Complexified Hamiltonian Symplectomorphisms and Solutions of the Homogeneous Complex Monge–Ampère Equation

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1. HCMA and imaginary time Hamiltonian flows

The Homogeneuous Complex Monge–Ampère (HCMA) equation is the following nonlinear equation on a complex $(n+1)$–dimensional manifold N

$$MA(\Phi) := \det\left(\frac{\partial^2 \Phi}{\partial z_j \partial z_{\bar{k}}}\right) = 0,$$

or, equivalently,

$$\left(\partial\bar{\partial}\Phi\right)^{n+1} = 0. \qquad (1)$$

It is a very difficult equation with very few (genuinly complex) solutions known.

Even for $n = 1$ it is very nontrivial.
For functions Φ on (open subsets of) $N = [0, T] \times S^1 \times M$, which are

(a) S^1–invariant and
(b) such that $g_{1\bar{1}} = \frac{\partial^2 \Phi}{\partial z \partial \bar{z}}(t, z, \bar{z}) > 0$

(i.e. $\phi_t = \Phi(t, \cdot)$ is a path of Kähler potentials on M), we have

$$\left(\partial_N \bar{\partial}_N \Phi\right)^2 = 0 \quad \Leftrightarrow$$

$$\frac{\partial^2 \Phi}{\partial t^2} \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} - |\frac{\partial^2 \Phi}{\partial t \partial \bar{z}}|^2 \quad = \quad 0 \Leftrightarrow$$

$$\frac{\partial^2 \Phi}{\partial t^2} = g^{1\bar{1}} |\frac{\partial^2 \Phi}{\partial t \partial \bar{z}}|^2 \quad \Leftrightarrow$$

$$\ddot{\phi}_t = ||\nabla \dot{\phi}_t||^2_{\phi_t} \qquad (2)$$

Ellaborating on an idea of Semmes and Donaldson we will show how to reduce the Cauchy problem for (2)

$$\begin{cases} \ddot{\phi}_t &= ||\nabla\dot{\phi}_t||^2_{\phi_t} \\ \phi_0 &= k, \\ \dot{\phi}_0 &= H, \end{cases} \qquad k \in C^\infty(U), H \in C^\infty(M). \tag{3}$$

to the problem of finding the integral curves of the Hamiltonian vector field X^ω_H, where $\omega = i\partial\bar{\partial}k$, followed by "rotating" t to the imaginary axis (in the complex t–plane)

$$\exp(sX^\omega_H) \rightsquigarrow \exp(\sqrt{-1}tX^\omega_H) \in Ham_{\mathbb{C}}(M,\omega) \overset{??}{\subset} Diff(M), \tag{4}$$

in a certain way.

To make sense of (4) we must be working on the symplectic picture (see section 3 below) in which ω is fixed and the complex structure J_t changes.

Then the imaginary time integral curves in (4) are solutions of the following coupled system

$$\begin{cases} \dot{x}_t &= J_t X_H^\omega = \nabla^{\gamma_t} H \\ J_t &= \left(\exp(\sqrt{-1}tX_H^\omega)\right)^* (J). \end{cases} \tag{5}$$

Then Donaldson shows that a solution of (3) is given by the Kähler potential ϕ_t of ω_t in

$$\omega_t = \left(\left(\exp(\sqrt{-1}tX_H^\omega)\right)^{-1}\right)^* (\omega) \,. \tag{6}$$

This is the so called Donaldson formal solution of the CHMA.

The problem is that to find the imaginary time flow $\exp(\sqrt{-1}\, tX_H^\omega)$ with (5) is equivalent to solving a complicated system of PDE (see [BLU]). So it is not clear what have we gaigned in going from the original HCMA (3) to the coupled system (5).

NO PDE needed!

In Section 3 we will describe a method, proposed in [MN], to integrate (5) by only finding the real time flow $\exp(tX_H^\omega)$ and then rotating to imaginary time and finding

$$\exp(\sqrt{-1}\, tX_H^\omega) \in \mathrm{Diff}(M)$$

without the need of solving any PDE.

This method will be used in Section 4 to find infinite dimensional spaces of new solutions of HCMA equations in different manifolds.

2. HCMA and the geometry of the space of Kähler metrics on M

The idea behind Semmes–Donaldson formal solution (6) comes from the geometry on the space $\mathcal{H}_0$ of Kähler metrics on (M, J) with fixed cohomology class $[\omega]$. The space of Kähler potentials on (M, J) with class $[\omega]$ is

$$\mathcal{H} = \left\{\phi \in C^\infty(M) \,:\, \omega_\phi = \omega + i\partial\bar\partial\,\phi > 0\right\},$$

so that also

$$T\mathcal{H} = \mathcal{H} \times C^\infty(M).$$

The space of Kähler metrics is

$$\mathcal{H}_0 = \left\{\omega_\phi = \omega + i\partial\bar\partial\,\phi\,, \phi \in \mathcal{H}\right\} \cong \mathcal{H}/\mathbb{R}\,.$$

Definition 1 *The Mabuchi metric on $\mathcal{H}$ is*

$$\langle f_1, f_2 \rangle_\phi = \int_M f_1 \, f_2 \, \frac{\omega_\phi^n}{n!}. \tag{7}$$

Theorem 2 (Donaldson) *The geodesics for the metric (7) are the stationary points of the energy functional*

$$E(\phi) = \int_0^1 \int_M \dot{\phi}_t^2 \, dt \, \frac{\omega_\phi^n}{n!},$$

which coincide with the solutions of (3) and therefore with the solutions of the HCMA satisfying the conditions (a) and (b).

Donaldson further shows in [Do1] that $\mathcal{H}_0$ with the Mabuchi metric is an infinite dimensional analogue of the symmetric spaces of non–compact type of the form

$$PSL(N, \mathbb{C})/PSU(N)\,,$$

with $PSL(N, \mathbb{C})$–invariant metric.

1. $\mathcal{H}_0$ as a quotient

Let

$$Ham_{\mathbb{C}}(M,\omega) := \left\{\varphi \in Diff(M) \,:\, \left(\varphi^{-1}\right)^{*}(\omega) \in \mathcal{H}_0\right\} \overset{\text{no subgroup}}{\subset} \text{Diff}(M), \tag{8}$$

we obtain, from Moser theorem,

$$\begin{aligned} Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega) &\cong \mathcal{H}_0 \\ \varphi &\mapsto \left(\varphi^{-1}\right)^{*}(\omega). \end{aligned}$$

2. $T_\omega \mathcal{H}_0 \cong C^\infty(M)/\mathbb{R}$

We have

$$\mathcal{L}_{JX_H^{\omega_\phi}}\left(\omega_\phi\right) = i\partial\bar{\partial}H,$$

3. Curvature formulas

Theorem 3 (Donaldson) *The curvature of the Mabuchi metric (7) and the sectional curvature read*

$$R_\phi(f_1, f_2)f_3 = -\frac{1}{4}\{\{f_1, f_2\}_\phi, f_3\}_\phi, \ K_\phi(f_1, f_2) = -\frac{1}{4}||\{f_1, f_2\}_\phi||^2_\phi.$$

for all $f_1, f_2, f_3 \in T_\phi \mathcal{H}_0$*, where*

$$T_\phi \mathcal{H}_0 = \left\{ f \in C^\infty(M) \, : \, \int_M f \frac{\omega_\phi}{n!} = 0 \right\} \cong Lie(Ham(M, \omega_\phi)).$$

The above expressions are in full agreement with the formulas for the curvature of the finite dimensional symmetric spaces $K_{\mathbb{C}}/K$,

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$

and

$$K(X,Y) = -\frac{1}{4}||[X,Y]||^2 .$$

for all $X, Y, Z \in T_0 K_{\mathbb{C}}/K \cong iLie(K) \cong Lie(K)$ and the Lie brackets are calculated in $Lie(K)$.

4. $\mathcal{H}_0 \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega) = \lim_{N\to\infty} PSL(N,\mathbb{C})/PSU(N)$

Let $L \to M$ be a very ample holomorphic line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$ and $\dim H^0(m, L^k) = d_k + 1$. Every ordered basis $\underline{s} = (s_0, \ldots, s_{k+1})$ defines an embedding $i_{\underline{s}} : M \to \mathbb{CP}^{d_k}$ and the $k-$th root of the pullback of the Fubini-Study hermitian structure defines an hermitian structure on $\mathcal{H}$,

$$FS_k(\underline{s}) = (i_{\underline{s}} h_{FS})^{1/k} = \frac{1}{(\sum_{j=0}^{d_k} |s_j(z)|^2)^{1/k}}$$

$$\mathcal{B}_k = \left\{ FS_k(\underline{s}) : \underline{s} \text{ a basis of } H^0(M, L^k) \right\} \cong GL(d_k+1)/U(d_k+1).$$

Every $h \in \mathcal{H}$ defines an inner product on $H^0(M, L^k)$

$$\langle s, \tilde{s} \rangle_k = \int_M (s, \tilde{s})_{h^k} \frac{\omega_h^m}{m!}$$

Let $\underline{s}(k)$ be an orthonormal basis for $\langle \cdot, \cdot \rangle_k$ and let

$$h(k) = FS_k(\underline{s}(k)) \in \mathcal{B}_k$$

Theorem 4 *(Tian, 1990)*

$$h = \lim_{k \to \infty} h(k) \, .$$

3. Explicit "rotation" of hamiltonian flows to imaginary time

The missing step to transform Donaldson formal solution of the Cauchy problem (3) for the HCMA given by (6) into an actual solution is the rotation (4)

$$\exp(sX_H^\omega) \rightsquigarrow \exp(\sqrt{-1}tX_H^\omega).$$

In the present section we will describe our solution to this problem obtained in [M-Nunes, IMNR2015]. One key technical tool to rotate the flow is the Gröbner theory of Lie series of vector fields (which is still very popular in numerical methods in astronomy – satelite motion, exoplanets, etc).

Theorem 5 *Let (M, J) be a compact complex manifold and $X \in \mathcal{X}(M)$ an analytic vector field. There exist local charts $((z_j), U)$ in neighourhoods of every point and $T > 0$ such that for all $\tau \in \mathbb{D}_T$ the functions*

$$z_j^\tau = e^{\tau X} z_j = u_j^\tau(x, y) + \sqrt{-1} v_j^\tau(x, y), \tag{9}$$

where $x_j = \Re(z_j), y_j = \Im(z_j), u_j^\tau(x, y) = \Re(z_j^\tau), v_j^\tau(x, y) = \Im(z_j^\tau)$, define on $V \subset U$ local J_τ–holomorphic charts for a unique complex structure J_τ and there exists a unique diffeomorphism $\varphi_\tau^{X,J}$ such that

$$\left(\varphi_\tau^{X,J}\right)^* J = J_\tau.$$

The complex time flow is then given explicitly locally by

$$\varphi_\tau^{X,J}(x, y) = (u^\tau(x, y), v^\tau(x, y)), \tag{10}$$

We see that, as expected, if $\tau = t \in \mathbb{R}$ the complex time flow is J–independent and coincides with the real time flow

$$\varphi_t^{X,J} = \varphi_t^X \, .$$

Theorem 6 (M-Nunes) *Consider the Cauchy problem for the HCMA (3) on $I \times M$ (where we are already supressing the angular coordinate of the first factor in $A \times M$). Then by replacing* exp$(\sqrt{-1} t X_H^\omega)$, *in the formal solution (6), by $\varphi_{it}^{X_H,J}$ obtained as in (10) one obtains a solution of (3).*

4. Infinite dimensional spaces of new solutions of the HCMA

4.1 Solutions with $\mathbb{R}^n$-invariant initial data, HRMA and Legendre transforms

Suppose that one can choose holomorphic charts with coordinates $z_j = x_j + iy_j$ such that the initial data in (3) do not depend on y

$$\begin{aligned} \phi_0(x,y) &= k(x,y) = \tilde{k}(x) \\ \dot{\phi}_0(x,y) &= H(x,y) = \tilde{H}(x). \end{aligned}$$

Particular cases of this situation are toric Kähler metrics on toric manifolds with also torus–invariant initial velocities $\dot{\phi}_0 = H$. Then the HCMA becomes the HRMA equation

$$\det_{n+1}\mathrm{Hess}_{\tilde{x}}(\tilde{\Phi}) = 0,$$

where $\tilde{x} = (t, x)$.

It is well known that the geodesic equation (3) in this case is linearized by the Legendre transform associated with $\tilde{\phi}$

$$\begin{aligned}
\tilde{\phi} &\mapsto g = \mathcal{L}(\phi) \\
u_j &= \frac{\partial \tilde{\phi}}{\partial x_j} \\
g(u) &= \sum_j x_j(u) u_j - \tilde{\phi}(x(u))
\end{aligned}$$

because in the space of the symplectic potentials g the geodesic equation becomes

$$\ddot{g}_t = 0,$$

and therefore the solutions are straight lines

$$g_t = \mathcal{L}(k) + t\widehat{H},$$

where $\widehat{H}(u) = \widetilde{H}(x(u))$.

In our formalism one obtains that

$$\omega = \sum_j du_j \wedge dy_j$$

and

$$z_j = x_j + iy_j = \frac{\partial g}{\partial u_j}(u) + iy_j.$$

Let now $H(x,y) = \tilde{H}(x) = \widehat{H}(u)$. Then

$$X_H^\omega = -\sum_j \frac{\partial \widehat{H}}{\partial u_j}\frac{\partial}{\partial y_j}$$

and therefore the real time flow reads

$$\varphi_t^{X_H^\omega}(u,y) = \left(u, y - t\frac{\partial \widehat{H}}{\partial u}\right).$$

Then, as in (9), to find the rotation of the flow to the imaginary axis we have to act with the real time flow on J–holomorphic coordinates and analytically continue t only for those functions. We have

$$z_j^{it} = \exp(itX_H^\omega)(z_j) = \frac{\partial g}{\partial u} + i\left(y - it\frac{\partial \widehat{H}}{\partial u}\right) = \frac{\partial}{\partial u}\left(g + t\widehat{H}\right) + iy.$$

Comparing with (10) we see that indeed g_t varies (affine) linearly with t and the rotation of the real time flow to the imaginary axis is the composition of two Legendre transforms

$$\varphi_{it}^{X_H^\omega}(x,y) = (\mathcal{L}_{g+t\widehat{H}} \circ \mathcal{L}_{\tilde{k}}(x), y).$$

4.2 New solutions of the HCMA on an elliptic curve

Let us now obtain an infinite dimensional family of nonsymmetric solutions of the HCMA on an elliptic curve $M = \mathcal{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with J^ϵ defined by the holomorphic coordinate $z = x + \epsilon \sin(x) + iy$, where $|\epsilon| < 1$ and (x, y) are the standard periodic coordinates on $\mathcal{T}^2$. We choose $\omega = dx \wedge dy$, which corresponds to choosing an initial Kähler potential $\phi_0 = k$. Let $\dot{\phi}_0(x, y) = H(y)$, a (periodic) function of y only.

Remark 7 The calculations remain simple if we consider tke more general initial Kähler structure

$$z = u(x, y) + iv(x, y)$$

but we keep H as a function of y (or x) alone. $\Diamond$

To solve the HCMA with the given initial conditions let us first find the real time hamiltonian flow of H. Since

$$X_H^\omega = H'(y)\,\frac{\partial}{\partial x},$$

we obtain

$$\varphi_t^{X_H^\omega}(x,y) = \big(x + tH'(y), y\big)\,.$$

To rotate the flow to the imaginary axis we find

$$z^{it} = \exp(itX_H^\omega)(z) = x+\epsilon\,\sin(x)\cosh(tH'(y))+i\left(y + tH'(y)\epsilon\,\cos(x)\sinh(tH'(y)\right) \tag{11}$$

We see that, as expected, though the evolution is linear in the geodesic (= imaginary hamiltonian) time t only in the symmetric (with respect to translations in x) case $\epsilon = 0$, the explicit expressions can be found also for $\epsilon \neq 0$ and for any function $H(y)$. From (10) and (11) we see that

$$\varphi_{it}^{X_H, J^\epsilon}(x,y) = \big(x + \epsilon\,\sin(x)\cosh(tH'(y), y + tH'(y) + \epsilon\,\cos(x)\sinh(tH'(y)\big)\,.$$

4.3 New solutions of the HCMA on $\mathbb{CP}^1$

Thank you!