsf{hor}}}$ is the (discrete) Fourier transform.

$$\begin{array}{rcl} L^2(S^1, dx) & \longrightarrow & \ell^2(\widehat{S}^1) = \ell^2(\mathbb{Z}) \\ f & \mapsto & \hat{f}_n = \int_{S^1} f(x)\, e^{-i2\pi n x} dx \end{array}$$

Let now $M = T^*K$, where K is a compact Lie group and $\mathcal{P}_{\text{mom}}$ be the mixed polarization obtained rescaling to infinity the fiber coordinates of T^*K

Theorem (Kirwin-Wu, [KW])
$B_{\mathcal{P}_{\text{Sch}}\mathcal{P}_{\text{mom}}}$ is (naturally equivalent to) the non-abelian Fourier transform or Peter-Weyl transform, $B_{\mathcal{P}_{\text{Sch}}\mathcal{P}_{\text{mom}}} \cong \widetilde{U}_{PW}$

$$\begin{array}{rccc} U_{PW} & : & L^2(K, dx) & \longrightarrow & \ell^2(\widehat{K}) \\ & & f & \mapsto & \hat{f}_\pi = \int_K f(x)\,\bar{\pi}(x)dx \end{array}$$

Remark

Importance of choice of polarization: Choosing a polarization is the same as choosing which observables (real or complex) act diagonally, i.e. as operators of multiplication by functions. This is known to lead to inequivalent quantum theories.

Degeneration of Kähler Polarizations

We will study the degeneration of Kähler polarizations to real polarizations

$$\mathcal{P}_{J_t} \to \mathcal{P}^{\mathbb{R}},$$

as a tool to do the two things mentioned in the summary of the lecture:

a) study the dependence of the quantum theory on the choice of polarization.

b) find a proper definition of real polarized sections in cases with singular leaves, i.e. a proper definition of $\mathcal{H}^{Q}_{\mathcal{P}^{\mathbb{R}}}$.

1.4. Integrable systems

A Liouville (completely) integrable system is a symplectic manifold (M^{2n}, ω) plus a nice, free, in a dense subset, Hamiltonian action of $K = \mathbb{R}^k \times \mathbb{T}^{n-k}$ with moment map

$$\mu = (H_1, \dots, H_n) \ : \ M^{2n} \to \mathbb{R}^n$$

defining, in a dense subset, a regular Lagrangian fibration.

If the regular level sets of μ are compact then they are n-dimensional torii or unions of n-dimensional torii.

Theorem[Arnold–Liouville]
Let $\eta_0 \in \mathbb{R}^n$ be a regular value of μ and $m_0 \in \mu^{-1}(\eta_0)$. Then there is a K–invariant neighborhood U and coordinates $\alpha = (\alpha_1, \ldots, \alpha_n)$ on the fibers of the moment map on U, canonically conjugate to $H = (H_1, \ldots, H_n)$, i.e. such that,

$$\omega = \sum_{j=1}^{n} dH_j \wedge d\alpha_j$$

Notice that $X_{H_j} = -\frac{\partial}{\partial \alpha_j}$ and $X_{\alpha_j} = \frac{\partial}{\partial H_j}$ so that, in particular, if

$h = ||\mu||^2/2 = \frac{1}{2}(H_1^2 + \cdots + H_n^2)$, then

$$X_h = -H_1 \frac{\partial}{\partial \alpha_1} - \cdots - H_n \frac{\partial}{\partial \alpha_n}$$

and

$$(\varphi_t^{X_h})^*(f)(H, \alpha) = f(H, \alpha - tH)$$

Naturally then, if f is analytic on α, we will be able to analytically continue this result and get

$$(\varphi_\tau^{X_h})^*(f)(H, \alpha) = f(H, \alpha - \tau H)$$

for sufficiently small $\tau \in \mathbb{C}$.

Remark. So some fibers of μ may be singular but μ is *usually* assumed to be a smooth map to $\mathbb{R}^n$ thus defining n smooth global functions $H_1, \ldots, H_n$. Cases in which one drops smoothness of μ are nevertheless very interesting and have been the focus of much recent interest.

Examples:

Ex 1. C^0–case, i.e. $H_j \in C^0(M)$, as the Gel'fand-Cetlin systems [GS2], [AM], [HaKo], [KMNW] and the integrable systems associated with Okounkov bodies [HaKa].

Ex 2. Discontinuous examples appear in torii $\mathbb{T}^{2n}$ and in moduli spaces of flat connections on Riemann surfaces [JW2], [Sikora].

1.5. Symplectic toric manifolds

Toric Kähler manifolds provide the simplest, yet quite rich examples of Liouville integrable systems $(M, \omega, \mu = (H_1, \ldots, H_n))$. These are the cases in which μ generates an effective action of $\mathbb{T}^n$. These manifolds correspond to (partial) equivariant compactifications of the complex torus $(\mathbb{C}^*)^n$ and a very convenient way of studying their infinite dimensional space of Kähler structures is with the help of the following diagram.

$$\begin{array}{rcccl} \psi_{\mathsf{st}} : & T^*\mathbb{T}^n & \longrightarrow & T\mathbb{T}^n \longrightarrow (\mathbb{C}^*)^n \\ & (\theta, H) & \mapsto & (\theta, H) \mapsto e^{i\theta}e^{H} \end{array} \quad (1)$$

Notice the following trivial, yet important facts:

1. $T^*\mathbb{T}^n$ has a standard symplectic structure and a free hamiltonian action of $\mathbb{T}^n$, $e^{i\beta} \cdot (\theta, H) = (\theta - \beta, H)$, with moment map $\mu(\theta, H) = H$.

2. $(\mathbb{C}^*)^n$ has a standard complex structure and a free action of $\mathbb{T}^n$ (as subgroup), $e^{i\beta} \cdot w = (e^{i\beta_1} w_1, \ldots, e^{i\beta_n} w_n)$.

3. The diffeomorphism ψ_{st} is $\mathbb{T}^n$–equivariant and induces a $\mathbb{T}^n$–invariant complex structure on $T^*\mathbb{T}^n$ and a $\mathbb{T}^n$–invariant symplectic structure on $(\mathbb{C}^*)^n$ making the $\mathbb{T}^n$ action Hamiltonian with moment map $\widehat{\mu}(w) = \mu \circ \psi_{\mathsf{st}}^{-1}(w) = (\log|w_1|, \ldots, \log|w_n|)$. The resulting structure on both sides is a $\mathbb{T}^n$–invariant Kähler structure.

$$\begin{array}{rcl} \psi_{\mathsf{st}} \ : \ (T^*\mathbb{T}^n, \omega_{\mathsf{st}}, \psi_{\mathsf{st}}^*(\widehat{J}_{\mathsf{st}})) & \longrightarrow & ((\mathbb{C}^*)^n, (\psi_{\mathsf{st}}^{-1})^*(\omega_{\mathsf{st}}), \widehat{J}_{\mathsf{st}}) \\ (\theta, H) & \mapsto & e^{i\theta}\, e^H \end{array} \qquad (2)$$

While the second map in (1) is standard (the inverse of the polar decomposition of complex numbers) the first can be changed, preserving $\mathbb{T}^n$ equivariance. We thus change the Kähler structure on either side of (2) by changing the fiber part of the map ψ_{st}. That freedom is equivalent to the choice of a $\mathbb{T}^n$–invariant Kähler potential on $(\mathbb{C}^*)^n$. The *standard* one corresponding to (2) reads

$$k_{\mathsf{st}}(w) = \frac{1}{2}\log^2|w_1| + \cdots + \frac{1}{2}\log^2|w_n|$$

The starting point to describe the other $\mathbb{T}^n$–invariant Kähler structures is then $((\mathbb{C}^*)^n, \widehat{J}_{\mathrm{st}})$ and a $\mathbb{T}^n$–invariant Kähler potential (i.e. a strictly plurisubharmonic function), k. Let $u = \mathrm{Log}(w) = (\log(|w_1|), \ldots, \log(|w_n|))$ so that, $k(w) = \widetilde{k}(u)$ and

$$\widehat{\omega} = \frac{i}{2}\partial\bar{\partial}\, k = \sum_{l,j} \mathrm{Hess}(\widetilde{k})_{lj}\; du_l \wedge d\theta_j$$

with $\mathrm{Hess}(\widetilde{k})$ positive definite. Introducing $H = (H_1, \ldots, H_n) = \frac{\partial \widetilde{k}}{\partial u}$, we obtain

$$\widehat{\omega} = \sum_j \; dH_j \wedge d\theta_j$$

and therefore the map

$$\begin{aligned} \psi^{-1} \; : \; ((\mathbb{C}^*)^n, \widehat{\omega}) \; &\longrightarrow \; (T^*\mathbb{T}^n, \omega_{\mathsf{st}}) \\ w = e^{i\theta} e^u \; &\mapsto \; (\theta, \frac{\partial \widetilde{k}}{\partial u}) \end{aligned} \tag{3}$$

is a symplectomorphism to its image and $\widehat{\mu}(w) = \frac{\partial \widetilde{k}}{\partial u} = H(w)$, is the moment map of the $\mathbb{T}^n$ action on $((\mathbb{C}^*)^n, \widehat{\omega})$. Having invariant divisors D_j to which the Kähler structure $((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\mathsf{st}})$ extends implies loss of surjectivity of the Legendre transform of the fibers in (3): we get $\ell_{F_j}(H) = \nu_{F_j} \cdot H + \lambda_{F_j} \geq 0$, where ν_{F_j} is the integral vector associated with D_j in the fan of the resulting toric variety $\widehat{X}$. If the corresponding Kähler manifold $(\widehat{X}, \widehat{\omega}, \widehat{J}_{\mathsf{st}}) \supset ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\mathsf{st}})$ is compact then the image of the fibers is a Delzant polytope P (Atiyah–Guillemin–Sternberg and Delzant theorems, [At], [De], [GS1]).

We get

$$
\begin{aligned}
(\widehat{X}_P, \widehat{\omega}, \widehat{J}_{\text{st}}) \supset ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\text{st}}) &\stackrel{\psi^{-1}}{\longrightarrow} (\mathbb{T}^n \times \breve{P}, \omega_{\text{st}}, J) \subset (X_P, \omega_{\text{st}}, J) \\
w = e^{i\theta} e^{u} &\mapsto (\theta, \frac{\partial \widetilde{k}}{\partial u}), \qquad (4)
\end{aligned}
$$

where $\breve{P}$ denotes the interior of P.
The map ψ^{-1} extends to a Kähler isomorphism, $\psi^{-1} : (\widehat{X}_P, \widehat{\omega}, \widehat{J}_{\text{st}}) \longrightarrow (X_P, \omega_{\text{st}}, J)$. The inverse ψ of the map in (4) is also given by a Legendre transform associated with the so called symplectic potential, $g(H) = u \cdot H - \widetilde{k}(u)$.

$$
\begin{aligned}
(X_P, \omega_{\text{st}}, J) \supset (\mathbb{T}^n \times \breve{P}, \omega_{\text{st}}, J) &\stackrel{\psi}{\longrightarrow} ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}_{\text{st}}) \subset (\widehat{X}_P, \widehat{\omega}, \widehat{J}_{\text{st}}) \\
(\theta, H) &\mapsto e^{i\theta} e^{\frac{\partial g}{\partial H}} \qquad (5)
\end{aligned}
$$

Then, Guillemin–Abreu theory [Ab] tells us that $\mathbb{T}^n$–invariant Kähler structures on $(X_P, \omega_{\text{st}})$ are "parametrized" by symplectic potentials

$$g(x) = g_P(x) + \phi(x) = \sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) + \phi(x),$$

where $P = \{x \in \mathbb{R}^n \ : \ \ell_F(x) = \nu_F(x) \cdot x + \lambda_F \geq 0, F \text{ facet of } P\}$ and $\phi \in C^\infty(P) \ : \ \text{Hess}(g_P + \phi) > 0$.

Notice that $g_P \in C^0(P)$.

We get the following diagram

$$
\begin{array}{ccc}
\left(\bar{X}_P, \omega, J_\phi\right) \cong \left(\mathbb{T}^n \times \bar{P}, \omega, J_\phi\right) & \xrightarrow{\psi_\phi} & ((\mathbb{C}^*)^n, \widehat{\omega}_\phi, \widehat{J}) \\
\uparrow & & \uparrow \\
\left(X_P, \omega, J_\phi\right) & \xrightarrow{\psi_\phi} & \left(\widehat{X}_P, \widehat{\omega}_\phi, \widehat{J}\right) \\
\downarrow \mu & \swarrow \widehat{\mu}_\phi & \\
P & &
\end{array}
$$

$$
\begin{aligned}
\psi_\phi(\theta, H) &= e^{u_\phi + i\theta} = e^{\partial g_\phi / \partial x + i\theta} \\
g_\phi(H) &= g_P(H) + \phi(H) = \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi \\
\mu(\theta, H) &= H \; - \text{moment map in the } \textit{symplectic} \text{ picture} \\
\widehat{\mu}_\phi(w) &= \mu \circ \psi_\phi^{-1}(w) \; - \text{moment map in the } \textit{complex} \text{ picture}
\end{aligned}
$$

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Lecture 2

Complex time evolution in Kähler geometry and in quantum physics

2.1. Summary of Lecture 2

$\{e^{itX_f}\,,\ t \in \mathbb{R}\}$ - Imaginary one parameter subgroups of $\text{Ham}_{\mathbb{C}}(M,\omega)$ $\rightsquigarrow$ geodesics in the space of (fixed cohomology class) Kähler metrics.

Gröbner Lie series: $e^{\tau X_f}(g) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_f^k(g)$

$$\begin{aligned} \text{Ham}_{\mathbb{C}}(M,\omega) \curvearrowright C^{\text{an}}(M) \ &\rightsquigarrow\ \text{Ham}_{\mathbb{C}}(M,\omega) \curvearrowright \{\mathcal{P}\} \\ &\rightsquigarrow\ \text{Ham}_{\mathbb{C}}(M,\omega) \xrightarrow{\Psi_{\mathcal{P}}} \widetilde{\text{Diff}}(M) \end{aligned}$$

key points

Under $\mathrm{Ham}_{\mathbb{C}}(M, \omega)$:

- polarizations may change type
- Polarized functions evolve as in Hamiltonian dynamics but with complex time

2.2. Imaginary time: introduction

As we will see, evolution in imaginary time enters the scene in the study of the dependence of quantization on the choice of complex structure/polarization.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

What we are studying is a new way of looking at imaginary time evolution in (some situations in) quantum mechanics giving it a precise geometric meaning (Thiemann [Th1], [Th2], Hall-Kirwin [HK1], [HK2], Kirwin-M-Nunes [KMN2], M-Nunes [MN1])

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in '96 in order to transform the spin $SU(2)$ connection to the Ashtekar $SL(2,\mathbb{C})$ connection

$$\Gamma_\mu \mapsto A_\mu^{\mathbb{C}}.$$

In Kähler geometry imaginary time evolution leads to geodesics and is used *e.g.* to study the stability of varieties (Semmes '92, [Se], and Donaldson '99, [Do1], [Do2]).

2.3. Complex time evolution in Kähler geometry

Let (M, ω, J_0) be a compact Kähler manifold.

$G = \mathrm{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with $\mathrm{Lie}(\mathrm{Ham}(M, \omega)) = C^\infty(M)$

$G_{\mathbb{C}} = \mathrm{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits [Do1], [Do2]. There are natural orbits of two types passing through a Kähler pair (ω, J_0).

Complex picture

Let us start with the smaller one, which we have already seen

$$\begin{aligned}\mathcal{H}(\omega, J_0) &= \left\{ f \in C^\infty(M) \ : \ \omega + \frac{i}{2}\partial_0\bar{\partial}_0 f > 0 \right\} /\mathbb{R} =: \\ &=: \ \{(\varphi^*\omega, J_0), \varphi \in G_{\mathbb{C}} \subset \mathrm{Diff}_0(M)\} = (G_{\mathbb{C}} \cdot \omega, J_0)\end{aligned}$$

So we get a orbit-dependent "definition" of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of Diff(M).

As mentioned above $\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space:
$\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ with constant negative curvature mimicking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \mathrm{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that M is simply connected and $\mathrm{Aut}_0(\mathrm{M}, \mathrm{J}_0)$ is trivial the second natural orbit is the total space of a principal G–bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0),\ \varphi \in G_{\mathbb{C}} \subset \mathrm{Diff}_0(\mathrm{M})\} \cong G_{\mathbb{C}}$$

with projection

$$\begin{aligned} \pi \ :\ \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) \ &\longrightarrow\ \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \cong G_{\mathbb{C}}/G \\ \pi(\omega, \varphi_* J_0) \ &=\ \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{aligned}$$

Gröbner Lie series

Let (M,ω) be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\mathrm{an}}(M)$

Theorem (Gröbner, [Gr])
Let $f \in C^{\mathrm{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} \; : \; f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{6}$$

converges absolutely to a (complex) real analytic function.

So $\mathrm{Ham}_{\mathbb{C}}(M,\omega)$ acts locally on $C^{\mathrm{an}}(M)$. To descend to an "action" on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local J_0-holomorphic coordinates. Then we use (6) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \tag{7}$$

or, more generally, a new polarization $\mathcal{P}_\tau$.

Theorem (Burns-Lupercio-Uribe, 2013 [BLU]; M-Nunes, 2013 [MN1])
Under natural conditions on J_0 the relations (7) define a unique map $\varphi_\tau^{X_h} : M \to M$ such that

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) = (\varphi_\tau^{X_h})^*(z_\alpha)$$

For τ sufficiently small $\varphi_\tau^{X_h}$ is a biholomorphism. If $\mathcal{P}_\tau$ is defined but is a mixed polarization then $\varphi_\tau^{X_h}$ is not a diffeomorphism.

Remark. In [BLU] the authors defined $\varphi_\tau^{X_h}$ by defining the flow of a complex hamiltonian in a complexification $M_{\mathbb{C}}$ of M and then projecting back to M. The relation between the two approaches is discussed in [MN1].

Example 1 - $M = \mathbb{R}^2$

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_0)$ and $h = y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then
$\varphi_t^{X_h}(x, y) = (x + ty, y)$
and
$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$

Indeed: $e^{ity\frac{\partial}{\partial x}}(x + iy) = x + i(1 + t)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow \nabla^{\gamma_{it}} h = J_{it} X_h = \frac{1}{1+t}\, y \frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t} dx^2 + (1+t) dy^2$.

Example 2 - Toric manifolds

For a toric manifold let h be $\mathbb{T}^n$–invariant, $h(H)$. Then, $X_h = -\sum_j \frac{\partial h}{\partial H_j} \frac{\partial}{\partial \theta_j}$ and

$$w_{it} = e^{itX_h}\, w = e^{itX_h}\, e^{\frac{\partial g}{\partial H} + i\theta} = e^{\frac{\partial g}{\partial H} + i(\theta - it\frac{\partial h}{\partial H})} = e^{\frac{\partial (g+th)}{\partial H} + i\theta} = e^{t\frac{\partial h}{\partial H}}\, w,$$

so that the change of the symplectic potential under imaginary time toric flow is that of a geodesic in flat space

$$g \rightsquigarrow g_t = g + th$$

Thus we get

$$
\begin{array}{ccccc}
\left(\bar{X}_P, \omega, J_t\right) \cong & \left(\mathbb{T}^n \times \bar{P}, \omega, J_t\right) & \xrightarrow{\quad\psi_t\quad} & & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}) \\
& \big\downarrow & & & \big\downarrow \\
& (X_P, \omega, J_t) & \xrightarrow{\quad\psi_t\quad} & & \left(\widehat{X}_P, \widehat{\omega}_t, \widehat{J}\right) \\
& {\scriptstyle \mu}\big\downarrow & & \swarrow {\scriptstyle \widehat{\mu}_t} & \\
& P & & &
\end{array}
$$

$$
\begin{array}{rcl}
\psi_t(\theta, x) & = & e^{u_t + i\theta} = e^{\partial g_t / \partial H + i\theta} \\
g_t(H) & = & \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi(H) + t\, h(H) \\
\mu(\theta, H) & = & H \; - \text{ moment map in the } \textit{symplectic} \text{ picture} \\
& & \varphi_{it}^{X_h} = \psi_0^{-1} \circ \psi_t
\end{array}
$$

Example 3 - Reductive groups $K_{\mathbb{C}}$ (and their compactifications)

The “adapted” Kähler structure (Guillemin-Stenzel [GS3], Lempert-Szöke [LS1]) on complex reductive groups is [HK1]

$$\begin{array}{rcl} (T^*K, \omega, J_0) & \xrightarrow{\psi_0} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_0) \\ (x, Y) & \mapsto & xe^{iY}, \end{array}$$

where we used $T^*K \cong TK \cong K \times \mathrm{Lie}K$

General bi-K-invariant Kähler structures on T^*K are in one-to-one correspondence with strictly convex Ad–invariant “symplectic potentials” $\breve{g}$ on Lie$K$, [Alexeev–Katzarkov], [BFMN2]

They define and are defined by their (Weyl–invariant) restriction g to the Cartan subalgebra. One has

$$\begin{array}{ccc} (T^*K, \omega, J_{\breve{g}}) & \xrightarrow{\psi_{\breve{g}}} & (K_{\mathbb{C}}, \hat{\omega}, \hat{J}_{\breve{g}}) \\ (x, Y) & \mapsto & xe^{iu(Y)}, \quad u = \dfrac{\partial \breve{g}}{\partial Y} \end{array} \tag{8}$$

The adapted Kähler structure corresponds to $\breve{g} = ||Y||^2/2$.

Choosing an h bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial y_j} X_j$, where $Y = \sum_j y_j T_j$. Then [HK1], [KMN2]

$$e^{itX_h} \cdot xe^{iY} = xe^{i(Y + t\frac{\partial h}{\partial Y})} = xe^{i\frac{\partial \breve{g}_t}{\partial Y}},$$

where $\breve{g}_t = \frac{||Y||^2}{2} + th$.

Notice that if instead of $g(Y) = ||Y||^2/2$ on the dual of the Cartan subalgebra $\mathcal{H}^*$ we choose a Guillemin–Abreu symplectic potential on a Weyl–invariant Delzant polytope $P \subset \mathcal{H}^*$, then the Kähler structure defined by (8) on $(K_{\mathbb{C}}, \hat{\omega}, \hat{J}_{\breve{g}})$ extends to an equivariant compactification $\overline{K}_{\mathbb{C}}$ of $K_{\mathbb{C}}$, having $\widehat{X}_P$ as a toric subvariety, corresponding to the compactification of the Cartan torus. In particular, if we take $K_{\mathbb{C}}$ of adjoint type and P has vertices given by the Weyl orbit of a dominant weight then $\overline{K}_{\mathbb{C}}$ is a wondeful compactification. The imaginary time flow of a bi–K–invariant function acts in a similar way

$$e^{itX_h} \cdot xe^{i\frac{\partial \breve{g}}{\partial Y}} = xe^{i\frac{\partial \breve{g}_t}{\partial Y}},$$

where $\breve{g}_t = \breve{g} + th$.

2.4 Complex time evolution in (geometric) quantization

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{prQ} \overset{J_0}{\rightsquigarrow} \mathcal{H}^{Q}_{J_0} = H^0_{J_0}(M, L) = \left\{ s \in \mathcal{H}^{prQ} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned} \mathcal{H}^{Q}_{J_0} &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow < \frac{\partial}{\partial \bar{z}^j_{J_0}} > = \mathcal{P}_{J_0} \\ &= \text{space of sections depending (essentially) only on } z_{J_0} \\ &\quad \text{(i.e. } J_0\text{-holomorphic)} \end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$\begin{aligned} \mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} < \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} > \\ &= e^{\tau \mathcal{L}_{X_h}} < X_{z^1}, \dots, X_{z^n} > = < X_{e^{\tau X_h}(z^1)}, \dots, X_{e^{\tau X_h}(z^n)} > = \\ &= < X_{z^1_\tau}, \dots, X_{z^n_\tau} > = (\varphi_\tau^{X_h})^* (\mathcal{P}_0) \end{aligned}$$

For some values of $\tau \neq 0$ the polarization may be real or mixed.

In our flat example, $h = y^2/2, \tau = it$,

$$\begin{aligned} \mathcal{P}_{it} &= < \frac{\partial}{\partial \bar{z}_{it}} >=< X_{z_{it}} >=< X_{x+i(1+t)y} >= \\ &= < -\frac{\partial}{\partial y} + i(1+t)\frac{\partial}{\partial x} > \stackrel{t \to -1}{\longrightarrow} < \frac{\partial}{\partial y} >= \mathcal{P}_{-i} \end{aligned}$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}^Q_{it}$ − holomorphic functions of z_{it} − Fock-like rep.

$t = -1 \Rightarrow \mathcal{H}^Q_{-i}$ − L^2-functions of x − Schrödinger representation

If two quantum quantum theories – with Hilbert spaces $\mathcal{H}_0^Q$ and $\mathcal{H}_\mathcal{T}^Q$ – are equivalent there must be a unitary operator

$$U_\mathcal{T} \ : \ \mathcal{H}_0^Q \longrightarrow \mathcal{H}_\mathcal{T}^Q$$

intertwining the representations of relevant observable algebras.

2.5. Decomplexification and definition of $\mathcal{H}^{Q}_{\mathcal{P}_\mu}$

One problem that motivated us initially was the problem that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic oscillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ is the moment map of a nice $\mathbb{R}^{n-k} \times \mathbb{T}^k$ action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu =< X_{H_1}, \ldots, X_{H_n} >$$

So that $\mathcal{H}^Q_{\mathcal{P}_\mu} =??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations $\mathcal{P}_t$ degenerating to $\mathcal{P}_\mu$ as $t \to \infty$.

Theorem (M-Nunes, 2013, [MN1])
For a class of completely integrable systems (M, ω, μ), $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

1. [proved for big class]

$$\lim_{t\to\infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0 \;=\; \mathcal{P}_\mu \quad \text{in a open dense subset}$$
$$\Gamma^\infty(\lim_{t\to\infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0) \;=\; \Gamma^\infty(\mathcal{P}_\mu)$$

2. (so far proved for much smaller class; to be increased soon with the Gelfand–Cetlin systems [KMNW])

The family $\mathcal{P}_t = (\varphi_{it}^{X_h})^* \mathcal{P}_0$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M, L)$, with a L^2-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty} \sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^Q_{\mathcal{P}_\mu} = \mathrm{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

Half-form corrected decomplexification of toric manifolds

[BFMN1], [KMN1]

Let us illustrate the above results in the case of the half-form corrected quantization of a symplectic toric manifold (X_P, ω, μ) with Delzant polytope P,

$$P = \{H \in \mathbb{R}^n \ : \ \ell_1(H) =< \nu_1, H > +\lambda_1 \geq 0, \ldots, \ell_r(H) =< \nu_r, H > +\lambda_r \geq 0\}$$

and

$$\mu^{-1}(\breve{P}) \cong \breve{P} \times T^n, \ \omega = \sum_{j=1}^{n} dH_j \wedge d\theta_j$$

As we have seen (example 2, p. 47, 48) the flow of $iX_{||\mu||^2/2}$ gives a one parameter family of compatible complex structures on X_P. These are given from the symplectic potential

$$g_t(H) = \sum_{j=1}^{r} \frac{1}{2}\ell_j(H)\log(\ell_j(H)) + \varphi(H) + t\frac{||\mu||^2}{2}.$$

Then

$$J_t \ : \ z_t = \frac{\partial g_t}{\partial H} + i\theta$$

and the family of Kähler polarizations converges indeed to the real toric polarization (on $\mu^{-1}(\bar{P})$) [BFMN1]

$$\mathcal{P}_t = \langle \frac{\partial}{\partial \overline{z}_t} \rangle \overset{t\to\infty}{\longrightarrow} \langle \frac{\partial}{\partial \theta} \rangle$$

Let ω be such that $\left[\frac{\omega}{2\pi}\right] - \frac{1}{2}c_1(X)$ is an integral cohomology class.

Consider a $U(1)$ equivariant complex line bundle $\widetilde{L}$ with $c_1(\widetilde{L}) = \left[\frac{\omega}{2\pi}\right] - \frac{1}{2}c_1(X)$ and an associated integral Delzant polytope P_L

$$P_L = \{H \in \mathbb{R}^n \ : < \nu_1, H > + \lambda_1^L \geq 0, \ldots, < \nu_r, H > + \lambda_r^L \geq 0\},$$

where $\lambda_j \in \mathbb{Z}$.

By adding constants to H_j we can choose the moment polytope P to be given by $\lambda_j = \lambda_j^L + \frac{1}{2}$, so that $P_L \subset P$ and the facets of P_L are obtained from the facets of P by shifts of $\frac{1}{2}$ along the corresponding inner pointing normals. This gives an ω in the appropriate cohomology class since $K_J \cong O(-D_1 - \cdots - D_r)$, and therefore, $c_1(X) = [D_1 + \cdots + D_r]$, where D_j denote the torus invariant divisors corresponding to the r facets of the polytope.

The full corrected connection in the dense orbit $U_0 = \mu^{-1}(\breve{P})$ and in the vertex charts U_v is

$$\widetilde{\Theta}_0 := \frac{\nabla^J \mathbf{1}^{U(1)}}{\mathbf{1}^{U(1)}} = -i\, H \cdot d\theta + \frac{i}{4}\left(\frac{\partial}{\partial H} \log\det G\right) \cdot G^{-1} d\theta$$

$$\widetilde{\Theta}_v := \frac{\nabla^J \mathbf{1}_v^{U(1)}}{\mathbf{1}_v^{U(1)}} = -i\, H_v \cdot d\theta_v + \frac{i}{2}\sum_{k=1} d\theta_v^k + \frac{i}{4}\left(\frac{\partial}{\partial H_v} \log\det G_v\right) \cdot G_v^{-1} d\theta_v$$

where $H_v = A_v H + \lambda_v, \theta_v =^t A_v^{-1}\theta, G = \mathrm{Hess}_H(g_t), G_v = \mathrm{Hess}_{H_v}(g_t)$.

For the polarized J_t-holomorphic sections in the dense orbit one then obtains

$$\sigma_m(w_t) = w_t^m e^{-k_t(H)} \mathbf{1}^{U(1)} \sqrt{|dz_t|},$$

where $w_t = e^{z_t}$ and $k_t = H \cdot \partial g_t / \partial H - g_t$.

Taking the limit $t \to \infty$ we obtain [BFMN1], [KMN1]:

$$\lim_{t\to\infty} \frac{\sigma_m}{||\sigma_m||_{L^2}} = \delta(H-m)e^{im\cdot\theta}\sqrt{dH}$$

which, due to the definition of $\mathcal{P}_\mu$, do correspond to distributional sections supported at the corrected BS orbits.

Simple but illustrative nontoric example

Let us consider the harmonic oscillator case

$$\left(\mathbb{R}^2, dx \wedge dy, \mu(x,y) = H(x,y) = \frac{1}{2}(x^2+y^2)\right)$$

but start with the nontoric Schrödinger or vertical polarization,

$$\mathcal{P}_0 = < X_x > = < \frac{\partial}{\partial y} >$$

For $h = \frac{1}{2}H^2 = \frac{1}{8}(x^2+y^2)^2$, we get

$$X_h = -H\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -H\,\frac{\partial}{\partial \theta}$$

and therefore

$$\begin{aligned} w_{it} = e^{itX_h}\, x &= e^{itX_h}\sqrt{2H}\,\mathrm{cos}(\theta) = \sqrt{2H}\,\mathrm{cos}(\theta - itH) = \\ &= \sqrt{\frac{H}{2}}\left(e^{tH}e^{i\theta} - e^{-tH}e^{-i\theta}\right) \end{aligned}$$

$$\lim_{t\to\infty} \mathcal{P}_{it} = \lim_{t\to\infty} < X_{w_{it}} >= \begin{cases} < \frac{\partial}{\partial\theta} > & \text{if } H > 0 \\ < \frac{\partial}{\partial y} > & \text{if } H = 0 \end{cases}$$

The monomial sections also converge to delta-functions supported on the corrected BS fibers as in the toric case.

2.6. Some references (and work in progress) for Lecture 2

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Lecture 3

Geometric tropicalization of manifolds and divisors

3.1. Summary of Lecture 3

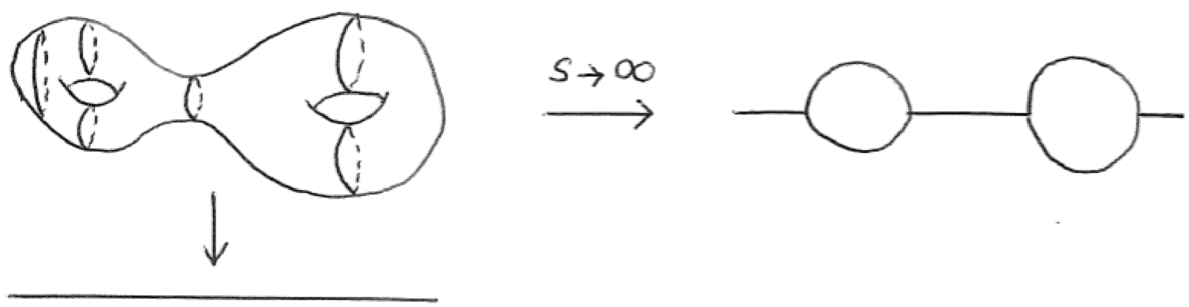
- Tropical amoebas in $(\mathbb{C}^*)^n$ and moment map/geodesics interpretation.

- Definition of geometric tropicalization of varieties and divisors, [MN2].

- Geometric tropicalization of toric manifolds, [BFMN1].

Main thesis for geometric tropicalization of manifolds

Imaginary time $t \to \infty$ limit

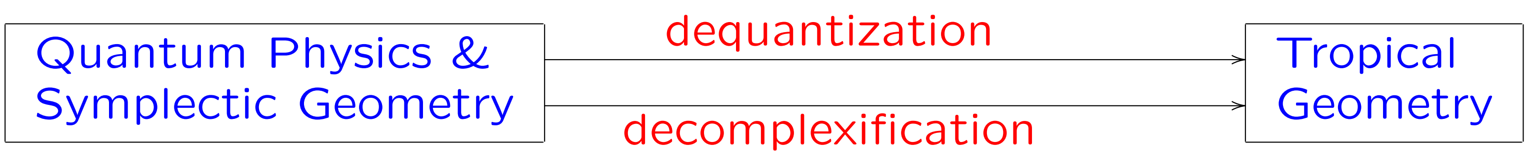


Main thesis for geometric tropicalization of divisors

The Log_T map of standard amoeba theory generalizes as the moment map $\mathrm{Log}_T = \widehat{\mu}_t$, where t is the geodesic time and $T = e^t$

New relation:

[for completely integrable systems $(M^{2n}, \omega, \mu \;:\; X \to \mathbb{R}^n)$]



Dequantization – $\hbar \to 0$

Decomplexifications – (Kähler) Geometric degenerations to tropical varieties:

:= Follow (to infinite time) a geodesic ray,
in the space of Kähler metrics,
generated by $h = ||\mu||^2 = H_1^2 + \cdots + H_n^2$,

= Do a Wick rotation, time $\rightsquigarrow it$, followed by $t \to \infty$
for $h = ||\mu||^2$

3.2. Tropicalization: flat case

Recall the origin of tropical geometry

$$Y = \left\{ w \in (\mathbb{C}^*)^n \ : \ \sum_{m \in P} c_m \, w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

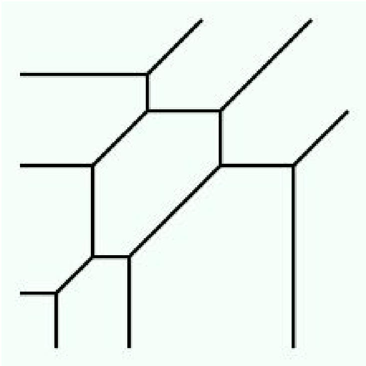
$$Y \xrightarrow{\mathrm{Log}_T} \mathbb{R}^n$$

$$\mathrm{Log}_T(w) = \frac{1}{t}(\log|w_1|, \ldots, \log|w_n|)$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T\to\infty} \mathrm{Log}_T(Y)$

An example for a generic degree 3 plane curve is:



Some geometric and enumerative properties become very simple in tropical geometry.

Simple examples for plane curves are the following. Let $C, \widetilde{C}$ be nonsingular plane curves.

Genus–degree formula

$$g(C) = \frac{1}{2}(d(C) - 1)(d(C) - 2)$$

Bezout Theorem

$$|C_1 \cap C_2| = d(C_1)\ d(C_2)$$

Let us make a geodesic interpretation of the $T \to \infty$ limit.

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J) & \xrightarrow{\psi} & ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}) \\ (\theta, H) & \mapsto & e^{H+i\theta} \end{array}$$

We see that the initial moment map $\widehat{\mu}_0 \ : \ (\mathbb{C}^*)^n \to \mathbb{R}^n$ of the torus action is

$$\widehat{\mu}_0 = \mathsf{Log}_1(w) = H$$

Now let us consider a geodesic starting at $\widehat{\omega}_0$ in the direction of $h = (H_1^2 + \cdots + H_n^2)/2$ so that $X_h = -H \cdot \frac{\partial}{\partial \theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h}\, e^{H+i\theta} = e^{(1+\tilde{t})H+i\theta} = e^{tH+i\theta}$$

with $t = \tilde{t} + 1$. We have

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \widehat{J}_0) \\ (\theta, H) & \mapsto & e^{tH+i\theta} \end{array}$$

As $\widehat{\omega}_t = \frac{1}{t} du \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t}\, u$ so that

$$\widehat{\mu}_t(w) = \mathsf{Log}_T(w) = H$$

We see that, in this interpretation, the T of amoeba theory is exponential of the geodesic time, $T = e^t$, and the Log_T map is the moment map of the complex picture.

3.3. Definition of Geometric Tropicalization

Definition

Tropicalization (Liouville integrable) Kähler manifolds

Let $(M, \omega_t, I, \gamma_t)$ be a $||\mu||^2$-geodesic ray. The geometric tropicalization of (M, ω_0, I_0), in the direction of $||\mu||^2$, is the Gromov-Hausdorff limit

$$
\begin{aligned}
(M_{\text{gtrop}}, d_{\text{gtrop}}) &= \lim_{t \to \infty} (M_t, \gamma_t) \\
M_{\text{gtrop}} &= \mu(M) \text{ if the fibers are connected}
\end{aligned}
$$

Tropicalization of divisors (for connected fibers)

The geometric tropicalization of an hypersurface $Y \subset M$, in the direction of $||\mu||^2$, is the Hausdorff limit of $\widehat{\mu}_t(Y)$ in $\mu(M) = M_{\text{gtrop}}$,

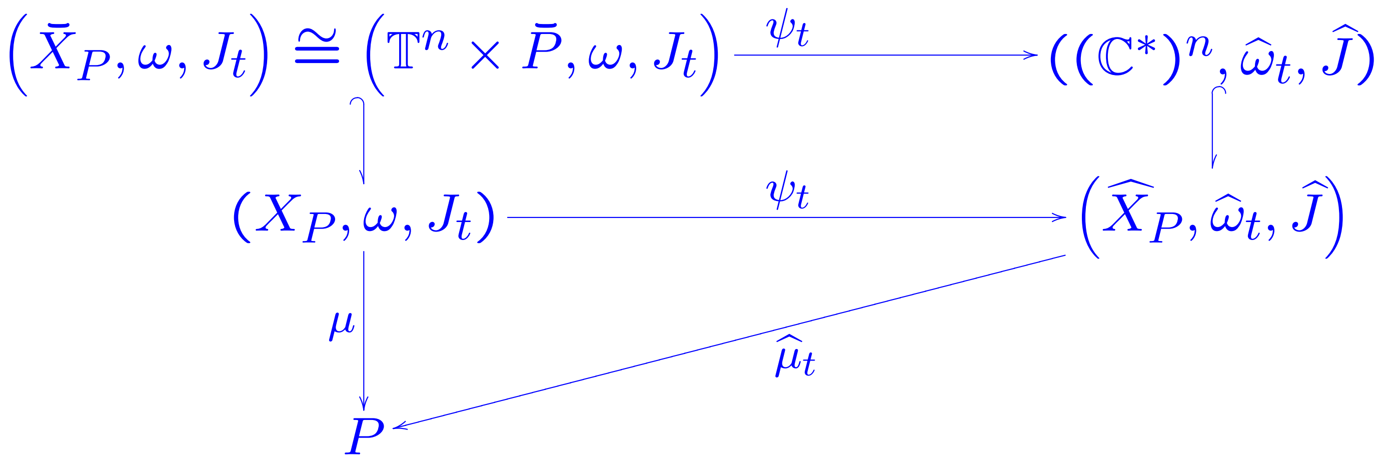
$$Y_{\text{gtrop}} = \lim_{t \to \infty} \widehat{\mu}_t(Y) \subset \mu(M)$$

3.4. Geometric tropicalization: toric case

What we did in [BFMN1] was (equivalent) to show that the flat picture extends to (nonflat) toric varieties

$$h = \frac{1}{2}||\mu||^2 = \frac{1}{2}(H_1^2 + \cdots + H_n^2) \Rightarrow w_t = e^{\frac{\partial}{\partial H}(g+th)+i\theta}$$

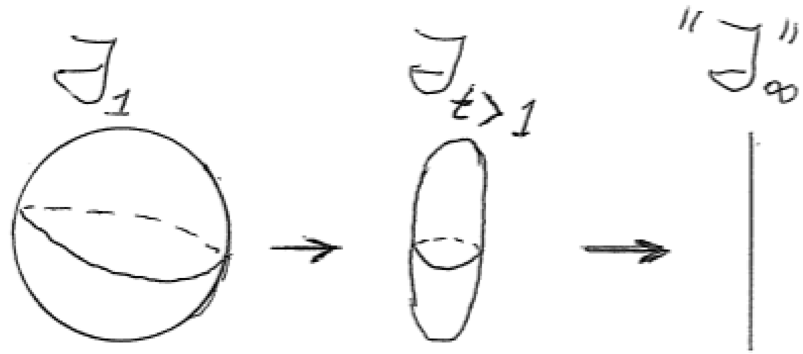
through the diagram we had before



$$
\begin{array}{lcl}
\psi_t(\theta, H) & = & e^{u_t + i\theta} = e^{\partial g_t/\partial H + i\theta} \\
g_t(H) & = & \sum_{F \subset P} \frac{1}{2} \ell_F(H) \log \ell_F(H) + \phi + t\, h(H) \\
\mu(\theta, H) & = & H \; - \text{moment map in the } \textit{symplectic} \text{ picture}
\end{array}
$$

Baier-Florentino-M-Nunes, [BFMN1]

- GH collapse or geometric tropicalization of toric manifolds: Metrically, as $t \to \infty$, the Kähler manifold (X_P, ω, J_t) collapses to P with metric Hess(h) on $\bar{P}$



$$\frac{1}{t}\gamma_t = \frac{1}{t}(\text{Hess}(g_t)dH^2 + \text{Hess}(g_t)^{-1}d\theta^2) \stackrel{t\to\infty}{\longrightarrow} \text{Hess}(h)\, dH^2$$

- Decomplexification: The complex (Kähler) structure degenerates to the real toric polarization

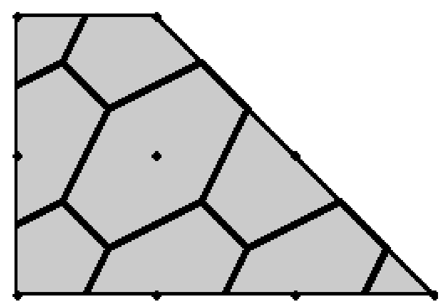
- Geometric Tropicalization of hypersurfaces: In the same limit (part of) the amoebas of divisors tropicalize

$$Y_t = \left\{ \sum_{m \in \mathcal{P}} c_m e^{m \cdot \frac{\partial g_P}{\partial H} + tm \cdot H + im \cdot \theta} = 0 \right\} \xrightarrow{\mu} P$$

Theorem (Baier-Florentino-M-Nunes, 2011, [BFMN1])

$$\lim_{t \to \infty} \widehat{\mu}_t(Y) = \lim_{t \to \infty} \mu(Y_t) = \pi(A_{\mathrm{trop}})$$

where π denotes the convex projection to P.



3.5. Some References (and work in progress) for Lecture 3

Tropical geometry, mirror symmetry and Gromov-Hausdorff limits

[BBI] D. Burago, Y. Burago, S. Ivanov. *A course in metric geometry*, GSM, American Math Soc., 2001.
[Gr] M. Gross, *Tropical geometry and mirror symmetry*, CBMS, American Math Soc., 2011.
[MS] D. Mclagan, B. Sturmfels, *Introduction to Tropical Geometry*, book in progress
[Mi] G. Mikhalkin, *Tropical geometry and its applications*, Proceedings ICM, Madrid 2006, pp. 827–852.
[ncl2] ncatlab, *Tropical geometry*

Decomplexification and Geometric Tropicalization
Lecture 2 references: [BFMN1], [KMN1], [MN2]

Thank you!