Geodesics on the space of Kähler metrics, geometric tropicalization and quantization

José Mourão

Técnico Lisboa, U Lisboa

Mini-Workshop on Geometry
Chinese University of Hong Kong

October, 16–17, 2013, Hong Kong

work in collaboration with
Will Kirwin (U Köln) and João P. Nunes (Técnico Lisboa)

Index

1. Summary

Three main topics of the seminar:

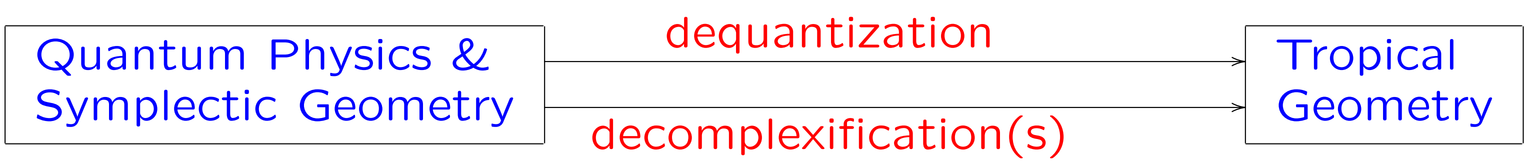
1. Hamiltonian complex time evolution in Kähler geometry and in quantum physics

2. Problem that motivated us initially: For a completely integrable system $(M, \omega, \mu = (H_1, \dots, H_n) \;:\; M \to \mathbb{R}^n)$ define its quantization in the (usually singular) real polarization $\mathcal{P}_\mu$ defined by μ,

$$\mathcal{H}^Q_{\mathcal{P}_\mu} = ??$$

3. New relation:

[for (some?) completely integrable systems $(M, \omega, \mu \ : \ M \to \mathbb{R}^n)$]



Dequantization – $\hbar \to 0$

Decomplexifications – (Kähler) Geometric degenerations to tropical varieties
= geometric tropicalization:
=: Follow (to infinite time) a geodesic ray,
in the space of Kähler metrics,
generated by $H = ||\mu||^2 = f_1^2 + \cdots + f_n^2$,
= Do a Wick rotation, time $\rightsquigarrow is$, followed by $s \to \infty$
for $H = ||\mu||^2$

2. Complex time evolution

2.1 Imaginary time: introduction

Geometric quantization

$$(M,\omega), \qquad \frac{1}{\hbar}[\omega] \in H^2(M,\mathbb{Z})$$

Prequantum data: (L,∇,h), $L \to M$

Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M,L) = \overline{\left\{ s \in \Gamma^\infty(M,L) : ||s||^2 = \int_M h(s,s)\,\frac{\omega^n}{n!} < \infty \right\}}$$

Quantum observables: $\widehat{f} = Q_\hbar(f) = -i\hbar\nabla_{X_f} + f$

Kähler quantization: fix a complex structure I on M such that (M, ω, I, γ) is a Kähler manifold.

$$\mathcal{H}_I^Q = \left\{ s \in \mathcal{H}^{\mathsf{prQ}} \; : \; \nabla_{\bar\partial}\, s = 0 \right\} = H^0(M, L_I)$$

Get the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T}$$

Need to study the dependence of quantization on the choice of the complex structure.

It is precisely to study the dependence of $Q_\hbar$ on the choice of the complex structure that evolution in imaginary time enters the scene.

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$

What we are studying is a new way of looking at imaginary time evolution in (some situations in) quantum mechanics giving it a precise geometric meaning.

In Kähler geometry imaginary time evolution leads to geodesics and is used to study the stability of varieties [Semmes '92 and Donaldson '99].

In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in '96 in order to transform the spin connection in the Ashtekar connections

$$\Gamma_\mu \mapsto A^{\mathbb{C}}_\mu.$$

2.2 Complex time evolution in Kähler geometry

$G = \mathrm{Ham}(M, \omega)$ - is an infinite dimensional analogue of a compact Lie group, with $\mathrm{Lie}(\mathrm{Ham}(M, \omega)) = C^\infty(M)$

$G_{\mathbb{C}} = \mathrm{Ham}_{\mathbb{C}}(M, \omega)$ - doesn't exist as a group. Donaldson: define its orbits. There are natural orbits of two types passing through a Kähler pair (ω, J_0).

Complex picture

Let us start with the smaller one

$$\begin{aligned}\mathcal{H}(\omega, J_0) &= \left\{f \in C^\infty(M)\ :\ \omega + i\bar{\partial}_0\partial_0 f > 0\right\}/\mathbb{R} =: \\ &=: \{(\varphi^*\omega, J_0), \varphi \in G_{\mathbb{C}}\} = (G_{\mathbb{C}} \cdot \omega, J_0)\end{aligned}$$

So we get a orbit-dependent "definition" of $G_{\mathbb{C}}$ (made possible by the Moser theorem) as a subset of Diff(M)

$\mathcal{H}(\omega, J_0)$ is naturally a infinite dimensional symmetric space: $\mathcal{H}(\omega, J_0) \cong G_{\mathbb{C}}/G$ and Donaldson shows that the Mabuchi metric has constant negative curvature mimiking the analogous situation for compact simple groups and their complexifications.

In both (finite and infinite dimensional) cases geodesics are given by one parameter imaginary time subgroups $\{e^{itX}, X \in \mathrm{Lie}(G)\}$ ($X = X_f, f \in C^\infty(M)$ in our case)

Symplectic picture

Assuming that $\mathrm{Aut}_0(\mathrm{M}, \mathrm{J}_0)$ is trivial the second natural orbit is the total space of a G principal bundle over the first

$$\mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) = \{(\omega, \varphi_* J_0),\ \varphi \in G_{\mathbb{C}}\}$$

with projection

$$\begin{aligned} \pi \ : \ \mathcal{G}(\omega, J_0) = (\omega, G_{\mathbb{C}} \cdot J_0) \ &\longrightarrow \ \mathcal{H}(\omega, J_0) = (G_{\mathbb{C}} \cdot \omega, J_0) \\ \pi(\omega, \varphi_* J_0) \ &= \ \varphi^*(\omega, \varphi_* J_0) = (\varphi^* \omega, J_0) \end{aligned}$$

Gröbner Lie series

Let (M, ω) be compact, real analytic. Can use the Gröbner theory of Lie series to make the complex one-parameter sugbroup $\{e^{\tau X_h}, \tau \in \mathbb{C}\}$ act (locally in the functions) on $C^{\text{an}}(M)$

Theorem [Gröbner]

Let $f \in C^{\text{an}}(M)$ Exists $T_f > 0$ the series

$$e^{\tau X_h} \ : \ f \mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(f) \tag{1}$$

converges absolutely to a (complex) real analytic function.

So $\text{Ham}_{\mathbb{C}}(M, \omega)$ acts locally on $C^{\text{an}}(M)$. To descend to an "action" on M we need to fix a complex structure.

Let (M, ω, J_0) be a Kähler structure and $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)$ be local J_0-holomorphic coordinates. Then we use (1) to define the holomorphic coordinates of a new complex structure J_τ

$$z_\alpha^\tau = e^{\tau X_h}(z_\alpha) \qquad (2)$$

Theorem [Burns-Lupercio-Uribe, 2013; M-Nunes, 2013]
Under natural conditions on J_0 the relations (2) define a unique map $\varphi_\tau^{X_h} \ : \ M \to M$ such that

$$z_\alpha^\tau = (\varphi_\tau^{X_h})^*(z_\alpha)$$

For τ sufficiently small $\varphi_\tau^{X_h}$ is a biholomorphism.

Example 1 - $M = \mathbb{R}^2$

Let $(M, \omega, J_0) = (\mathbb{R}^2, dx \wedge dy, J_0)$ and $h = y^2/2 \Rightarrow X_h = y\frac{\partial}{\partial x}$

Then
$\varphi_t^{X_h}(x, y) = (x + ty, y)$
and
$\varphi_{it}^{X_h}(x, y) = (x, (1 + t)y)$

Indeed: $e^{ity\frac{\partial}{\partial y}}(x + iy) = x + i(1 + t)y = (\varphi_{it}^{X_h})^*(x + iy)$

Geometrically, $iX_h \leftrightarrow \nabla^{\gamma_{it}} h = J_{it} X_h = \frac{1}{1+t}\, y\frac{\partial}{\partial y}$

Also, $\gamma_{it} = \frac{1}{1+t} dx^2 + (1 + t)dy^2$.

Example 2 - Toric Varieties

P − Delzant polytope

$P = \{x \in \mathbb{R}^n \ : \ \ell_F(x) = \nu_F(x) \cdot x + \lambda_F \geq 0, F \text{ facet of } P\}$.

Let X_P the associated toric variety,

$\breve{X}_P \cong \breve{P} \times \mathbb{T}^n \ni (x, \theta)$

Toric complex structures are are in one-to-one correspondence with the space of symplectic potentials (Guillemin-Abreu theory)

$$g(x) = g_P(x) + \phi(x) = \sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) + \phi(x)$$

via the Legendre transform

$$\left(\bar{X}_P, \omega, J_\phi\right) \cong \left(\mathbb{T}^n \times \bar{P}, \omega, J_\phi\right) \xrightarrow{\psi_\phi} ((\mathbb{C}^*)^n, \widehat{\omega}_\phi, \widehat{J})$$

where

$$\psi_\phi(x, \theta) = e^{z_\phi} = e^{\frac{\partial g}{\partial x} + i\theta}$$

For a toric $h(x)$ we have $X_h = -\sum_j \frac{\partial g}{\partial x_j} \frac{\partial}{\partial \theta_j}$ and

$$e^{itX_h} e^{\frac{\partial g}{\partial x} + i\theta} = e^{\frac{\partial g}{\partial x} + i(\theta - it\frac{\partial h}{\partial x})} = e^{\frac{\partial (g+th)}{\partial x} + i\theta}$$

Thus we get

$$\begin{array}{ccc}
\left(\bar{X}_P,\omega,J_t\right)\cong\left(\mathbb{T}^n\times\bar{P},\omega,J_t\right) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n,\widehat{\omega}_t,\widehat{J}) \\
\downarrow & & \downarrow \\
(X_P,\omega,J_t) & \xrightarrow{\psi_t} & \left(\widehat{X}_P,\widehat{\omega}_t,\widehat{J}\right) \\
\mu\downarrow & \swarrow \widehat{\mu}_t & \\
P & &
\end{array}$$

$$\begin{aligned}
\psi_t(\theta,x) &= e^{y_t+i\theta}=e^{\partial g_t/\partial x+i\theta} \\
g_t(x) &= g_P(x)+\phi+th(x)=\textstyle\sum_{F\subset P}\frac{1}{2}\ell_F(x)\log\ell_F(x)+\phi+th(x) \\
\mu(\theta,x) &= x-\text{moment map in the } \textit{symplectic} \text{ picture} \\
& \quad\ \varphi_{it}^{X_h}=\psi_0^{-1}\circ\psi_t
\end{aligned}$$

Example 3 - Complex reductive groups $K_{\mathbb{C}}$

The "adapted" Kähler structure is [Guillemin-Stenzel, Lempert-Szöke, 1991]

$$\begin{array}{ccc} (T^*K, \omega, J_0) & \xrightarrow{\psi_0} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_0) \\ (x, Y) & \mapsto & xe^{iY}, \end{array}$$

where we used $T^*K \cong TK \cong K \times \mathrm{Lie}K$

General bi-K-invariant Kähler structures on T^*K are in one-to-one correspondence with strictly convex Ad–invariant "symplectic potentials" $\breve{g}$ on Lie$K$ [Kirwin-M-Nunes, 2013].

They define and are defined by their (Weyl–invariant) restriction g to the Cartan subalgebra. One has

$$\begin{array}{ccc}(T^*K, \omega, J_{\breve{g}}) & \xrightarrow{\psi_{\breve{g}}} & (K_{\mathbb{C}}, \widehat{\omega}, \widehat{J}_{\breve{g}}) \\ (x, Y) & \mapsto & xe^{iu(Y)}, \quad u = \dfrac{\partial \breve{g}}{\partial Y}\end{array}$$

The adapted Kähler structure corresponds to $\breve{g} = ||Y||^2/2$.

Choosing an h bi-K-invariant we get, $X_h = \sum_j \frac{\partial h}{\partial y_j} X_j$, where $Y = \sum_j y_j T_j$. Then

$$e^{itX_h} \cdot xe^{iY} = xe^{i(Y + t\frac{\partial h}{\partial Y})} = xe^{i\frac{\partial \breve{g}_t}{\partial Y}},$$

where $\breve{g}_t = \frac{||Y||^2}{2} + th$.

2.3 Complex time evolution in quantum mechanics

What does all this have to do with quantum mechanics?

$$\mathcal{H}^{prQ} \overset{J_0}{\rightsquigarrow} \mathcal{H}^Q_{J_0} = H^0_{J_0}(M, L) = \left\{ s \in \mathcal{H}^{prQ} : \nabla_{\bar{\partial}_{J_0}} s = 0 \right\}$$

$$\begin{aligned} \mathcal{H}^Q_{J_0} &= \text{Kernel of Cauchy-Riemann operators } \bar{\partial}_{J_0} \Leftrightarrow < \frac{\partial}{\partial \bar{z}^j_{J_0}} > = \mathcal{P}_{J_0} \\ &= \text{space of sections depending only on } z_{J_0} \\ &\quad (= \text{i.e. } J_0\text{-holomorphic}) \end{aligned}$$

When we evolve in complex time $\mathcal{P}_0 \mapsto \mathcal{P}_\tau$

$$\begin{aligned} \mathcal{P}_\tau &= e^{\tau \mathcal{L}_{X_h}} \mathcal{P}_0 = e^{\tau \mathcal{L}_{X_h}} < \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} > \\ &= e^{\tau \mathcal{L}_{X_h}} < X_{z^1}, \dots, X_{z^n} > = < X_{e^{\tau X_h}(z^1)}, \dots, X_{e^{\tau X_h}(z^n)} > = \\ &= < X_{z_\tau^1}, \dots, X_{z_\tau^n} > = (\varphi_\tau^{X_h})^* (\mathcal{P}_0) \end{aligned}$$

For some values of $\tau \neq 0$ the polarization may be real or mixed. Recall that mixed polarizations are spanned by real vector fields at every point.

In our flat example, $h = y^2/2, \tau = it$,

$$\mathcal{P}_{it} = < \frac{\partial}{\partial \bar{z}_{it}} >=< X_{z_{it}} >=< X_{x+i(1+t)y} >=$$
$$= < -\frac{\partial}{\partial y} + i(1+t)\frac{\partial}{\partial x} > \overset{t\to -1}{\longrightarrow} < \frac{\partial}{\partial y} >= \mathcal{P}_{-i}$$

We see that the quantum Hilbert spaces are

$t > -1 \Rightarrow \mathcal{H}^Q_{it}$ − holomorphic functions of z_{it} − Fock-like rep.

$t = -1 \Rightarrow \mathcal{H}^Q_{-i}$ − L^2-functions of x − Schrödinger representation

If two quantum quantum theories – with Hilbert spaces $\mathcal{H}_0^Q$ and $\mathcal{H}_\tau^Q$ – are equivalent there must be a unitary operator

$$U_\tau \ : \ \mathcal{H}_0^Q \longrightarrow \mathcal{H}_\tau^Q$$

intertwining the representations of relevant observable algebras.

3. Decomplexification and defintion of $\mathcal{H}^Q_{\mathcal{P}_\mu}$

One problem that motivated us initially was the problem that geometric quantization was ill defined for real polarizations associated with singular Lagrangian fibrations.

Even for the harmonic orcillator the fibration is singular!

The setting is that of a completely integrable Hamiltonian system

$$(M, \omega, \mu)$$

where $\mu = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ is the moment map of a $\mathbb{R}^n$ action.

The associated real polarization is usually singular

$$\mathcal{P}_\mu =< X_{H_1}, \ldots, X_{H_n} >$$

So that $\mathcal{H}^Q_{\mathcal{P}_\mu} =??$

Our approach has been to find a one parameter (continuous) family of Kähler polarizations $\mathcal{P}_t$ degenerating to $\mathcal{P}_\mu$ as $t \to \infty$.

Theorem [M-Nunes, 2013]
For a class of completely integrable systems (M, ω, μ), $h = ||\mu||^2$, and starting Kähler polarizations $\mathcal{P}_0$

1.(proved for big class)
$\lim_{t\to\infty} (\varphi_{it}^{X_h})^* \mathcal{P}_0 = \mathcal{P}_\mu.$

2. (so far proved for much smaller class)
The family $\mathcal{P}_t = (\varphi_{it}^{X_h})^* \mathcal{P}_0$ selects a basis of holomorphic sections of $H^0_{\mathcal{P}_t}(M, L)$, with a L^2-normalized holomorphic section $\sigma_{t,\mathcal{L}_{BS}}$ for every Bohr-Sommerfeld fiber of the (singular) real polarization $\mathcal{P}_\mu$. Then

$$\lim_{t\to\infty} \sigma_{t,\mathcal{L}_{BS}} = \delta_{\mathcal{L}_{BS}} \quad \text{and} \quad \mathcal{H}^Q_{\mathcal{P}_\mu} = \text{span}_{\mathbb{C}}\{\delta_{\mathcal{L}_{BS}}\}$$

4. Geometric tropicalization of varieties and divisors

Recall the origin of tropical geometry

$$Y = \left\{ w \in (\mathbb{C}^*)^n \ : \ \sum_{m \in P} c_m \, w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{Log}_T} & \mathbb{R}^n \\ \mathrm{Log}_T(w) & = & \frac{1}{t}(\log|w_1|, \ldots, \log|w_n|) \end{array}$$

where $T = e^t$.

Tropical amoebas are: $\mathcal{A}_\infty = \lim_{T \to \infty} \mathrm{Log}_T(Y)$

Let us make a geodesic interpretation of this picture.

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J) & \xrightarrow{\psi} & ((\mathbb{C}^*)^n, \widehat{\omega}, \widehat{J}) \\ (\theta, x) & \mapsto & e^{x+i\theta} \end{array}$$

We see that the initial moment map $\widehat{\mu}_0 \ : \ ((\mathbb{C}^*)^n \to \mathbb{R}^n$ of the torus action is

$$\widehat{\mu}_0 = \mathrm{Log}_1(w) = x$$

Now let us consider a geodesic starting at $\widehat{\omega}_0$ in the direction of $h = x^2/2$ so that $X_h = -x \cdot \frac{\partial}{\partial \theta}$. The complex structure flows in the symplectic picture, $J_0 \rightsquigarrow J_t$

$$e^{i\tilde{t}X_h}\, e^{x+i\theta} = e^{(1+\tilde{t})x+i\theta} = e^{tx+i\theta}$$

with $t = \tilde{t} + 1$.

We have

$$\begin{array}{ccc} (T^*\mathbb{T}^n, \omega, J_t) & \xrightarrow{\psi_t} & ((\mathbb{C}^*)^n, \widehat{\omega}_t, \hat{J}_0) \\ (\theta, x) & \mapsto & e^{tx+i\theta} \end{array}$$

As $\widehat{\omega}_t = \frac{1}{t} dy \wedge d\theta$ the moment map in the complex picture also changes $\widehat{\mu}_t(w) = \frac{1}{t} y$ so that

$$\widehat{\mu}_t(w) = \mathsf{Log}_T(w) = x$$

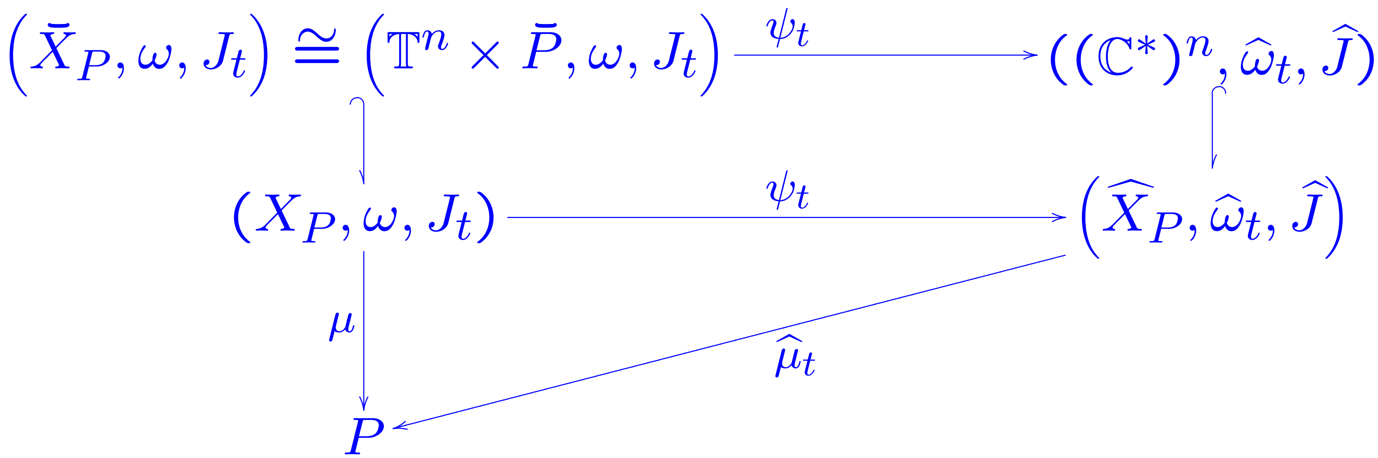
We see that, in this interpretation, the T of amoeba theory is exponential of the geodesic time, $T = e^t$, and the Log_T map is the moment map of the complex picture.

Toric case

What we did in our JDG paper was (equivalent) to show that this picture extends to toric varieties

$$h = \frac{1}{2}||\mu||^2 = \frac{1}{2}x^2 \Rightarrow w_t = e^{\frac{\partial}{\partial x}(g+th)+i\theta}$$

through the diagram we had before



$$
\begin{aligned}
\psi_t(\theta, x) &= e^{y_t + i\theta} = e^{\partial g_t/\partial x + i\theta} \\
g_t(x) &= g_P(x) + \phi + th(x) = \textstyle\sum_{F \subset P} \frac{1}{2} \ell_F(x) \log \ell_F(x) + \phi + th(x) \\
\mu(\theta, x) &= x - \text{moment map in the } \textit{symplectic} \text{ picture}
\end{aligned}
$$

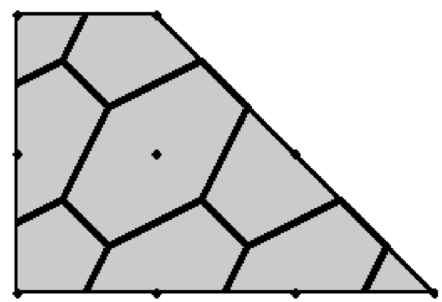
The metric γ_t collapses along the fibers of μ

$$\lim_{t\to\infty} \gamma_t = \gamma_\infty$$

For the tropical amoebas, in this context (with geodesic time $t \to \infty$ rather than deforming the field $\mathbb{C}$ to the tropical semiring, i.e. $\hbar \to 0$), we obtained:

Theorem[Baier-Florentino-M-Nunes, JDG, 2011]

$$\lim_{t\to\infty} \mu(Y_t) = \pi(A_{\mathrm{trop}})$$



6. References

References

- J.Mourão and J.P. Nunes, *On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics*, arXiv:1310.4025.

- W. Kirwin, J.Mourão and J.P. Nunes, *Complex time evolution in geometric quantization and generalized coherent state transforms*, J. Funct. Anal. **265** (2013) 1460–1493.

- W. Kirwin, J.Mourão and J.P. Nunes, *Degeneration of Kaehler structures and half-form quantization of toric varieties*, Journ. Sympl. Geom. **11** (2013) 603–643.

- T. Baier, C.Florentino, J.Mourão and J.P. Nunes, *Toric Kaehler metrics seen from infinity, quantization and compact tropical amoebas*, Journ. Diff. Geom. **89** (2011) 411–454.

- T. Baier, J.Mourão and J.P. Nunes, *Quantization of Abelian Varieties: distributional sections and the transition from Kaehler to real polarizations*, Journ. Funct. Anal. **258** (2010) 3388–3412.

Work in Progress

- J.Mourão and J.P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*

- W. Kirwin, J.Mourão and J.P. Nunes, *Decomplexification of Riemann surfaces and quantization*

Thank you!