Complex time evolution in geometric quantization

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1. Introduction

1.1 Ambiguity of quantization

The dream of the founders of quantum mechanics was to have quantization as a well defined process assigning a quantum system to every classical system in a well defined way and satisfying the correspondence principle

$$(M,\omega) \mapsto Q_\hbar(M,\omega) \overset{\hbar\to 0}{\mapsto} (M,\omega)$$

It was soon realized that this can never be the case even for the simplest systems.

Particle in the line

Classical

$(M, \omega) = (\mathbb{R}^2, dx \wedge dp)$

Quantum

$$Q_\hbar(M, \omega):$$

$$\begin{array}{rcl}
\mathcal{H}_{Sch} & = & L^2(\mathbb{R}, dx) \\
x & \mapsto & \widehat{x} = x \\
p & \mapsto & \widehat{p} = -i\hbar\dfrac{d}{dx} \\
f(x,p) & \mapsto & ?? \\
\dfrac{1}{2}p^2 + V(x) & \mapsto & -\hbar^2\dfrac{d}{dx^2} + V(x)
\end{array}$$

van Hove no go Thm:

It is impossible to quantize all observables exactly as Dirac hoped

$$\begin{aligned} Q(f) &= \widehat{f} \\ [Q(f), Q(h)] &= i\hbar Q(\{f, g\}) \end{aligned}$$

In order to quantize one needs to add additional data to the classical system. *eg* choose a (Lie) subalgebra of the algebra of all observables

$$\mathcal{A} = \mathsf{Span}_{\mathbb{R}}\{1, x, p\}$$

Then we have to study the dependence of the quantum theory on the additional data.

Geometric quantization is mathematically perhaps the best defined quantization

$$(M, \omega), \quad \frac{1}{\hbar}[\omega] \in H^2(M, \mathbb{Z})$$

Prequantum data: (L, ∇, h), $L \to M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^\infty(M, L) : ||s||^2 = \int_M h(s, s)\, \frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f} = Q(f) = -i\hbar \nabla_{X_f} + f$

This almost works! But the Hilbert space is too large, the representation is reducible. We need a smaller Hilbert space:
Prequantization $\Rightarrow$ Quantization

Kähler quantization: fix a complex structure I on M such that (M, ω, I, γ) is a Kähler manifold.

$$\mathcal{H}_I^Q = \left\{ s \in \mathcal{H}^{\mathrm{prQ}} \ : \ \nabla_{\bar{\partial}} s = 0 \right\} = H^0(M, L_I)$$

Get the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

Need to study the dependence of quantization on the choice of the complex structure.

1.2 Imaginary time: introduction

It is precisely to study the dependence of $Q_{\hbar}$ on the choice of the complex structure that evolution in imaginary time enters the scene.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

What we are studying is a new way of looking at imaginary time evolution in (some situations in) quantum mechanics giving it a precise geometric meaning.

In Kähler geometry imaginary time evolution leads to geodesics in the (infinite dimensional) space of Kähler metrics in a given cohomology class, and is used to study the stability of varieties [Semmes '92 and Donaldson '99].

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in '96 in order to transform the spin connection in the Ashtekar connections.

$$\Gamma_\mu \mapsto A_\mu^{\mathbb{C}}.$$

2. Imaginary time in Kähler geometry

2.1 Compact Lie groups G and their duals G'

Compact simple Lie groups have a natural unique bi-K-invariant metric of constant positive curvature.
Example: $K = SU(2) \cong (SU(2) \times SU(2))/SU(2) \cong S^3$
The dual is $K' = K_{\mathbb{C}}/K$. It has a natural unique K-invariant metric of constant negative curvature.
Example: $SU(2)' = SL(2, \mathbb{C})/SU(2) \cong \mathbb{H}^3$

sectional curvature

G: $$K_G(\xi, \eta) = \frac{1}{4}||[\xi, \eta]||^2$$

G': $$K_{G'}(\xi, \eta) = -\frac{1}{4}||[\xi, \eta]||^2$$

$$T_e(G \times G) = \mathcal{G} \oplus \mathcal{G} \quad \Rightarrow \quad T_{[e]}(G) \cong \mathcal{G}$$
$$T_e(G_{\mathbb{C}}) = \mathcal{G} \oplus i\mathcal{G} \quad \Rightarrow \quad T_{[e]}(G) \cong i\mathcal{G}$$

2.2 Ham(M, ω) **versus** Ham$_{\mathbb{C}}(M, \omega)$

M-Nunes, arXiv, this week

$G \rightsquigarrow$ Ham(M, ω) - is an infinite dimensional analogue of a compact Lie group, with Lie(Ham$(M, \omega)) = C^{\infty}(M)$

$G_{\mathbb{C}} \rightsquigarrow$ Ham$_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits.

Let (M, ω) be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\mathrm{an}}(M)$

Theorem [Gröbner]

Let $f \in C^{\text{an}}(M)$. Exists $T_f > 0$ the series

$$e^{\tau X_h} \;:\; f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{1}$$

is absolutely convergent, for $|\tau| < T_f$, to a (complex) real analytic function.

So $\text{Ham}_{\mathbb{C}}(M, \omega)$ acts locally on $C^{\text{an}}(M)$.

To descend to an “action” on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local J_0-holomorphic coordinates. Then we use (1) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \tag{2}$$

Theorem [Burns-Lupercio-Uribe, 2013; M-Nunes, 2013]
Under natural conditions on J_0 the relations (2) define a unique map $\varphi_\tau^{X_h} \ : \ M \to M$ such that

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

For τ sufficiently small $\varphi_\tau^{X_h}$ is a biholomorphism.

Example

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_0)$ and $h = y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then
$\varphi_t^{X_h}(x, y) = (x + ty, y)$
and
$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$

Indeed: $e^{ity\frac{\partial}{\partial x}}(x + iy) = x + i(1 + t)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow \nabla^{\gamma_{it}} h = J_{it} X_h = \frac{1}{1+t}\, y\frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t} dx^2 + (1 + t)dy^2$.

$h = \frac{y^2}{2}$, $X_h = y\frac{\partial}{\partial x}$ $\left\{e^{itX_h},\ t \in \mathbb{R}\right\} \subset \mathrm{Ham}_{\mathbb{C}}(\mathbb{R}^2, dx \wedge dy)$

For $t > -1$, $\varphi_{it}^{X_h}$ are orientation preserving diffeomorphisms (linear isomorphisms).

So, for $t = -1$, $\varphi_{-i}^{X_h}$ colapses $\mathbb{R}^2$ to the horizontal axis and there is also a corresponding metric collapse.

For $t < -1$, $\varphi_{it}^{X_h}$ are orientation reversing linear isomorphisms.

3. Imaginary time in geometric quantization

3.1 Changing the representation (= polarization) of wave functions

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{\mathrm{prQ}} \overset{J_0}{\rightsquigarrow} \mathcal{H}^{\mathrm{Q}}_{J_0} = H^0_{J_0}(M, L) = \left\{ s \in \mathcal{H}^{\mathrm{prQ}} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned} \mathcal{H}^{\mathrm{Q}}_{J_0} &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow < \frac{\partial}{\partial \bar{z}^j_{J_0}} > = \mathcal{P}_{J_0} \\ &= \text{space of sections depending only on } z_{J_0} \\ &\quad (= \text{i.e. } J_0\text{-holomorphic}) \end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$\begin{aligned}
\mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} < \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} > \\
&= e^{\tau \mathcal{L}_{X_h}} < X_{z^1}, \ldots, X_{z^n} > = < X_{e^{\tau X_h}(z^1)}, \ldots, X_{e^{\tau X_h}(z^n)} > = \\
&= < X_{z_\tau^1}, \ldots, X_{z_\tau^n} > = (\varphi_\tau^{X_h})^* (\mathcal{P}_0)
\end{aligned}$$

For some values of $\tau \neq 0$ the polarization may be real or mixed. Recall that real polarizations are spanned by real vector fields at every point.

In our flat example, $h = y^2/2, \tau = it$,

$$\mathcal{P}_{it} = <\frac{\partial}{\partial \bar{z}_{it}}>=<X_{z_{it}}>=<X_{x+i(1+t)y}>=$$
$$= <-\frac{\partial}{\partial y}+i(1+t)\frac{\partial}{\partial x}>\stackrel{t\to -1}{\longrightarrow}<\frac{\partial}{\partial y}>=\mathcal{P}_{-i}$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}^{\mathrm{Q}}_{it}-$ holomorphic functions of z_{it} − Fock-like rep.

$t = -1 \Rightarrow \mathcal{H}^{\mathrm{Q}}_{-i}-$ L^2-functions of x − Schrödinger representation

If two quantum quantum theories – with Hilbert spaces $\mathcal{H}_0^{\mathsf{Q}}$ and $\mathcal{H}_\tau^{\mathsf{Q}}$ – are equivalent there must be a unitary operator

$$U_\tau \ : \ \mathcal{H}_0^{\mathsf{Q}} \longrightarrow \mathcal{H}_\tau^{\mathsf{Q}}$$

intertwining the representations of relevant observable algebras.

3.2 From L^2 functions to holomorphic functions and CST

Some beautiful harmonic analysis coming up in this context.

$$A = (A\,(A^\dagger A)^{-1/2})\,(A^\dagger A)^{1/2} = XH = Xe^{iY}$$

We get the diffeomorphisms (for $t \neq -1$)

$$\begin{array}{rcl} SL(n,\mathbb{C}) & \cong & SU(n) \times \mathbb{H} \cong SU(n) \times isu(n) \cong T^*SU(n) \\ A & \mapsto & (X,H) \mapsto (X,Y) \mapsto (X, \dfrac{1}{1+t} Y) \end{array}$$

Let ψ_t denote their inverses and consider the diagram

$$\begin{array}{ccc} T^*SU(n) & \xrightarrow{\psi_0} & SL(n,\mathbb{C}) \\ \uparrow & \nearrow_{\psi_t} & \\ T^*SU(n) & & \end{array}$$

For $T^*SU(n)$ with standard symplectic structure and with (initial) complex structure J_0 pulled back with ψ_0 from $SL(n,\mathbb{C})$ we get

Theorem [Hall-Kirwin] Let $h(X,Y) = ||Y||^2/2$. Then

$$\varphi_{it}^{X_h} = \psi_0^{-1} \circ \psi_t$$

Similarly to the flat example we saw before we get for the $\{e^{itX_h}, t \in \mathbb{R}\}$ "orbit" through J_0 the following

For $t \neq -1$, $\varphi_{it}^{X_h}$ are diffeomorphisms of $T^*SU(n) \cong SL(n, \mathbb{C})$

So, for $t = -1$, $\varphi_{-i}^{X_h}$ colapses $SL(n, \mathbb{C})$ to the maximal compact subgroup $SU(n)$ with a corresponding metric collapse and $\varphi_{-i_*}^{X_h} \mathcal{P}_0 =< \frac{\partial}{\partial Y} >$

Accordingly we have

$$\begin{aligned} \mathcal{H}^{\mathbb{Q}}_{-i} &\cong L^2(SU(n), dX) \\ \mathcal{H}^{\mathbb{Q}}_{it} &\cong \mathcal{H}L^2(SL(n,\mathbb{C}), d\nu_{t+1}) \end{aligned}$$

where dX denotes the Haar measure on $SU(n)$ and $d\nu_{t+1}$ is the heat-kernel measure

$$d\nu_{t+1} = e^{-\frac{||Y||^2}{t+1}} \eta(Y) dX dY,$$

and η is the Ad–invariant function on $su(n)$ determined, on a chosen Cartan subalgebra, by $\eta(Y) = \prod_{\alpha\in\Delta^+} \frac{\sinh(\alpha(Y))}{\alpha(Y)}$

These quantizations of $(T^*SU(n), \omega)$ are all equivalent.

Theorem [Hall 1994, 2002]
The unitary equivalence of the quantizations for $t = -1$ and $t = 0$ is given by the following Segal-Bargman (or coherent states) transform (CST):

$$\begin{aligned} U \ : \ L^2(SU(n), dX) &\longrightarrow \mathcal{H}L^2(SL(n,\mathbb{C}), d\nu_1) \\ f &\mapsto (\varphi_{-i}^{X_h})^* \left[e^{\frac{\Delta}{2}}(f) \right] \end{aligned}$$

Note: $(\varphi_{-i}^{X_h})^*$ is equivalent to the operator of analytic continuation $\mathcal{C} : C^{\mathrm{an}}(SU(n)) \to \mathcal{H}(SL(n,\mathbb{C}))$.

4. Decomplexification of completely integrable systems

One problem that motivated us initially was the problem that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic orcillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ is the moment map of a $\mathbb{R}^n$ action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu =< X_{H_1}, \ldots, X_{H_n} >$$

So that $\mathcal{H}^{Q}_{\mathcal{P}_\mu} =??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations $\mathcal{P}_t$ degenerating to $\mathcal{P}_\mu$ as $t \to \infty$.

Theorem [M-Nunes, 2013]
For a large class of completely integrable systems (M, ω, μ), $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

1. $\lim_{t\to\infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0 = \mathcal{P}_\mu$.

2. The family $\mathcal{P}_t = (\varphi_{it}^{X_h})^* \mathcal{P}_0$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M, L)$, with a L^2-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty} \sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^{Q}_{\mathcal{P}_\mu} = \text{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

Conjecture

This theorem is valid for all completely integrable systems.

6. References

References

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- T. Baier, J.Mourão and J.P. Nunes, *Quantization of Abelian Varieties: distributional sections and the transition from Kaehler to real polarizations*, Journ. Funct. Anal. **258** (2010) 3388–3412.

Work in Progress

- J.Mourão and J.P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*

- W. Kirwin, J.Mourão and J.P. Nunes, *Decomplexification of Riemann surfaces and quantization*

Thank you!