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Coherent State Transforms for Spaces of Connections

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The Segal–Bargmann transform plays an important role in quantum theories of linear fields. Recently, Hall obtained a non-linear analog of this transform for quantum mechanics on Lie groups. Given a compact, connected Lie group G with its normalized Haar measure μ_H, the Hall transform is an isometric isomorphism from $L^2(G, \mu_H)$ to $\mathscr{H}(G^{\mathbb{C}}) \cap L^2(G^{\mathbb{C}}, \nu)$, where $G^{\mathbb{C}}$ the complexification of G, $\mathscr{H}(G^{\mathbb{C}})$ the space of holomorphic functions on $G^{\mathbb{C}}$, and ν an appropriate heat-kernel measure on $G^{\mathbb{C}}$. We extend the Hall transform to the infinite dimensional context of non-Abelian gauge theories by replacing the Lie group G by (a certain extension of) the space $\mathscr{A}/\mathscr{G}$ of connections modulo gauge transformations. The resulting "coherent state transform" provides a holomorphic representation of the holonomy C^* algebra of real gauge fields. This representation is expected to play a key role in a non-perturbative, canonical approach to quantum gravity in 4 dimensions.

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1. Introduction

In the early sixties, Segal [1, 2] and Bargmann [3] introduced an integral transform that led to a holomorphic representation of quantum states of linear, Hermitian, Bose fields. (For a review of the holomorphic —or, coherent-state—representation, see Klauder [4].) The purpose of this paper is to extend that construction to non-Abelian gauge fields and, in particular, to general relativity. The key idea is to combine two ingredients: (i) A non-linear analog of the Segal-Bargmann transform due to Hall [5] for a system whose configuration space is a compact, connected Lie group; and, (ii) A calculus on the space of connections modulo gauge transformations based on projective techniques [6-15].

Let us begin with a brief summary of the overall situation. Recall first that, in theories of connections, the classical configuration space is given by $\mathscr{A}/\mathscr{G}$, where $\mathscr{A}$ is the space of connections on a principal fibre bundle $P(\Sigma, G)$ over a ("spatial") manifold Σ, and $\mathscr{G}$ is the group of vertical automorphisms of P. In this paper, we will assume that Σ is an analytic n-manifold, G is a compact, connected Lie group, and elements of $\mathscr{A}$ and $\mathscr{G}$ are all smooth. In field theory the quantum configuration space is, generically, a suitable completion of the classical one. A candidate, $\overline{\mathscr{A}/\mathscr{G}}$, for such a completion of $\mathscr{A}/\mathscr{G}$ was recently introduced [6]. This space will play an important role throughout our discussion. It first arose as the Gel'fand spectrum of a C^* algebra constructed from the so-called Wilson loop functions, the traces of holonomies of smooth connections around (piecewise analytic) closed loops. It is therefore a compact, Hausdorff space. However, it was subsequently shown [10, 14] that, using a suitable projective family, $\overline{\mathscr{A}/\mathscr{G}}$ can also be obtained as the projective limit of topological spaces G^n/Ad, the quotient of G^n by the adjoint action of G. Here, we will work with this characterization of $\overline{\mathscr{A}/\mathscr{G}}$.

It turns out that $\overline{\mathscr{A}/\mathscr{G}}$ is a very large space: there is a precise sense in which it can be regarded as the "universal home" for measures[1] that define quantum quange theories in which the Wilson loop operators are well-defined [12]. However, it is small enough to admit various notions from differential geometry such as forms, vector fields, Laplacians and heat kernels [13]. In Yang-Mills theories, one expects the physically relevant measures to have support on a "small" subspace of $\overline{\mathscr{A}/\mathscr{G}}$. The structure of quantum general relativity, on the other hand, is quite different. In the canonical approach, each quantum state arises as a measure and there are strong indications that measures with support on all of $\overline{\mathscr{A}/\mathscr{G}}$ will be physically significant [16].

Now, as in linear theories [1], for non-Abelian gauge fields, it is natural to first construct a "Schrödinger-type" representation in which the Hilbert space of states arises as $L^2(\overline{\mathscr{A}/\mathscr{G}}, \mu)$ for a suitable measure μ on $\overline{\mathscr{A}/\mathscr{G}}$. This will be our point of departure. The projective techniques referred to above enable us to define measures as well as integrals over $\overline{\mathscr{A}/\mathscr{G}}$ as projective limits of measures and integrals over G^n/Ad. We would, however, like to construct a "holomorphic representation". Thus, we need to complexity $\overline{\mathscr{A}/\mathscr{G}}$, consider holomorphic functions thereon and introduce suitable measures to integrate these functions. It is here that we use the techniques introduced by Hall [5]. Given any compact Lie group G, Hall considers its complexification $G^{\mathbb{C}}$, defines holomorphic functions on $G^{\mathbb{C}}$, and, using heat-kernel methods, introduces measures ν with appropriate fall-offs (for the scalar products between holomorpic functions to be well-defined). Finally, he provides a transform C_ν, from $L^2(G, \mu_H)$ to the space of ν-square-integrable holomorphic functions over $G^{\mathbb{C}}$. Since Hall's transform is of a geometric rather than algebraic or representation-theoretic nature, it can be readily combined with the projective techniques. Using it, we will construct the appropriate Hilbert spaces of holomorphic functions on $\overline{\mathscr{A}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}}$—an appropriate complexification of $\overline{\mathscr{A}/\mathscr{G}}$—and obtain isometric isomorphisms between this space and $L^2(\overline{\mathscr{A}/\mathscr{G}}, \mu)$. For gauge theories—such as the 2-dimensional Yang-Mills theory—our results provide a new, coherent state representation of quantum states which is well suited to analyze a number of issues.

The main motivation for this analysis comes, however, from quantum general relativity: the holomorphic representation serves as a key step in the canonical approach to quantum gravity. Let us make a brief detour to explain this point. The canonical quantization program for general relativity was initiated by P. A. M. Dirac and P. Bergmann already in the

[1] While we will be mostly concerned here with Hilbert spaces of quantum states, the space $\overline{\mathscr{A}/\mathscr{G}}$ is also useful in the Euclidean approach to quantum gauge theories. In particular, the 2-dimensional Yang-Mills theory can be constructed on $\mathscr{R}^2$ or on $S^1 \times \mathscr{R}$ by defining the appropriate measure on $\overline{\mathscr{A}/\mathscr{G}}$ [15].

late fifties, and developed further, over the next two decades, by a number of researchers including R. Arnowitt, S. Deser, C. W. Misner and J. A. Wheeler and his co-workers. The first step is a reformulation of general relativity as a Hamiltonian system. This was accomplished using 3-metrics as configuration variables rather early. While these variables are natural from a geometrical point of view, it turns out that they are not convenient for discussing the dynamics of the theory. In particular, the basic equations are *non-polynomial* in these variables. Therefore, a serious attempty at making mathematical sense of their quantum analogs has never been made and the work in this area has remained heuristic.

In the mid-eighties, however, it was realized [17] that a considerable simplification occurs if one uses self-dual connections as dynamical variables. In particular, the basic equations become low order polynomials. Furthermore, since the configuration variables are now connections, one can take over the sophisticated machinery that has been used to analyze gauge theories. Consequently, over the last few years, considerable progress could be made in this area. (For a review, see, e.g., [18]). However, in the Lorentzian signature, self-dual connections are complex and provide a *complex* coordinatization of the *phase space* of general relativity rather than a real coordinatization of its configuration space. Therefore, if one is to base one's quantum theory on these variables, it is clear heuristically that the quantum states must be represented by *holomorphic* functionals of self-dual connections. (Detailed considerations show that they should in fact be complex measures rather than functionals.) Given the situation in the classical theory, this is the representation in which one might expect the quantum dynamics to simplify considerably. Indeed, heuristic treatments have yielded a variety of results in support of this belief [19, 18]. Furthermore, they have brought out a potentially deep connection between knot theory and quantum gravity [20]. To make these results precise, one first needs to construct the holomorphic representation rigorously. The coherent state transform of this paper provides a solution to this problem. In particular, it has already led to a rigorous understanding of the relation between knots and states of quantum gravity [16, 21].

The paper is organized as follows. In Section 2, we recall the definition and properties of the Hall transform. Section 3 summarizes the relevant results from calculus on the space of connections. In particular, in Section 3, we will: (i) construct, using projective techniques, the spaces $\overline{\mathscr{A}}$ of generalized connections, $\overline{\mathscr{G}}$ of generalized automorphisms of P and their quotient $\overline{\mathscr{A}/\mathscr{G}}$ and complexifications $\overline{\mathscr{A}^{\mathbb{C}}}$ and $\overline{\mathscr{G}^{\mathbb{C}}}$; (ii) see that the space $\overline{\mathscr{A}}$ is equipped with a natural measure μ_0 which is faithful and invariant under the induced action of the diffeomorphism group of the underlying manifold Σ; and (iii) show that it also admits a family of diffeomorphism invariant measures $\mu^{(m)}$, introduced by Baez. All these measures project down unambiguously to $\overline{\mathscr{A}/\mathscr{G}}$. Section 4 contains a precise formulation of the main problem of this paper and summary of our strategy. In Section 5 using heat kernel methods, we construct a family of (cylindrical) measures v_t^l on $\overline{\mathscr{A}^{\mathbb{C}}}$, and a family of transforms Z_t^l from $L^2(\mathscr{A}, \mu_0)$ to (the Cauchy completion of) the intersection $\mathscr{H}_{\mathscr{C}} \cap L^2(\overline{\mathscr{A}^{\mathbb{C}}}, v_t^l)$ of the space of cylindrical holomorphic functions on $\overline{\mathscr{A}^{\mathbb{C}}}$ with the space of v-square integrable functions. These transforms provide isometric isomorphisms between the two spaces. Furthermore, the transforms are gauge-covariant so that they map $\overline{\mathscr{G}}$-invariant functions on $\mathscr{A}$ to $\overline{\mathscr{G}^{\mathbb{C}}}$ invariant functions on $\overline{\mathscr{A}^{\mathbb{C}}}$. However, these transforms are not diffeomorphism covariant: Although the measure μ_0 on $\bar{\mathscr{A}}$ is diffeomorphism invariant, to define the corresponding heat kernel one is forced to introduce an additional structure which fails to be diffeomorphism invariant [13]. The Baez measures $\mu^{(m)}$, on the other hand, are free of this difficulty. That is, using $\mu^{(m)}$ in place of μ_0, one can obtain coherent state transforms which are both gauge *and* diffeomorphism covariant. This is the main result of Section 6. The Appendix provides the explicit expression of one of these transforms for the case when the gauge group is Abelian.

2. Hall Transform for Compact Groups G

In this section we recall from [5] those aspects of the Hall transform which will be needed in our main analysis. Let $G^{\mathbb{C}}$ be the complexification of G in the sense of [22] and v be a bi-G-invariant measure on $G^{\mathbb{C}}$ that falls off rapidly at infinity (see (2) below). The Hall transform C_v is an isometric isomorphism from $L^2(G, \mu_H)$, where μ_H denotes the normalized Haar measure on G, onto the space of v-square integrable holomorphic functions on $G^{\mathbb{C}}$

$$C_v \colon L^2(G, \mu_H) \to \mathscr{H}(G^{\mathbb{C}}) \cap L^2(G^{\mathbb{C}}, v(g^{\mathbb{C}})). \tag{1}$$

Such a transform exists whenever the Radon-Nikodym derivative $dv/d\mu_H^{\mathbb{C}}$ exists, is locally bounded away from zero, and falls off at infinity in such a way that the integral

$$\sigma_\pi^v = \frac{1}{dim\ V_\pi} \int_{G^{\mathbb{C}}} \|\pi(g^{\mathbb{C}^{-1}})\|^2\, dv(g^{\mathbb{C}}) \tag{2}$$

is finite for all π. Here, $\mu_H^{\mathbb{C}}$ is the Haar measure on $G^{\mathbb{C}}$, π denotes (one representative of) an isomorphism class of irreducible representations of G on the complex linear spaces V_π, and all, $\|A\| = \sqrt{Tr(A^\dagger A)}$ for $A \in End\ V_\pi$ and $A^\dagger$ the adjoint of A with respect to a G-invariant inner product on V_π. For a v satisfying (2), the Hall transform is given by

$$[C_\nu(f)](g^{\mathbb{C}}) = (f \star \rho_\nu)(g^{\mathbb{C}}) = \int_G f(g)\, \rho_\nu(g^{-1}g^{\mathbb{C}})\, d\mu_H(g), \tag{3}$$

where $\rho_\nu(g^{\mathbb{C}})$ is the kernel of the transform given in terms of ν by

$$\rho_\nu(g^{\mathbb{C}}) = \sum_\pi \frac{\dim V_\pi}{\sqrt{\sigma_\pi^\nu}}\, Tr(\pi(g^{\mathbb{C}^{-1}})). \tag{4}$$

The transform C_ν takes a particularly simple form for the (real analytic) functions $k_{\pi, A}$ on G corresponding to matrix elements of $\pi(g)$,

$$k_{\pi, A}(g) = Tr(\pi(g)A).$$

This is significant because, according to the Peter-Weyl Theorem the matrix elements $k_{\pi, A}$, for all π and all $A \in End\, V_\pi$, span a dense subspace in $L^2(G, d\mu_H)$. The image of these functions $k_{\pi, A}$ under the transform is (see [5])

$$\begin{aligned}[C_\nu(k_{\pi, A})](g^{\mathbb{C}}) &= [k_{\pi, A} \star \rho_\nu](g^{\mathbb{C}}) \\ &= \frac{1}{\sqrt{\sigma_\pi^\nu}}\, k_{\pi, A}(g^{\mathbb{C}}).\end{aligned} \tag{5}$$

The evaluation of the Hall transform of a generic function f, $f \in L^2(G, d\mu_H)$, can be naturally divided into two steps. In the first, one obtains a real analytic function on the original group G,

$$f \mapsto f \star \rho_\nu.$$

In the second step the function $f \star \rho_\nu$ is analytically continued to $G^{\mathbb{C}}$. It follows from (4) that

$$f \star \rho_\nu = \rho_\nu \star f. \tag{6}$$

A natural choice for the measure ν on $G^{\mathbb{C}}$ is the "averaged" heat kernel measure ν_t [5]. This measure is defined by

$$d\nu_t(g^{\mathbb{C}}) = \left[\int_G \mu_t^{\mathbb{C}}(gg^{\mathbb{C}})\, d\mu_H(g)\right] d\mu_H^{\mathbb{C}}(g^{\mathbb{C}}), \tag{7}$$

where $\mu_t^{\mathbb{C}}$ is the heat kernel on $G^{\mathbb{C}}$, i.e., the solution to the equations

$$\begin{aligned}\frac{\partial}{\partial t}\mu_t^{\mathbb{C}} &= \frac{1}{4}\Delta_{G^{\mathbb{C}}}\mu_t^{\mathbb{C}} \\ \mu_0^{\mathbb{C}}(g^{\mathbb{C}}) &= \delta(g^{\mathbb{C}}, 1_{G^{\mathbb{C}}}).\end{aligned} \tag{8}$$

Here the Laplacian $\Delta_{G^{\mathbb{C}}}$ is defined by a left $G^{\mathbb{C}}$-invariant, bi-G-invariant metric on $G^{\mathbb{C}}$, $1_{G^{\mathbb{C}}}$ denotes the identity of the group $G^{\mathbb{C}}$, and δ is the delta function corresponding to the measure $\mu_H^{\mathbb{C}}$. If we take for ν the averaged heat kernel measure ν_t then in (2) we have

$$\sigma_\pi^{\nu_t} = e^{t\delta_\pi}, \tag{9}$$

here δ_π denotes the eigenvalue of the Laplacian Δ_G on G corresponding to the eigenfunction $k_{\pi, A}$. Notice that Δ_G gives the representation on $L^2(G, d\mu_H)$ of a (unique up to a multiplicative constant if G is simple) quadratic Casimir element. The result (9) follows from (4) and the fact that the kernel $\rho_{\nu_t} \equiv \rho_t$ of the transform $C_{\nu_t} \equiv C_t$ is the (analytic extension of) the fundamental solution of the heat equation on G:

$$\frac{\partial}{\partial t} \rho_t = \frac{1}{2} \Delta_G \rho_t. \tag{10}$$

Therefore, in this case one obtains

$$\rho_t(g^{\mathbb{C}}) = \sum_\pi dim\ V_\pi e^{-t\delta_\pi/2}\ Tr(\pi(g^{\mathbb{C}^{-1}})). \tag{11}$$

These results will be used in Sections 4 and 5 to define infinite dimensional generalizations of the Hall transform.

3. Measures on Spaces of Connections

In this section, we will summarize the construction of certain spaces of generalized connections and indicate how one can introduce interesting measures on them. Since the reader may not be familiar with any of these results, we will begin with a chronological sketch of the development of these ideas.

Recall that, in field theories of connections, a basic object is the space $\mathscr{A}$ of smooth connections on a given smooth principal fibre bundle $P(\Sigma, G)$. (We will assume the base manifold Σ to be analytic and G to be a compact, connected Lie group.) The classical configuration space is then the space $\mathscr{A}/\mathscr{G}$ of orbits in $\mathscr{A}$ generated by the action of the group $\mathscr{G}$ of smooth vertical automorphisms of P. In quantum mechanics, the domain space of quantum states coincides with the classical configuration space. In quantum field theories, on the other hand, the domain spaces are typically larger; indeed the classical configuration spaces generally form a set of zero measure. In gauge theories, therefore, one is led to the problem of finding suitable extensions of $\mathscr{A}/\mathscr{G}$. The problem is somewhat involved because $\mathscr{A}/\mathscr{G}$ is a rather complicated, *non-linear* space.

One avenue [6] towards the resolution of this problem is offered by the Gel'fand-Naimark theory of commutative C^*-algebras. Since traces of holonomies of connections around closed loops are gauge invariant, one can use them to construct a certain Abelian C^*-algebra with identity, called the *holonomy algebra*. Elements of this algebra separate points of $\mathscr{A}/\mathscr{G}$, whence, $\mathscr{A}/\mathscr{G}$ is densely embedded in the spectrum of the algebra. The spectrum is therefore denoted by $\overline{\mathscr{A}/\mathscr{G}}$. This extension of $\mathscr{A}/\mathscr{G}$ can be taken to be the domain space of quantum states. Indeed, in every cyclic representation of the holonomy algebra, states can be identified as elements of $L^2(\overline{\mathscr{A}/\mathscr{G}}, \mu)$ for some regular Borel measure μ on $\overline{\mathscr{A}/\mathscr{G}}$.

One can characterize the space $\overline{\mathscr{A}/\mathscr{G}}$ purely algebraically [6, 7] as the space of all homomorphisms from a certain group (formed out of piecewise analytic, based loops in Σ) to the structure group G. Another—and, for the present paper more convenient—characterization can be given using certain projective limit techniques [10, 14]: $\overline{\mathscr{A}/\mathscr{G}}$ with the Gel'fand topology is homeomorphic to the projective limit, with Tychonov topology, of an appropriate projective family of finite dimensional compact spaces. This result simplifies the analysis of the structure of $\overline{\mathscr{A}/\mathscr{G}}$ considerably. Furthermore, it provides an extension of $\mathscr{A}/\mathscr{G}$ also in the case when the structure group G is *non-compact*. Projective techniques were first used in [10, 14] for measure-theoretic purposes and then extended in [13] to introduce "differential geometry" on $\overline{\mathscr{A}/\mathscr{G}}$.

The first example of a non-trivial measure on $\overline{\mathscr{A}/\mathscr{G}}$ was constructed in [7] using the Haar measure on the structure group G. This is a natural measure in that it does not require any additional input; it is also faithful and invariant under the induced action of the diffeomorphism group of Σ. Baez [8] then proved that every measure on $\overline{\mathscr{A}/\mathscr{G}}$ is given by a suitably consistent family of measures on the projective family. He also replaced the projective family labeled by loops on Σ [10, 14] by a family labeled by graphs (see also [9, 11]) and introduced a family of measures which depend on characteristics of vertices. Finally, he provided a diffeomorphism invariant construction which, given a family of preferred vertices and almost any measure on G, produces a diffeomorphism invariant measure on $\overline{\mathscr{A}/\mathscr{G}}$.

We will now provide the relevant details of these constructions. Our treatment will, however, differ slightly from that of the papers cited above.

3.1. *Spaces $\overline{\mathscr{A}}$, $\overline{\mathscr{G}}$ and $\overline{\mathscr{A}/\mathscr{G}}$*

Let Σ be a connected analytic n-manifold and G be a compact, connected Lie group. Consider the set $\mathscr{E}$ of all oriented, unparametrized, embedded, analytic intervals (edges) in Σ. We introduce the space $\overline{\mathscr{A}}$ of (generalized) connections on Σ as the space of all maps $\bar{A}\colon \mathscr{E} \to G$, such that

$$\bar{A}(e^{-1}) = [\bar{A}(e)]^{-1} \qquad \text{and} \qquad \bar{A}(e_2 \circ e_1) = \bar{A}(e_2)\, \bar{A}(e_1) \tag{12}$$

whenever two edges $e_2, e_1 \in \mathscr{E}$ meet to form an edge. Here, $e_2 \circ e_1$ denotes the standard path product and e^{-1} denotes e with opposite orientation.

The group $\overline{\mathscr{G}}$ of (generalized) gauge transformations acting on $\overline{\mathscr{A}}$ is the space of maps $\bar{g}: \Sigma \to G$ or equivalently the Cartesian product group

$$\overline{\mathscr{G}} := \mathop{\times}_{x \in \Sigma} G. \tag{13}$$

A gauge transformation $\bar{g} \in \overline{\mathscr{G}}$ acts on $\bar{A} \in \overline{\mathscr{A}}$ through

$$[\bar{g}(\bar{A})](e_{p_1, p_2}) = \bar{g}_{p_1} \bar{A}(e_{p_1, p_2})(\bar{g}_{p_2})^{-1}, \tag{14}$$

where e_{p_1, p_2} is an edge from $p_1 \in \Sigma$ to $p_2 \in \Sigma$ and $\bar{g}_{p_i}$ is the group element assigned to p_i by $\bar{g}$. The spece $\overline{\mathscr{G}}$ equipped with the product topology is a compact topological group. Note also that $\overline{\mathscr{A}}$ is a closed subset of

$$\mathscr{A} \subset \mathop{\times}_{e \in \mathscr{E}} \mathscr{A}_e, \tag{15}$$

where the space $\mathscr{A}_e$ of all maps from the one point set $\{e\}$ to G is homeomorphic to G. $\overline{\mathscr{A}}$ is then compact in the topology induced from this product.

It turns out that the space $\overline{\mathscr{A}}$ (and also $\overline{\mathscr{G}}$) can be regarded as the projective limit of a family labeled by graphs in Σ in which each member is homeomorphic to a finite product of copies of G [10, 14]. Since this fact will be important for describing measures on $\overline{\mathscr{A}}$ and for constructing the integral transforms we will now recall this construction briefly. Let us first define what we mean by graphs.

Definition 1. *A graph on Σ is a finite subset $\gamma \in \mathscr{E}$ such that* (i) *two different edges, $e_1, e_2 : e_1 \neq e_2$ and $e_1 \neq e_2^{-1}$, of γ meet, if at all, only at one or both ends and* (ii) *if $e \in \gamma$ then $e^{-1} \in \gamma$.*

The set of all graphs in Σ will be denoted by $\mathrm{Gra}(\Sigma)$. In $\mathrm{Gra}(\Sigma)$ there is a natural relation of partial ordering $\geqslant$,

$$\gamma' \geqslant \gamma \tag{16}$$

whenever every edge of γ is a path product of edges associated with γ'. Furthermore, for any two graphs γ_1 and γ_2, there exists a γ such that $\gamma \geqslant \gamma_1$ and $\gamma \geqslant \gamma_2$, so that $(\mathrm{Gra}(\Sigma), \geqslant)$ is a directed set.

Given a graph γ, let $\mathscr{A}_\gamma$ be the associated space of assignments ($\mathscr{A}_\gamma = \{A_\gamma \mid A_\gamma : \gamma \to G\}$) of group elements to edges of γ, satisfying $A_\gamma(e^{-1}) = A_\gamma(e)^{-1}$ and $A_\gamma(e_1 \circ e_2) = A_\gamma(e_1)\, A_\gamma(e_2)$, and let $p_\gamma: \overline{\mathscr{A}} \to \mathscr{A}_\gamma$ be the projection which restricts $\bar{A} \in \overline{\mathscr{A}}$ to γ. Notice that p_γ is a surjective map. For every ordered pair of graphs, $\gamma' \geqslant \gamma$, there is a naturally defined map.

$$p_{\gamma\gamma'}: \mathscr{A}_{\gamma'} \to \mathscr{A}_\gamma, \qquad \text{such that} \quad p_\gamma = p_{\gamma\gamma'} \circ p_{\gamma'}. \tag{17}$$

With the same graph γ, we also associate a group $\mathscr{G}_\gamma$ defined by

$$\mathscr{G}_\gamma := \{g_\gamma \mid g_\gamma : V_\gamma \to G\} \tag{18}$$

where V_γ is the set of *vertices* of γ; that is, the set V_γ of points lying at the end of edges of γ. There is a natural projection $\overline{\mathscr{G}} \to \mathscr{G}_\gamma$ which will also be denoted by p_γ and is again given by restriction (from Σ to V_γ). As before, for $\gamma' \geqslant \gamma$, p_γ factors into $p_\gamma = p_{\gamma\gamma'} \circ p_{\gamma'}$ to define

$$p_{\gamma\gamma'}: \mathscr{G}_{\gamma'} \to \mathscr{G}_\gamma. \tag{19}$$

Note that the group $\mathscr{G}_\gamma$ acts naturally on $\mathscr{A}_\gamma$ and that this action is equivariant with respect to the action of $\overline{\mathscr{G}}$ on $\overline{\mathscr{A}}$ and the projection p_γ. Hence, each of the maps $p_{\gamma\gamma'}$ projects to new maps also denoted by

$$p_{\gamma\gamma'}: \mathscr{A}_{\gamma'}/\mathscr{G}_{\gamma'} \to \mathscr{A}_\gamma/\mathscr{G}_\gamma. \tag{20}$$

We collect the spaces and projections defined above into a (triple) projective family $(\mathscr{A}_\gamma, \mathscr{G}_\gamma, \mathscr{A}_\gamma/\mathscr{G}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$. It is not hard to see that $\overline{\mathscr{A}}$ and $\overline{\mathscr{G}}$ as introduced above are just the projective limits is the projective limit of the compact quotients [10, 14],

$$\overline{\mathscr{A}}/\overline{\mathscr{G}} = \overline{\mathscr{A}/\mathscr{G}}. \tag{21}$$

Note however that the projections $p_{\gamma\gamma'}$ in (17), (19) and (20) are different from each other and that the same symbol $p_{\gamma\gamma'}$ is used only for notational simplicity; the context should suffice to remove the ambiguity. In particular, the properties of $p_{\gamma\gamma'}$ in (19) allow us to introduce a group structure in the projective limit $\overline{\mathscr{G}}$ of $(\mathscr{G}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$ while the same is not possible for the projective limits $\overline{\mathscr{A}}$ and $\overline{\mathscr{A}/\mathscr{G}}$ of $(\mathscr{A}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$ and $(\mathscr{A}_\gamma/\mathscr{G}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$ respectively.

The ★-algebra of *cylindrical functions* on $\overline{\mathscr{A}}$ is defined to be the following subalgebra of continuous functions

$$\mathrm{Cyl}(\overline{\mathscr{A}}) = \bigcup_{\gamma \in \mathrm{Gra}(\Sigma)} (p_\gamma)^* \, C(\mathscr{A}_\gamma). \tag{22}$$

$\mathrm{Cyl}(\overline{\mathscr{A}})$ is dense in the C^*-algebra of all continuous functions on $\overline{\mathscr{A}}$. The ★-algebra $\mathrm{Cyl}(\overline{\mathscr{A}/\mathscr{G}})$ of cylindrical functions on $\overline{\mathscr{A}/\mathscr{G}}$ coincides with the subalgebra of $\overline{\mathscr{G}}$-invariant elements of $\mathrm{Cyl}(\overline{\mathscr{A}})$.

Finally, let us turn to the analytic extensions. Since the projections $p_{\gamma\gamma'}$ (in (17) and (19)) are analytic, the complexification $G^{\mathbb{C}}$ of the gauge group G leads to the complexified projective family $(\mathscr{A}^{\mathbb{C}}_\gamma, \mathscr{G}^{\mathbb{C}}_\gamma, p^{\mathbb{C}}_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$. Note that the projections $p^{\mathbb{C}}_\gamma: \overline{\mathscr{A}^{\mathbb{C}}} \to \mathscr{A}^{\mathbb{C}}_\gamma$ maintain surjectivity. The projective limits $\overline{\mathscr{A}^{\mathbb{C}}}$ and $\overline{\mathscr{G}^{\mathbb{C}}}$ are characterized as in (12) and (13) with the group G replaced by $G^{\mathbb{C}}$. Since $G^{\mathbb{C}}$ is non-compact, so will be the spaces $\overline{\mathscr{A}^{\mathbb{C}}}$ and $\overline{\mathscr{G}^{\mathbb{C}}}$. The algebra of cylindrical functions is defined as above with $\mathscr{A}^{\mathbb{C}}_\gamma$ substituted for $\mathscr{A}_\gamma$. However these functions may now be unbounded and $C(\overline{\mathscr{A}^{\mathbb{C}}})$ is not a C^* algebra.

There is a natural notion of an analytic cylindrical function on $\bar{\mathscr{A}}$ and a holomorphic cylindrical function on $\overline{\mathscr{A}^{\mathbb{C}}}$:

DEFINITION 2. *A cylindrical function $f = f_\gamma \bullet p_\gamma$ ($f^{\mathbb{C}} = f^{\mathbb{C}}_\gamma \bullet p^{\mathbb{C}}_\gamma$) defined on $\bar{\mathscr{A}}$ ($\overline{\mathscr{A}^{\mathbb{C}}}$ is real analytic (holomorphic) if f_γ ($f^{\mathbb{C}}_\gamma$) is real analytic (holomorphic).*

In the complexified case the formula $\overline{\mathscr{A}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}} = \overline{\mathscr{A}^{\mathbb{C}}}/\overline{\mathscr{G}^{\mathbb{C}}}$ has not (to the authors' knowledge) been verified, but the natural isomorphism between $\mathrm{Cyl}(\overline{\mathscr{A}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}})$ and the algebra of all the $\overline{\mathscr{G}^{\mathbb{C}}}$ invariant elements of $\mathrm{Cyl}(\overline{\mathscr{A}^{\mathbb{C}}})$ continues to exist. We shall extend it to define cylindrical holomorphic (analytic) functions on $\overline{\mathscr{A}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}}$ ($\overline{\mathscr{A}/\mathscr{G}}$) to be all the $\overline{\mathscr{G}^{\mathbb{C}}}$ ($\bar{\mathscr{G}}$)-invariant cylindrical holomorphic (analytic) functions on $\overline{\mathscr{A}^{\mathbb{C}}}$ ($\bar{\mathscr{A}}$).

3.2. *Measures on $\bar{\mathscr{A}}$*

We will now apply to $\bar{\mathscr{A}}$ the standard method of constructing measures on projective limit spaces using consistent families of measures (see e.g. [23]).

Let us consider the projective family

$$(\mathscr{A}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)} \tag{23}$$

discussed in the last section and let

$$(\mathscr{A}_\gamma, \mu_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)} \tag{24}$$

be a projective family of measure spaces associated with (23); i.e., such that the measures μ_γ are (signed) Borel measures on $\mathscr{A}_\gamma$ and satisfy the consistency conditions

$$(p_{\gamma\gamma'})_* \mu_{\gamma'} = \mu_\gamma \qquad \textit{for} \quad \gamma' \geqslant \gamma. \tag{25}$$

Every projective family of measure spaces defines a cylindrical measure. To see this, recall first that a set C_B in $\bar{\mathscr{A}}$ is called a cylinder set with base $B \subset \mathscr{A}_\gamma$ if

$$C_B = p_\gamma^{-1}(B), \tag{26}$$

where B is a Borel set in $\mathscr{A}_\gamma$. Hence, given a projective family μ_γ of measures, we can define a cylindrical measure μ on $(\bar{\mathscr{A}}, \mathscr{C}_{\bar{\mathscr{A}}})$, through

$$\mu : p_{\gamma *} \mu = \mu_\gamma, \tag{27}$$

where $\mathscr{C}_{\bar{\mathscr{A}}}$ denotes the algebra of cylinder sets on $\bar{\mathscr{A}}$. For a consistent family of measures $\mu = (\mu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ to define a cylindrical measure μ that is extendible to a regular (σ-additive) Borel measure on the Borel σ-algebra $\mathscr{B} \supset \mathscr{C}_{\bar{\mathscr{A}}}$ of $\bar{\mathscr{A}}$ it is necessary and sufficient that the functional

$$f \mapsto \int d\mu\, f, \qquad f \in \mathrm{Cyl}(\bar{\mathscr{A}}), \tag{28}$$

be bounded. This integral is bounded if and only if the family of measures $(\mu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ is uniformly bounded [8]; i.e., if and only if μ_γ considered as linear functionals on $C(\mathcal{A}_\gamma)$ satisfy

$$\|\mu_\gamma\| \leqslant M \tag{29}$$

for some $M > 0$ independent of γ. (If all the measures μ_γ are positive then (29) automatically holds [7, 8]).

From now on, all measures μ on $\bar{\mathcal{A}}$ will be assumed to be regular Borel measures unless otherwise stated. It follows from Section 3.1 that every such measure μ on $\bar{\mathcal{A}}$ induces a (regular Borel) measure μ' on $\overline{\mathcal{A}/\mathcal{G}}$

$$\mu' = \pi_* \mu, \tag{30}$$

where π denotes the canonical projection, $\pi\colon \bar{\mathcal{A}} \to \overline{\mathcal{A}/\mathcal{G}}$.

The C^ω-diffeomorphisms φ of Σ have a natural action on $\bar{\mathcal{A}}$ induced by their action on graphs. This defines an action on $C(\bar{\mathcal{A}})$ and on the space of measures on $\bar{\mathcal{A}}$ (equal to the topological dual $C'(\bar{\mathcal{A}})$ of $C(\bar{\mathcal{A}})$). Diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$ were studied in [6]–[8]. We will denote the group of C^ω-diffeomorphisms of Σ by $\mathrm{Diff}(\Sigma)$.

A natural solution of conditions (25) is the one obtained by taking μ_γ to be the pushward of the normalized Haar measure $\mu_H^{E_\gamma}$ on G^{E_γ} with respect to ψ_γ^{-1} where $\psi_\gamma\colon \mathcal{A}_\gamma \to G^{E_\gamma}$ is a diffeomorphism

$$\psi_\gamma\colon A_\gamma \mapsto (A_\gamma(e_1), ..., A_\gamma(e_{E_\gamma})) \tag{31}$$

and $\{e_1, ..., e_{E_\gamma}\}$ are edges of γ, such that if (and only if) $e \in \{e_j\}_{j=1}^{E_\gamma}$ then $e^{-1} \notin \{e_j\}_{j=1}^{E_\gamma}$ [7]. By choosing a different set $\{\tilde{e}_j\}_{j=1}^{E_\gamma}$ ($\tilde{e}_j = e_j^\varepsilon$, $\varepsilon = 1, -1$) we obtain a different diffeomorphism ψ'_γ. Notice, however, that μ_γ is well defined since the map $g \mapsto g^{-1}$ preserves the Haar measure μ_H of G. We will refer to the choice of this ψ_γ as a choice of orientation for the graph γ. The family of measures $(\mu_\gamma)_{\gamma \in \mathrm{Gra}}$ leads to the measure on $\overline{\mathcal{A}/\mathcal{G}}$ denoted in the literature by μ_0 and for which all edges are treated equivalently. We will use this measure in Section 5.

A method for finding new diffeomorphism invariant measures on $\bar{\mathcal{A}}$ —and therefore also on $\overline{\mathcal{A}/\mathcal{G}}$—was proposed by Baez in [8]. Since these measures will play an important role in our analysis, we now recall some aspects of this method.

Definition 3 (Baez [8]). A family $(\mu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ of measures on $\mathcal{A}_\gamma$ is called (diffeomorphism) covariant if, for every $\varphi \in \mathrm{Diff}(\Sigma)$ and γ, γ' such that $\varphi(\gamma) \leqslant \gamma'$, we have

$$(p_{\varphi(\gamma)\gamma'})_* \mu_{\gamma'} = \varphi_* \mu_\gamma. \tag{32}$$

As shown in [8] (Theorem 2), diffeomorphism invariant measures μ on $\bar{\mathscr{A}}$ are in 1-to-1 correspondence with uniformly bounded covariant families $(\mu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$. Note that a covariant family is automatically consistent; i.e., it satisfies (25).

Baez's strategy is to solve the covariance conditions by appropriately choosing measures m_v associated with different vertex types v. (Each vertex type is an equivalence class of vertices where two are equivalent if they are related by an analytic diffeomorphism of Σ.) The number n_v of edge ends incident at v is called the valence of the vertex. Thus, any edge with both ends at v is counted twice. For each vertex v, the measure m_v is a measure for n_v G-valued random variables $(g_{v1}, ..., g_{un_v})$, one for each of the n_v edge ends at v. When applied to the entire graph, this procedure assigns two random variables (g_{ea}, g_{eb}) to each of the E_γ edges $e \in \gamma$, where the variable g_{ea} (g_{eb}) corresponds to the vertex at the beginning (end) of the edge. We will find it convenient to alternately label the random variables by their association with vertices and their association with oriented edges and to denote the map induces by this relabelling as $r_\gamma : G^{2E_\gamma} \to G^{2E_\gamma}$. Given m_v for every vertex type v, we define μ_γ as (for a more detailed explanation see [8])

$$\int_{A_\gamma} f_\gamma(A_\gamma)\, d\mu_\gamma(A_\gamma) := \int_{G^{2E_\gamma}} (f_\gamma \circ \psi_\gamma^{-1} \circ \phi_\gamma) \prod_{v \in V_\gamma} dm_v(g_{v1}, ..., g_{vn_v}), \quad (33)$$

where ψ_γ is as in (31) and ϕ_γ: $G^{E_\gamma} \times G^{E_\gamma} \to G^{E_\gamma}$ is the map

$$\phi_\gamma : [(g_{1a}, ..., g_{E_\gamma a}), (g_{1b}, ..., g_{E_\gamma b})] \mapsto (g_{1a} g_{1b}^{-1}, ..., g_{E_\gamma a} g_{E_\gamma b}^{-1}). \quad (34)$$

We will refer to the associated family of measures $\prod_{v \in V_\gamma} dm_v(g_{v1}, ..., g_{vn_v})$ on G^{2E_γ} as $d\mu'_\gamma$. Notice that (33) is well defined because the map (with labelling given by the association of the random variables with the vertices (!))

$$\psi_\gamma^{-1} \circ \phi_\gamma \circ r_\gamma \colon G^{2E_\gamma} \to \mathscr{A}_\gamma \quad (35)$$

does not depend on the orientation chosen on the graph, even though ψ_γ, ϕ_γ and r_γ do.

The measure m_v has then to satisfy:

(i) If some diffeomorphism induces an inclusion i of v into the vertex w, then there is an associated projection π_i: $G^{n_w} \to G^{n_v}$ acting on the corresponding random variables. The measure m_v should coincide with the pushforward of m_w:

$$\pi_i^* m_w = m_v. \quad (36)$$

(ii) In order to consider embeddings of graphs

$$\gamma' \geqslant \varphi(\gamma),$$

for which several edges of γ' may join to form in a single edge of $\varphi(\gamma)$, Baez defines an *arc* to be a valence 2 vertex for which the two incident edges join at the arc to form an analytic edge. He then proposes the condition that for each valence-1 vertex v connected to an arc a by an edge e (for which the associated random variables $(g_{ve}, g_{ae}, g_{ae'})$ have the distribution $m_v \otimes m_a$), we have

$$p_{a*}(m_v \otimes m_a) = m_v, \tag{37}$$

where $p_a(g_{ve}, g_{ae}, g_{ae'}) = g_{ve}^{-1} g_{ae} g_{ae'}^{-1}$.

In [8] new solutions to conditions (36) and (37) were found that distinguish edges as follows. Let m be an arbitrary but fixed probability measure on G. If a pair of edges e and f meet at an arc a included in the vertex v, set the corresponding random variables equal:

$$g_{a1} = g_{a2}. \tag{38}$$

Otherwise the random variables g_{vi} are distributed according to the measure m. Thus,

$$m_v = \prod_{i=1}^{n_v} dm(g_{vi}) \prod_{j=1}^{A_v} \delta(g_{vj}, g_{v(n_v - j + 1)}), \tag{39}$$

where A_v denotes the number of arcs included in v and the edge ends have been labeled so that the arcs are associated with the random variable pairs $(g_{vi}, g_{v(n_v - i + 1)})$. The δ-functions in (39) correspond to the measure m. This procedure defines a measure $\mu^{(m)}$ on $\bar{\mathscr{A}}$ for each probability measure m on G and we will refer to such $\mu^{(m)}$ as the *Baez measures* on $\bar{\mathscr{A}}$. These measures distinguish various n-valent vertices v by the number of arcs they include. Additional diffeomorphism-invariant measures would be expected to distinguish vertices by using other diffeomorphism invariant characteristics.

Because $\overline{\mathscr{A}^{\mathbb{C}}}$ is not compact, it is more difficult to define σ-additive measures on this space than on $\bar{\mathscr{A}}$. Thus, we content ourselves with cylindrical measures μ on $(\overline{\mathscr{A}^{\mathbb{C}}}, \mathscr{C}_{\mathscr{A}^{\mathbb{C}}})$. Cylindrical measures $\mu^{\mathbb{C}}$ on $\overline{\mathscr{A}^{\mathbb{C}}}$ are in one-to-one correspondence with consistent families of measures $(\mu_\gamma)^{\mathbb{C}}_{\gamma \in \mathrm{Gra}(\Sigma)}$ exactly as in (27)

$$p^{\mathbb{C}}_{\gamma *} \mu^{\mathbb{C}} = \mu^{\mathbb{C}}. \tag{40}$$

The consistency conditions (25) and diffeomorphism covariance conditions (32)

$$(p_{\varphi(\gamma)\gamma'})_* \mu_{\gamma'} = \varphi_* \mu_\gamma. \tag{41}$$

also preserve their forms

$$(p^{\mathbb{C}}_{\gamma\gamma'})_* \mu^{\mathbb{C}}_{\gamma'} = \mu^{\mathbb{C}}_\gamma \qquad \textit{for} \quad \gamma' \geqslant \gamma \tag{42}$$

and

$$(p^{\mathbb{C}}_{\varphi(\gamma)\gamma'})_* \mu^{\mathbb{C}}_{\gamma'} = \varphi_* \mu^{\mathbb{C}}_\gamma \qquad \textit{for} \quad \gamma' \geqslant \varphi(\gamma), \tag{43}$$

respectively. Therefore, diffeomorphism invariant Baez measures $\mu^{(m)}$ can be constructed in the same way starting with an arbitrary probability measure $m^{\mathbb{C}}$ on $G^{\mathbb{C}}$. We will use these measures in Section 6.

Coherent State Transforms for Theories of Connections

The rest of the paper is devoted to the task of constructing coherent state transforms for functions defined on the projective limit $\bar{\mathscr{A}}$. The discussion contained in the last two sections makes our overall strategy clear: we shall attempt to "glue" coherent state transforms defined on the components $\mathscr{A}_\gamma$ of $\bar{\mathscr{A}}$ into a consistent family. However, since the measure-theoretic results are not as strong for a non-compact projective family, we must first state under what conditions a map

$$Z: L^2(\bar{\mathscr{A}}, d\mu) \to \mathscr{C}\{\mathscr{H}_{\mathscr{C}}(\overline{\mathscr{A}^{\mathbb{C}}} \cap L^2(\mathscr{A}^{\mathbb{C}}, d\nu)\} \tag{44}$$

is to be regarded as a coherent state transform. Here, $\mathscr{C}$ indicates completion with respect to the L^2 inner product and $\mathscr{H}_{\mathscr{C}}$ is the space of holomorphic cylindrical functions. The definition of the space $L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \nu)$ also requires some care as ν is not necessarily σ-additive.

We first introduce two definitions:

Definition 4. A transform (44) is $\bar{\mathscr{G}}$-covariant if it commutes with the action of $\bar{\mathscr{G}}$, that is, if

$$Z((L_{\bar{g}})^* (f)) = (L^{\mathbb{C}}_{\bar{g}})^* (Z(f)), \tag{45}$$

where $(A, \bar{g}) \mapsto L_{\bar{g}} \bar{A} := \overline{gA}$ stands for the action of $\bar{\mathscr{G}}$ on $\bar{\mathscr{A}}$ with the superscript $\mathbb{C}$ denoting the corresponding action on $\overline{\mathscr{A}^{\mathbb{C}}}$,

$$(L^{\mathbb{C}}_{\bar{g}} \overline{\mathscr{A}^{\mathbb{C}}})(e_{p_1, p_2}) = \bar{g}_{p_1} \overline{\mathscr{A}^{\mathbb{C}}}(e_{p_1, p_2})\, \bar{g}_{p_2}^{-1}, \tag{46}$$

and where *, as usual, denotes the pullback.

Note that in (45) and (46), we have used the inclusion of $\overline{\mathscr{G}}$ in $\overline{\mathscr{G}^{\mathbb{C}}}$.

DEFINITION 5. A family $(Z_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ of transforms $Z_\gamma\colon L^2(\mathscr{A}_\gamma, d\mu_\gamma) \to \mathscr{H}(\mathscr{A}_\gamma^{\mathbb{C}})$ is consistent if for every pair of ordered graphs, $\gamma' \geqslant \gamma$,

$$Z_{\gamma'}(f_\gamma \circ p_{\gamma\gamma'}) = Z_\gamma(f_\gamma) \circ p_{\gamma\gamma'}^{\mathbb{C}}. \tag{47}$$

Notice that the consistency condition is equivalent to requiring that

$$p_\gamma^* f_\gamma = p_{\gamma'}^* f_{\gamma'} \Rightarrow p_\gamma^{\mathbb{C}*} Z_\gamma(f_\gamma) = p_{\gamma'}^{\mathbb{C}*} Z_\gamma(f_{\gamma'}). \tag{48}$$

Definitions 4 and 5 allow us to use:

DEFINITION 6. For a measure[2] $\mu = (\mu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ on $\overline{\mathscr{A}}$ and a cylindrical measure $\nu = (\nu_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ on $\overline{\mathscr{A}^{\mathbb{C}}}$, a map (44) is a coherent transform on $\overline{\mathscr{A}}$ if there is a consistent family $(Z_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ of coherent transforms (see Section 2)

$$Z_\gamma\colon L^2(\mathscr{A}_\gamma, d\mu_\gamma) \to \mathscr{H}(\mathscr{A}_\gamma^{\mathbb{C}}) \cap L^2(\mathscr{A}_\gamma^{\mathbb{C}}, d\nu_\gamma) \tag{49}$$

such that, for every cylindrical function of the form $f = f_\gamma \circ p_\gamma$ with $f_\gamma \in L^2(\mathscr{A}_\gamma, d\mu_\gamma)$,

$$Z(f) = Z_\gamma(f_\gamma) \circ p_\gamma^{\mathbb{C}}. \tag{50}$$

When Z is an isometric coherent transform, it associates with every representation π of the holonomy algebra on $L_2(\overline{\mathscr{A}/\mathscr{G}}, \mu)$ a representation $\pi^{\mathbb{C}}$ on $L_2(\overline{\mathscr{A}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}}, \nu)$ by

$$\pi^{\mathbb{C}}(\alpha^{\mathbb{C}}) = Z\pi(\alpha) Z^{-1}, \tag{51}$$

where α is an arbitrary element of the holonomy algebra. Such $\pi^{\mathbb{C}}$ are the desired “holomorphic representations”.

Several important remarks concerning the properties of the analytic extensions are now in order. Suppose that we are given a family of transforms $(Z_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ as in Definition 5, but that equation (47) is only known to be satisfied when the functions are restricted to $\mathscr{A}_\gamma \subset \mathscr{A}_\gamma^{\mathbb{C}}$ (for every possible γ). Then, because both functions in (47) are holomorphic on $\mathscr{A}_\gamma^{\mathbb{C}}$, (47) holds on the entire $\mathscr{A}_\gamma^{\mathbb{C}}$.

In other words, in order to construct a family of transforms

$$Z_\gamma\colon L^2(\mathscr{A}_\gamma, d\mu_\gamma) \to \mathscr{H}(\mathscr{A}_\gamma^{\mathbb{C}}),$$

[2] Here we identify measures on $\overline{\mathscr{A}}$ and $\overline{\mathscr{A}^{\mathbb{C}}}$ with the corresponding consistent families of measures.

which is consistent in the sense of Definition 5, it is sufficient to find a family of maps $R_\gamma\colon L^2(\mathscr{A}_\gamma, d\mu_\gamma) \to \mathscr{H}(\mathscr{A}_\gamma)$ which satisfies (47) ($\mathscr{H}(\mathscr{A}_\gamma)$ denotes the space of real analytic functions on $\mathscr{A}_\gamma$). The analyticity of each function $R_\gamma(f_\gamma)$ guarantees the consistent holomorphic extension.

Let $R\colon L^2(\bar{\mathscr{A}}, d\mu) \to L^2(\bar{\mathscr{A}}, d\mu)$ be the transform defined by restricting $Z(f)$ to $\bar{\mathscr{A}} \subset \overline{\mathscr{A}^{\mathbb{C}}}$. Note that $\bar{\mathscr{G}}$ acts analytically on the components of the projective family. Thus, the image of the subspace of $\bar{\mathscr{G}}$-invariant functions, with respect to a coherent state transform on $\bar{\mathscr{A}}$, consists of $\overline{\mathscr{G}^{\mathbb{C}}}$-invariant functions on $\overline{\mathscr{A}^{\mathbb{C}}}$.

5. Gauge Covariant Coherent State Transforms

We now construct a family Z_t^l (parametrized by $t \in \mathbb{R}$ and a function l of edges) of gauge covariant isometric coherent state transforms when the measure μ on $\bar{\mathscr{A}}$ is taken to be the natural measure μ_0 (see Section 3.2). The corresponding $Z_{t,\gamma}^l$ will be coherent state transforms given by appropriately chosen heat kernels on $\mathscr{A}_\gamma \cong G^{E_\gamma}$. The measures ν_γ on the right hand side of (49) are averaged heat kernel measures on $(G^{\mathbb{C}})^{E_\gamma}$ (see Section 2).

The idea is to use a Laplace operator Δ^l on $\bar{\mathscr{A}}$ [13]. Our transform will then be defined through convolution with the fundamental solution of the corresponding heat equation.

The ingredients used to define the Laplacian are the following:

(i) a bi-invariant metric on G which defines the Laplace-Beltrami operator Δ;

(ii) a function l defined on the space $\mathscr{E}$ (see Subsection 3.1) of (analytic) edges in Σ, such that

$$l(e^{-1}) = l(e),\ l(e) \geqslant 0, \qquad l(e_2 \circ e_1) = l(e_2) + l(e_1), \tag{52}$$

whenever $e_2 \circ e_1$ exists an belongs to $\mathscr{E}$ and the intersection of e_1 with e_2 is a single point.

Elementary examples of functions l satisfying (52) are given by: (a) the intersection number of e with some fixed collection of points and/or surfaces in Σ; (b) the length with respect to a given metric on Σ.

To each graph γ we assign an operator acting on functions on $\mathscr{A}_\gamma$ as

$$\Delta_\gamma^l := l(e_1)\,\Delta_{e_1} + \cdots + l(e_{E_\gamma})\,\Delta_{e_{E_1}}, \tag{53}$$

where e_i, $i = 1, ..., E_\gamma$ are the edges of γ and Δ_{e_i} denotes the pull back, with respect to ψ_γ^* (see (31)), of the operator which is the tensor product of Δ, acting on the ith copy of G, with identity operators acting on the remaining copies. Because Δ is a quadratic Casimir operator, Δ^l_γ is independent of the choice of orientation for γ. The condition (52) implies that the family of operators $(\Delta^l_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ is consistent with the projective family [13] and therefore defines an operator Δ^l acting on cylindrical functions. In other words of f is a cylindrical function represented by a twice differentiable function f_γ on $\mathcal{A}_\gamma$, $f_\gamma \in C^2(\mathcal{A}_\gamma)$, then

$$\Delta^l f := (\Delta^l_\gamma f_\gamma) \circ p_\gamma \tag{54}$$

and the right hand side does not depend on the choice of the representative f_γ of f. (This would not have been the case if we had followed a more obvious strategy and attempted to define the Laplacian without the factors $l(e_i)$ in (53).)

5.1. *The Transform and the Main Result*

Given a function l on $\mathcal{E}$, the gauge covariant coherent state transform will be defined with the help of the fundamental solutions to the heat equation on $\overline{\mathcal{A}}$, associated with Δ^l:

$$\frac{\partial}{\partial t} F_t = \frac{1}{2} \Delta^l F_t. \tag{55}$$

The fundamental solution of (55) is given by the family $(\rho^l_{t,\gamma})_{\gamma \in \mathrm{Gra}(\Sigma)}$ of heat kernels for the operators Δ^l_γ on $\mathcal{A}_\gamma (\cong G^{E_\gamma})$,

$$\rho^l_{t,\gamma}(A_\gamma) = \rho_{s_1}(A_\gamma(e_1)) \cdots \rho_{s_{E_\gamma}}(A_\gamma(e_{E_\gamma})), \tag{56}$$

where $s_i = tl(e_i)$ and each of the functions $\rho_s(g)$ being the heat kernel of the Laplace-Beltrami operator on G. In fact the solution of (55) with cylindrical initial condition

$$F_{t=0} = f^{(0)}_\gamma \circ p_\gamma$$

is given by

$$F_t = \rho^l_{t,\gamma} \star f^{(0)}_\gamma, \tag{57}$$

where the convolution is

$$(\rho^l_{t,\gamma} \star f_{\gamma)(A^\gamma}) := \int_{G^{E_\gamma}} \rho^l_{t,\gamma}(A^h_\gamma) \times (f_\gamma \circ \psi_\gamma^{-1})(h_1, ..., h_{E_\gamma})\, d\mu_H(h_1) \cdots d\mu_H(h_{E_\gamma}), \tag{58}$$

and $A^h_\gamma\colon e_i \mapsto h_i^{-1} A_\gamma(e_i)$. Notice that (56) is well defined since the r.h.s. is invariant with respect to the change $e_i \mapsto e_i^{-1}$. It is also easy to verify, using the identity

$$\int_G \rho_t(g'^{-1})\, f(g'^{-1})\, d\mu_H(g') = \int_G \rho_t(g'^{-1} g^{-1})\, f(g')\, d\mu_H(g'), \tag{59}$$

that the r.h.s. of (58) does not depend on the orientation chosen for γ (see discussion after (31). Equality (59) follows from the following properties of the heat kernel [5]

$$\rho_t(g^{-1}) = \rho_t(g) \qquad \text{and} \qquad \rho_t(g_1 g_2) = \rho_t(g_2 g_1). \tag{60}$$

Let us consider the family of transforms $R^l_{t,\gamma}$:

$$R^l_{t,\gamma}(f_\gamma) = \rho^l_{t,\gamma} \star f_\gamma. \tag{61}$$

Our main result in the present Section will be:

THEOREM 1. *The map*

$$Z^l_t\colon L^2(\overline{\mathscr{A}}, \mu) \to \mathscr{C}\{\mathscr{H}_{\mathscr{C}}(\overline{\mathscr{A}^{\mathbb{C}}}) \cap L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \nu^l_t)\}, \tag{62}$$

defined on cylindrical functions $f = f_\gamma \circ p_\gamma$ as the analytic continuation of $R^l_{t,\gamma}(f_\gamma)$ and extended to the whole of $L^2(\overline{\mathscr{A}}, \mu)$ by continuity is a gauge covariant isometric coherent state transform.

The measure ν^l_t in (62) is defined below in Subsection 5.3. We will establish Theorem 1 with the help of several Lemmas proved in the following three subsections.

5.2. CONSISTENCY

Let us first show that the family of transforms (61) defines a map of cylindrical functions on $\overline{\mathscr{A}}$.

LEMMA 1. *The family $(R^l_{t,\gamma})_{\gamma \in \mathrm{Gra}(\Sigma)}$ in (61) is consistent.*

The proof follows from

$$f_\gamma \circ p_\gamma = f_{\gamma'} \circ p_{\gamma'} \Rightarrow (\rho^l_{\gamma,t} \star f_\gamma) \circ p_\gamma = (\rho^l_{\gamma',t} \star f_{\gamma'}) \circ p_{\gamma'}. \tag{63}$$

For convenience of the reader we recall from [13] the proof of (63). Since for every pair of graphs γ_1, γ_2 there exists a graph $\gamma_3 \geqslant \gamma_1, \gamma_2$, it is enough to prove (63) for

$$\gamma_2 \geqslant \gamma_1. \tag{64}$$

The graph γ_2 can be formed from γ_1 by adding additional edges, and subdividing edges—each of these steps being applied some finite number of times.

Thus, we need only to verify the consistency conditions for each of the following two cases: the graph γ_2 differs from γ_1 by (i) adding an extra edge to γ_1, and (ii) cutting an edge of γ_1 in two.

It follows from the construction of the projective family $(\mathscr{A}_\gamma, p_{\gamma\gamma'})_{\gamma, \gamma' \in \mathrm{Gra}(\Sigma)}$ and from formula (56), that (63) is equivalent to the equality

$$\int_{G^2} \rho_r(g'^{-1}g)\, \rho_s(h'^{-1}h)\, f(g'h')\, d\mu_H(g')\, d\mu_H(h')$$
$$= \int_G \rho_{r+s}(g'^{-1}gh)\, f(g')\, d\mu_H(g') \tag{65}$$

for any $r, s \geqslant 0$. Eq. (65) follows from (59), from the fact that L_g^* and R_g^* commute with $\rho_t \star$ for all $g \in G$ and from the composition rule

$$\rho_r \star \rho_s \star f = \rho_{r+s} \star f. \tag{66}$$

We have

$$\int_{G^2} \rho_r(g'^{-1}g)\, \rho_s(h'^{-1}h)\, f(g'h')\, d\mu_H(g')\, d\mu_H(h')$$
$$= \int_G \rho_r(g'^{-1}g)(\rho_s \star L_{g'}^* f)(h)\, d\mu_H(g')$$
$$= \int_G \rho_r(g'^{-1}g)(\rho_s \star R_h^* f)(g')\, d\mu_H(g')$$
$$= (R_h^* \rho_r \star \rho_s \star f)(g)$$
$$= (\rho_r \star \rho_s \star f)(gh) = (\rho_{r+s} \star f)(gh)$$
$$= \int_G \rho^{\mathbb{C}}_{r+s}(g'^{-1}gh)\, f(g')\, d\mu_H(g'). \tag{67}$$

This completes the proof of (63) and therefore also of Lemma 1.

According to Lemma 1, given a cylindrical function $f=f_\gamma\circ p_\gamma$ we have a well-defined "heat evolution,"

$$R_t^l(f):=R_{t,\gamma}^l(f_\gamma)\circ p_\gamma. \tag{68}$$

Notice that from Section 2 it follows that for any $f_\gamma\in L^2(\mathcal{A}_\gamma, d\mu_{0,\gamma})$ the convolution $\rho_{t,\gamma}^l\star f_\gamma=f_\gamma\star\rho_{\gamma,t}^l$ is a real analytic function.

We define a coherent state transform on each $\mathcal{A}_\gamma$ through

$$(Z_{t,\gamma}^l f_\gamma)(A_\gamma^{\mathbb{C}}):=(\rho_{t,\gamma}^{l\mathbb{C}}\star f_\gamma)(A_\gamma^{\mathbb{C}}), \tag{69}$$

where $\rho_{t,\gamma}^{l\mathbb{C}}$ is the analytic continuation of $\rho_{t,\gamma}^l$ from $\mathcal{A}_\gamma$ to $\mathcal{A}_\gamma^{\mathbb{C}}$ [5]. According to Lemma 1 and the remarks after Definition 6, the family of transforms $(Z_{t,\gamma}^l)_{\gamma\in\mathrm{Gra}(\Sigma)}$ is consistent in the sense of Definition 5. Hence, we may define the transform for each square-integrable cylindrical function $f=f_\gamma\circ p_\gamma\in\mathrm{Cyl}(\overline{\mathcal{A}})$

$$Z_t^l(f):=Z_{t,\gamma}^l[f_\gamma]\circ p_\gamma^{\mathbb{C}}, \tag{70}$$

which maps the space of μ_0-square integrable cylindrical functions on $\overline{\mathcal{A}}$ into the space of cylindrical holomorphic functions on $\overline{\mathcal{A}^{\mathbb{C}}}$.

5.3. *Measures on $\overline{\mathcal{A}^{\mathbb{C}}}$*

Consider the averaged heat kernel measure ν_t (7) defined on the complexified group $G^{\mathbb{C}}$ and the associated family of measures $(\nu_{t,\gamma}^l)_{\gamma\in\mathrm{Gra}(\Sigma)}$ on the spaces $\mathcal{A}_\gamma^{\mathbb{C}}$:

$$d\nu_{t,\gamma}^l(A_\gamma^{\mathbb{C}}):=d\nu_{l(e_1)t}(A_\gamma^{\mathbb{C}}(e_1))\otimes\cdots\otimes d\nu_{l(e_{E_\gamma})t}(A_\gamma^{\mathbb{C}}(e_{E_\gamma})). \tag{71}$$

It follows automatically from [5] that the transform $Z_{t,\gamma}^l\colon L^2(\mathcal{A}_\gamma, d\mu_{\gamma,AL})\to\mathcal{H}(\mathcal{A}_\gamma)\cap L^2(\mathcal{A}_\gamma^{\mathbb{C}}, d\nu_{t,\gamma}^l)$ is isometric. Isometry of the transforms $Z_{t,\gamma}^l$ implies the following equality for all square-integrable holomorphic functions $f_{1_\gamma}, f_{2_\gamma}$ and all $\gamma'\geqslant\gamma$

$$\int_{\mathcal{A}_\gamma^{\mathbb{C}}}\overline{f_{1_\gamma}(A_\gamma^{\mathbb{C}})}\,f_{2_\gamma}(A_\gamma^{\mathbb{C}})\,d\nu_{t,\gamma}^l=\int_{\mathcal{A}_{\gamma'}^{\mathbb{C}}}\overline{(f_{1_\gamma}\circ p_{\gamma\gamma'}^{\mathbb{C}})(A_{\gamma'}^{\mathbb{C}})}(f_{2_\gamma}\circ p_{\gamma\gamma'}^{\mathbb{C}})(A_{\gamma'}^{\mathbb{C}})\,d\nu_{t,\gamma'}^l. \tag{72}$$

From the arbitrariness of f_{1_γ} and f_{2_γ} we will conclude that the family $\{\nu_{t,\gamma}^{l,\mathbb{C}}\}_{\gamma,\gamma'\in\mathrm{Gra}(\Sigma)}$ is consistent and therefore defines a cylindrical measure on $\overline{\mathcal{A}^{\mathbb{C}}}$ which will be denoted by ν_t^l.

To see this let $i\colon\hat{G}^{\mathbb{C}}\to\mathbb{C}^N$ be an analytic immersion of $\hat{G}^{\mathbb{C}}:=G^{\mathbb{C}}\times\cdots\times G^{\mathbb{C}}$ into $\mathbb{C}^N$ for sufficiently large N. A Borel probability measure

$\mu^{\mathbb{C}}$ on $G^{\mathbb{C}}$ defines a Borel probability measure $i_* \mu^{\mathbb{C}}$ on $\mathbb{C}^N$ (supported on $i(\hat{G}^{\mathbb{C}})$) through

$$\int_{\mathbb{C}^N} fd(i_* \mu^{\mathbb{C}}) := \int_{\hat{G}^{\mathbb{C}}} i^*(f)\, d\mu^{\mathbb{C}}. \tag{73}$$

Consider the analytic functions $i^*(F_l)$ on $\hat{G}^{\mathbb{C}}$, where

$$F_l(z) = e^{lz}, \qquad l,\, z \in \mathbb{C}^N, \quad lz := \sum_{j=1}^{N} l_j z_j. \tag{74}$$

For every $\delta_1,\, \delta_2 \in \mathbb{R}^N$ we choose $l_1 = -1/2(\delta_2 + i\delta_1)$ and $l_2 = -l_1$ so that

$$(\bar{F}_{l_1} F_{l_2})(x, y) = e^{i(\delta_1 x + \delta_2 y)}, \tag{75}$$

where $z = x + iy$. Then

$$\chi_{\mu^{\mathbb{C}}}(\delta_1, \delta_2) := \int_{\mathbb{R}^{2N}} e^{i(\delta_1 x + \delta_2 y)} d(i_* \mu^{\mathbb{C}}) = \int_{\hat{G}^{\mathbb{C}}} \overline{i^*(F_{l_1})}\, i^*(F_{l_2})\, d\mu^{\mathbb{C}} \tag{76}$$

is the Fourier transform of the measure $i_* \mu^{\mathbb{C}}$ on $\mathbb{R}^{2N}$, which, according to the Bochner theorem, completely determines $i_* \mu^{\mathbb{C}}$ and therefore also $\mu^{\mathbb{C}}$. Thus (72) implies that $(p^{\mathbb{C}}_{\gamma\gamma'})_* \, \nu^l_{t,\gamma'}$ and $\nu^l_{t,\gamma}$ in fact agree as Borel measures on $\mathscr{A}^{\mathbb{C}}_{\gamma}$.

5.4. *Gauge Covariance*

Here we complete the proof of Theorem 1.
We only need to establish:

LEMMA 2. *R commutes with action of $\bar{\mathscr{G}}$ on $L^2(\bar{\mathscr{A}}, d\mu_0)$.*

In the proof, g, g_a, g_b, and $\psi_\gamma(A_\gamma)$ will be elements of G^{E_γ} and we define multiplication of E_γ-tuples component-wise; i.e., $(g_a g_b)_i = (g_a)_i\, (g_b)_i$.

Proof of Lemma 2. For cylindrical $f = f_\gamma \circ p_\gamma$ and $\bar{g} \in \bar{\mathscr{G}}$, let $g_a,\, g_b \in G^{E_\gamma}$ be given by $(g_a)_i := \bar{g}(p_{ia})$ and $(g_b)_i := \bar{g}(p_{ib})$, where p_{ia} and p_{ib} are the initial and final points of the edge e_i associated with a fixed choice of orientation on γ. Then,

$$\begin{aligned} R^l_t[f](\bar{g}[A_\gamma]) &= (\rho_{t,\gamma} \star f_\gamma)(\bar{g}[A_\gamma]) \\ &= \int_{G^{E_\gamma}} (f_\gamma \circ \psi_\gamma^{-1})(g g_a \psi_\gamma(A_\gamma)\, g_b^{-1}) \prod (\rho_t\, d\mu_H)(g) \\ &= \int_{G^{E_\gamma}} (f_\gamma \circ \psi_\gamma^{-1})(g_a g \psi_\gamma(A_\gamma)\, g_b^{-1}) \prod (\rho_t\, d\mu_H)(g) \\ &= R^l_t[\bar{g}^*(f)](\bar{A}), \end{aligned} \tag{77}$$

since the measure is conjugation invariant. Note that this is a consequence of the $\overline{\mathcal{G}}$-invariance of Δ^l.

Finally, note that since the transform (70) depends on the path function l, it fails to be diffeomorphism covariant.

6. Gauge and Diffeomorphism Covariant Coherent State Transforms

In this Section, we introduce a coherent state transform that is both gauge and diffeomorphism covariant. This new transform will be based on techniques associated with the Baez measures and we recall from Subsection 3.2 that, given any Baez measure $\mu^{(m)}$ on $\overline{\mathcal{A}^{\mathbb{C}}}$ and the corresponding measures $\mu^{(m)}_\gamma$ on $\mathcal{A}_\gamma$, we may write (33) as

$$\int_{\mathcal{A}_\gamma} f_\gamma \, d\mu^{(m)}_\gamma = \int_{G^{E_\gamma} \times G^{E_\gamma}} f_\gamma \circ \psi_\gamma^{-1} \circ \phi_\gamma \, d\mu^{(m)\prime}_\gamma. \tag{78}$$

From (39), each $d\mu^{(m)\prime}_\gamma$ is a product of measures dm on G and delta functions with respect to these measures. The arguments of the delta functions are pairs of coordinates and no coordinate appears in more than one delta function. Specifically, this is true for the Baez measure $\tilde{\mu}_0 \equiv \mu^{(\mu_H)}$ constructed from the Haar measure $m = \mu_H$ on G.

6.1. *The Transform and the Main Result*

Let us fix a measure ν on $G^{\mathbb{C}}$ that satisfies the conditions listed in Section 2 for the existence of the Hall transform C_ν. Given ν we have on G a generalized heat-kernel measure $d\rho = \rho_\nu \, d\mu_H$ used in the Hall transform (3) from $L^2(G, \mu_H)$ to $L^2(G^{\mathbb{C}}, \nu) \cap \mathcal{H}(G^{\mathbb{C}})$.

Our transform will be defined as follows. Given some $\bar{A}_0 \in \overline{\mathcal{A}}$ and the corresponding $A_{0,\gamma} \in \mathcal{A}_\gamma$, let $\phi_{\bar{A}_{0,\gamma}}: G^{E_\gamma} \times G^{E_\gamma} \to G^{E_\gamma}$ be the map

$$\begin{aligned} \phi_{\bar{A}_{0,\gamma}}: & [(g_{1a}, ..., g_{E_\gamma a}), (g_{1b}, ..., g_{E_\gamma b})] \\ & \mapsto (g_{1a} \bar{A}_0(e_1) \, g_{1b}^{-1}, ..., g_{E_\gamma a} \bar{A}_0(e_{E_\gamma}) \, g_{E_\gamma b}^{-1}). \end{aligned} \tag{79}$$

Note that $\phi_{\bar{A}_{0,\gamma}}$ depends on $\bar{A}_0$ only through $A_{0,\gamma}$ and that if $\bar{A}_0$ is the trivial connection $\bar{1}$ (for which $\bar{1}(e) = 1_G$ for any $e \in \mathcal{E}$) then $\phi_{\bar{1},\gamma} = \phi_\gamma$ of (34).

For $f: \overline{\mathcal{A}} \to \mathbb{C}$ such that $f = f_\gamma \circ p_\gamma$, we would like to define $R(f): \overline{\mathcal{A}} \to \mathbb{C}$ through $R(f) = R_\gamma(f_\gamma) \circ p_\gamma$, where

$$R_\gamma(f_\gamma)(A_{0,\gamma}) = \int_{G^{2E_\gamma}} f_\gamma \circ \psi_\gamma^{-1} \circ \phi_{\bar{A}_{0,\gamma}} \, d\rho'_\gamma. \tag{80}$$

In (80) $d\rho'_\gamma$ is the measure on $G^{E_\gamma} \times G^{E_\gamma}$ associated with the Baez measure $\rho = \mu^{(\rho)}$. Thus, $d\rho'_\gamma$ is a product of generalized heat kernel measures $d\rho$ and delta-functions with respect to this measure. We will show that the map R is well defined. Our main result will be

THEOREM 2. *For each* v, *there exists a unique isometric map*

$$Z: L^2(\bar{\mathscr{A}}, \tilde{\mu}_0) \to \mathscr{C}\{\mathscr{H}_{\mathscr{C}}() \cap L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \mu^{(v)})\}, \tag{81}$$

such that, for every $f \in \mathrm{Cyl}(\bar{\mathscr{A}})$ *and any holomorphic* (L^2-) *representative of* $Z(f)$ *with restriction to* $\mathscr{A}$ *denoted by* $\tilde{f}$, *the real-analytic function* $\tilde{f}$ *coincides* $\tilde{\mu}_0$*-everywhere with* $R(f)$. *The map* Z *is a gauge and diffeomorphism covariant isometric coherent state transform.*

6.2. *Consistency*

As before, it is convenient to break the proof of our theorem into several parts. We begin with

LEMMA 3. *The family* $(R_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$,

$$R_\gamma(f_\gamma)(A_{0,\gamma}) = \int_{G^{2E_\gamma}} f_\gamma \circ \psi_\gamma^{-1} \circ \phi_{\bar{A}_{0,\gamma}} \, d\rho'_\gamma, \tag{82}$$

is consistent.

Proof. Suppose that $f: \bar{\mathscr{A}} \to \mathbb{C}$ is cylindrical with $f = f_{\gamma_1} \circ p_{\gamma_1}$ and $f = f_{\gamma_2} \circ p_{\gamma_2}$. As in Section 5, it is enough to consider the case $\gamma_2 \geqslant \gamma_1$.

We must now establish the conditions (i), (ii) listed in the proof of Lemma 1 in Subsection 5.2. The first case is straightforward. Indeed $f_{\gamma_2} = f_{\gamma_1} \circ p_{\gamma_1 \gamma_2}$ depends only on those edges that actually lie in γ_1. Integration over the other variables in the measure $d\rho'_{\gamma_2}$ simply yields the measure $d\rho'_{\gamma_1}$ as in the usual Baez construction. Thus, $R_{\gamma_2}(f_{\gamma_2}) = R_{\gamma_1}(f_{\gamma_1}) \circ p_{\gamma_1 \gamma_2}$.

We now address (ii). Suppose that γ_2 is just γ_1 with the edge $e_0 \in \gamma_1$ split into e_1 and e_2 at the vertex v. Let e_1, e_2 have orientations induced by e_0. Without loss of generality, let $e_1 \circ e_2 = e_0$. Then we have

$$R_{\gamma_2}(f_{\gamma_2})(A_{0,\gamma_2}) = \int_{G_a^{E_{\gamma_2}} \times G_b^{E_{\gamma_2}}} (f_{\gamma_2} \circ \psi_{\gamma_2}^{-1})(g_{1a} \bar{A}_0(e_1)\, g_{1b}^{-1}, \ldots)\, d\rho'_{\gamma_2}, \tag{83}$$

where the g_{ia} are coordinates on $G_a^{E_{\gamma_2}}$ and the g_{ib} are coordinates on $G_b^{E_{\gamma_2}}$. Since $f_{\gamma_2} = f_{\gamma_1} \circ p_{\gamma_1 \gamma_2}$, $(f_{\gamma_2} \circ \psi_{\gamma_2}^{-1})(g_1, \ldots, g_{E_{\gamma_2}}) = (f_{\gamma_1} \circ \psi_{\gamma_1}^{-1})(g_1 g_2, g_3, \ldots, g_{E_{\gamma_2}})$, it follows that

$$R_{\gamma_2(f_{\gamma_2})}(A_{0,\gamma_2})$$

$$= \int_{G_a^{E_{\gamma_2}} \times G_b^{E_{\gamma_2}}} (f_{\gamma_1} \circ \psi_{\gamma_1}^{-1})(g_{1a} \bar{A}_0(e_1)\, g_{1b}^{-1} g_{2a} \bar{A}_0(e_2)\, g_{2b}^{-1},\, g_{3a} \bar{A}_0(e_3)\, g_{3b}^{-1}, ...,$$

$$g_{E_\gamma a} \bar{A}_0(e_{E_\gamma})\, g_{E_\gamma b}^{-1}) \times \delta(g_{1b}, g_{2a})\, d\rho(g_{1b})\, d\rho(g_{2a})$$

$$d\rho'_{\gamma_1[(g^{1a}}, g_{3a}, ..., g_{E_\gamma a}), (g_{2b}, g_{3b}, ..., g_{E_\gamma b})]$$

$$= (R_{\gamma_1}(f_{\gamma_1}) \circ \psi_{\gamma_1}^{-1})(\bar{A}(e_1)\, \bar{A}(e_2), \bar{A}(e_3), ..., \bar{A}(e_{E_\gamma}))$$

$$= (R_{\gamma_1}(f_{\gamma_1}) \circ p_{\gamma_1 \gamma_2})(A_{0,\gamma_2}).$$

This is enough to show consistency so that the family $(R_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ defines unambiguously a map R: $\mathrm{Cyl}(\bar{\mathscr{A}}) \cap L^2(\bar{\mathscr{A}}, \tilde{\mu}_0) \to \mathrm{Cyl}(\bar{\mathscr{A}})$.

6.3. *Extension and Isometry*

For a general $f \in \mathrm{Cyl}(\bar{\mathscr{A}}) \cap L^2(\bar{\mathscr{A}}, \tilde{\mu}_0)$, the function $R(f)$ may not be real-analytic on $\bar{\mathscr{A}}$. However, there still exists a natural "analytic extension" of $R(f)$ to a unique element of $L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \mu^{(v)})$ that can be briefly defined as follows. The function $R(f)$ is real-analytic when restricted to a subspace of $\bar{\mathscr{A}}$ carrying the support of the Baez measure; on the other hand, the complexification of this subspace contains the support of the Baez measure in $\overline{\mathscr{A}^{\mathbb{C}}}$. This is sufficient for the extension of $R(f)$ to exist and be unique (in the sense of L^2 spaces).

To define the extension more precisely, let us first express the Baez intregral in a more convenient form. Given an oriented graph γ, consider $\mathscr{A}_\gamma$, $\mathscr{A}_\gamma^{\mathbb{C}}$ and the corresponding maps $\psi_\gamma^{-1} \circ \phi_\gamma$: $G^{E_\gamma} \times G^{E_\gamma} \to \mathscr{A}_\gamma$ as well as the complexification $\psi_\gamma^{\mathbb{C}\,-1} \circ \phi_\gamma^{\mathbb{C}}$: $G^{\mathbb{C}E_\gamma} \times G^{\mathbb{C}E_\gamma} \to \mathscr{A}_\gamma^{\mathbb{C}}$. In what follows, all the functions on $\mathscr{A}_\gamma$ ($\mathscr{A}_\gamma^{\mathbb{C}}$) shall be identified with their pullbacks to the corresponding $G^{E_\gamma} \times G^{E_\gamma} (G^{\mathbb{C}E_\gamma} \times G^{\mathbb{C}E_\gamma})$. Since the delta-functions in the Baez measure identify some pairs (g_{ia}, g_{jb}) of variables, for some $E_\gamma \leqslant k_\gamma \leqslant 2E_\gamma$, they define embeddings

$$\begin{aligned} \lambda_\gamma&: G^{k_\gamma} \to G^{E_\gamma} \times G^{E_\gamma} \\ \lambda_\gamma^{\mathbb{C}}&: G^{\mathbb{C}k_\gamma} \to G^{\mathbb{C}E_\gamma} \times G^{\mathbb{C}E_\gamma}, \end{aligned} \tag{85}$$

where $\lambda_\gamma^{\mathbb{C}}$ is the complexification of λ_γ and both are insensitive to the choice of measure on G used to define the Baez measure. (Note that the maps λ and $\psi^{-1} \circ \phi \circ \lambda$ do not depend on the choice of an orientation of γ.)

Suppose that we wish to compute the integral of some $f = f_\gamma \circ p_\gamma \in \mathrm{Cyl}(\overline{\mathscr{A}^{\mathbb{C}}})$ with respect to $\tilde{\mu}_0 = \mu^{(\mu_H)}$ or $f = f_\gamma \circ p_\gamma^{\mathbb{C}} \in \mathrm{Cyl}(\overline{\mathscr{A}^{\mathbb{C}}})$ with respect to $\mu^{(v)}$. Then, we may use these embeddings to write the integrals as

$$\int_{\bar{\mathscr{A}}} f\, d\tilde{\mu}_0 = \int_{G^{k_\gamma}} f_\gamma \circ \psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma \prod d\mu_H \tag{86}$$

$$\int_{\bar{\mathscr{A}}^{\mathbb{C}}} f\, d\mu^{(\nu)} = \int_{G^{\mathbb{C}k_\gamma}} f_\gamma \circ \psi_{ю}^{\mathbb{C}\Gamma 1} \circ \phi_\gamma^{\mathbb{C}} \circ \lambda_\gamma^{\mathbb{C}} \prod d\nu. \tag{87}$$

The above formulas show the following statement.

Lemma 4. *Let $f_1, f_2 \in \mathrm{Cyl}(\bar{\mathscr{A}})$; $f_1 = f_2$ tilde μ_0-everywhere if and only if for a graph γ such that $f_i = p_\gamma^* f_{i\gamma}$ $i = 1, 2$, we have $(\psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma)^* f_{1\gamma} = (\psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma)^* f_{2\gamma} \prod d\mu_H$-everywhere (and analogously for the complexified case). The natural maps*

$$\begin{aligned} (\psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma)^* &: L^2(\mathscr{A}_\gamma, \tilde{\mu}_{0_\gamma}) \to L^2\left(G^{k_\gamma}, \prod \mu_H^{\mathbb{C}}\right), \\ (\psi_\gamma^{\mathrm{C}-1} \circ \phi_\gamma^{\mathbb{C}} \circ \lambda_\gamma^{\mathbb{C}})^* &: L^2(\mathscr{A}_\gamma^{\mathbb{C}}, \mu_\gamma^{(\nu)}) \to L^2\left(G^{\mathbb{C}k_\gamma}, \prod \nu\right), \end{aligned} \tag{88}$$

are isometric.

Further, let $C_{(k_\gamma)}$ be the coherent state transform defined by Hall from $L^2(G^{k_\gamma}, \prod_{i=1}^{k_\gamma} d\mu_H(g_i))$ to $L^2(G^{\mathbb{C}^{k_\gamma}}, \prod_{i=1}^{k_\gamma} d\nu(g_i^{\mathbb{C}}))$. It follows from (86, 87) that

$$\begin{aligned} &[R_\gamma(f_\gamma) \circ \psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma](g^*) \\ &\quad = \int_{G^{k_\gamma}} [f_\gamma \circ \psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma](g^{-1} g^*) \prod d\rho(g) \\ &\quad = (C_{(k_\gamma)}[f_\gamma \circ \psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma])(g^*), \end{aligned} \tag{89}$$

where g, g^*, $g^* \in G^{k_\gamma}$ and $(gg^*)_i = g_i\, g_i^*$. Re-expressing the last result less precisely, the restriction of $R_\gamma(f)$ to G^{k_γ} embedded in $G^{E_\gamma} \times G^{E_\gamma}$ coincides with the usual Hall transform. The following Lemma then follows from the results of [5].

Lemma 5. *Let f_γ be a measurable function on $\mathscr{A}_\gamma$ with respect to the Baez measure $\tilde{\mu}_{0_\gamma}$; the function $R_\gamma(f_\gamma)$ restricted to $\psi_\gamma^{-1} \circ \phi_\gamma \circ \lambda_\gamma(G^{k_\gamma})$ is real-analytic.*

The function $R_\gamma(f_\gamma)$ can thus be analytically extended to a holomorphic function defined on $\psi_\gamma^{\mathbb{C}-1} \circ \phi_\gamma^{\mathbb{C}} \circ \lambda_\gamma^{\mathbb{C}}(G^{k_\gamma \mathbb{C}})$ which, according to Lemma 4, uniquely determines an element $Z_\gamma(f_\gamma)$ in $L^2(\mathscr{A}_\gamma^{\mathbb{C}}, \nu_\gamma)$. We have defined a map Z_γ

$$Z_\gamma \colon L^2(A_\gamma, \tilde{\mu}_{0_\gamma}) \to L^2(A_\gamma^{\mathbb{C}}, \mu_\gamma^{(\nu)}). \tag{90}$$

The consistency of the family of maps $(Z_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ easily follows from the consistency of $(R_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$. Another advantage of relating, through (89), Z_γ with the usual Hall transform $C_{(k_\gamma)}$ is that we may again consult Hall's results and note that the map (90) is an isometry. Thus, we have verified the following Lemma.

LEMMA 6. (i) *The family of maps* $(Z_\gamma)_{\gamma \in \mathrm{Gra}(\Sigma)}$ (90) *is consistent*;

(ii) *The map*

$$Z\colon L^2(\bar{\mathscr{A}}, \tilde{\mu}_0) \cap \mathrm{Cyl}(\bar{\mathscr{A}} \to L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \mu^{(v)}) \tag{91}$$

is an isometry, where $Z(p_\gamma^* f_\gamma) := Z_\gamma(f_\gamma)$.

Since cylindrical functions are dense in $L^2(\bar{\mathscr{A}}, \tilde{\mu}_0)$, it follows that our transform Z extends to

$$Z\colon L^2(\bar{\mathscr{A}}, \tilde{\mu}_0) \to \mathscr{C}\{L^2(\overline{\mathscr{A}^{\mathbb{C}}}, \mu^{(v)})\} \tag{92}$$

as an isometry.

6.4. *Analyticity*

We have seen that the pullback of $Z_\gamma(f_\gamma)$ through the map $(\psi_\gamma^{\mathbb{C}})^{-1} \circ \phi_\gamma^{\mathbb{C}} \circ \lambda_\gamma^{\mathbb{C}}$ may be taken to be holomorphic. However, we will now show that this is the case for $Z(f)$ itself.

LEMMA 7. *If* $f \in \mathrm{Cyl}(\bar{\mathscr{A}} \cap L^2(\bar{\mathscr{A}}, \tilde{\mu}_0)$ *then*;

(i) *Any cylindrical function* $f = f_\gamma \circ p_\gamma$ *differs only on a set of tilde* μ_0 *measure zero from some* $f^0 = f_\gamma^0 \circ p_\gamma$ *such taht* $R_\gamma(f_\gamma^0)$ *is real analytic.*

(ii) $Z(f)$ *may be represented by a holomorphic function on* $\overline{\mathscr{A}^{\mathbb{C}}}$.

Note that the second part of the Lemma follows automatically from part (i).

For this Lemma, we will use the concept of the *Baez-equivalence graph* γ_E corresponding to a graph γ. This γ_E is an abstract graph (a collection of "edges" and "vertices" not embedded in any manifold) formed from the edges of γ. However, two edges in γ_E meet at a vertex if and only if the corresponding edges join to form an analytic path in γ. Since each edge of γ can, at a given vertex, meet at most one other edge analytically, each vertex in γ_E connects at most two edges. Thus, γ_E consists of a finite set of line segments and closed loops that do not intersect. Let us orient the edges of γ_E so that, at each vertex, one edge flows in and one edge flows out. We will assume that the edges of γ are oriented in the corresponding way.

A graph γ for which γ_E contains no cycles will be called *Baez-simple*. To derive Lemma 7, we will also need the following Lemma:

LEMMA 8. *Any cylindrical function $f\colon \bar{\mathcal{A}} \to \mathbb{C}$ is identical to a function f^0 that is cylindrical over a Baez-simple graph γ_s, except on sets of $\tilde{\mu}_0$ measure zero.*

To see this, we construct the Baez-simple graph γ_s from γ by removing one edge e^i_0 from the ith cycle in γ_E. Let $\zeta\colon \mathcal{A}_\gamma \to \mathcal{A}_\gamma$ be the map such that

$$[\zeta(A_\gamma)](e^i_0) = \left[\prod_{j=1}^{N_i} A_\gamma(e^i_j)\right]^{-1}, \tag{93}$$

where e^i_j are the other edges in the ith cycle and are numbered from 1 to N_i in a manner consistent with their orientations. For any other edge e, let $[\zeta(A_\gamma)](e) = A_\gamma(e)$.

Proof of Lemma 8. If $f = f_\gamma \circ p_\gamma$, let $f^0_\gamma = f_\gamma \circ \zeta$ and $f^0 = f^0_\gamma \circ p_\gamma$ so that f^0 is in fact cylindrical over γ_s ($f^0 = f^0_{\gamma_s} \circ p_{\gamma_s}$). Note that $d\mu'_\gamma$ is a product of measures associated with the connected components of γ_E and recall that f^0_γ differs from f_γ only in its dependence on edges in cycles of γ_E. For simplicity, let us assume for the moment that f_γ in fact depends only on edges that lie in one cycle α in γ_E so that $f_\gamma = f_\alpha \circ p_{\alpha\gamma}$ for some $f_\alpha\colon \mathcal{A}_\alpha \to \mathbb{C}$. Furthermore,

$$\begin{aligned}
&\|f - f^0\|^2_{L^2, \tilde{\mu}_0} \\
&\quad = \int_{G_a^{E_\alpha} \times G_b^{E_\alpha}} \Bigg| [f_\alpha \circ \psi_\alpha^{-1}](g_{0a} g_{0b}^{-1}, g_{1a} g_{1b}^{-1}, \ldots, g_{(E_\alpha - 1)a} g_{(E_\alpha - 1)b}^{-1}) \\
&\qquad - [f_\alpha \circ \psi_\alpha^{-1}]\left(\left(\prod_{i=1}^{E_\alpha - 1} (g_{ia} g_{ib}^{-1})\right)^{-1}, g_{1a} g_{1b}^{-1}, \ldots, g_{(E_\alpha - 1)a} g_{(E_\alpha - 1)b}^{-1}\right)\Bigg|^2 \\
&\qquad \times \prod_{i=1}^{E_\alpha - 1} \delta(g_{(i-1)b}, g_{ia})\, \delta(g_{(E_\alpha - 1)b}, g_{0a}) \prod_{j=0}^{E_\alpha - 1} d\mu_H(g_{ia})\, d\mu_H(g_{ib}) = 0,
\end{aligned} \tag{94}$$

so that f and f^0 differ only on sets of $\tilde{\mu}_0$ measure zero. The same is true when f_γ depends on several cycles α_i.

We can also use γ_E to introduce a convenient labelling of the edges in γ_s. Let $e_{(i,j)}$ be the jth edge of the ith connected component of γ_E, where we again assume that the edges in the ith component are numbered consistently with their orientations. Note that since γ_s is Baez-simple these components form open chains with well-defined initial edges ($e_{(i,1)}$) and final edges ($e_{(i,N_i)}$).

Proof of Lemma 7. Suppose that there are N_{γ_s} components of γ_s. Then, from (33), (34), and (39) we have

$$d\rho'_{\gamma_s} = \prod_{i=1}^{N_{\gamma s}} \left[d\rho(g_{(i,1)a})\, d\rho(g_{(i,1)b}) \right.$$
$$\left. \times \prod_{j=2}^{N_i} \delta(g_{(i,j-1)b}, g_{(i,j)a})\, d\rho(g_{(i,j)a})\, d\rho(g_{(i,j)b}) \right]. \qquad (95)$$

For $k_{\gamma_s} = \sum_{i=1}^{N_{\gamma_s}1} (N_i + 1)$, let us now introduce the map $\sigma_{\bar{A},\gamma_s}\colon G^{k_{\gamma_s}} \to G^{E_{\gamma_s}}$ through

$$[\sigma_{\bar{A},\gamma_s}(g)]_{i,j} = g_{(i,j)}\, \bar{A}(e_{(i,j)})\, g^{-1}_{(i,j+1)} \qquad (96)$$

for $g \in G^{k_\gamma}$, where we have set $g(i,1) = g_{(i,1)a}$ and $g_{(i,j)} = g_{(i,j-1)b}$ for $j \geqslant 2$. Thus, we may write

$$R_{\gamma_s}(f^0_{\gamma_s})(A_{\gamma_s}) = \int_{Gk\gamma_s} [f^0_{\gamma s} \circ \psi^{-1}_{\gamma_s} \circ \sigma_{\bar{A},\gamma_s}] \prod_{i=1}^{N_{\gamma_s}} \prod_{j=1}^{N_i+1} \rho(g_{(i,j)})\, d\mu_H(g_{(i,j)}). \quad (97)$$

Analyticity of (97) can now be shown by making the change of integration variables

$$g'_{(i,j)} = g_{(i,j)} \prod_{k=j}^{N_i+1} A(e_{(i,k)}) \qquad (98)$$

so that, using the invariance of μ_H, we may write

$$R_{\gamma_s}(f^0_{\gamma_s})(A_{\gamma_s}) = \int_{Gk\gamma_s} [f^0_{\gamma s} \circ \psi^{-1}_{\gamma_s} \circ \sigma_{\bar{1},\gamma_s}]$$
$$\times \prod_{i=1}^{N_{\gamma_s}} \prod_{j=1}^{N_i+1} \rho(g'_{(i,j)}) \left[\prod_{k=j}^{N_i+1} A(e_{(i,k)}) \right]^{-1} \Bigg)\, d\mu_H(g'_{(i,j)}). \quad (99)$$

From the analyticity of ρ [7] and the compactness of $G^{k_{\gamma_s}}$ it follows that $R_{\gamma_s}(f^0_{\gamma_s})$ is a real-analytic function. This concludes the proof of Lemma 5.

6.5. *Gauge Covariance*

We now derive

LEMMA 9. *Z is a $\bar{\mathscr{G}}$-covariant transform.*

In particular, this will show that R maps gauge invariant functions to gauge invariant functions.

Proof. For cylindrical $f = f_\gamma \circ p_\gamma$,

$$R_\gamma(f_\gamma)(A_\gamma) = \int_{G^{2E_\gamma}} (f_\gamma \circ \psi_\gamma^{-1})(g_{1a}\bar{A}(e_1)\, g_{1b}^{-1}, ..., g_{E_\gamma a}\bar{A}(e_{E_\gamma})\, g_{E_\gamma b}^{-1})\, d\rho'_\gamma \tag{100}$$

and

$$\begin{aligned} R_\gamma(f_\gamma)(g_\gamma[A_\gamma]) = \int_{G^{2E_\gamma}} & (f_\gamma \circ \psi_\gamma^{-1})(g_{1a}\,\bar{g}_{p_{1a}}\bar{A}(e_1)(\bar{g}_{p_{1b}})^{-1}\, g_{1b}^{-1}, ..., \\ & g_{E_\gamma a}\,\bar{g}_{p_{E_\gamma a}}\bar{A}(e_{E_\gamma})(\bar{g}_{p_{E_\gamma b}})^{-1}\, g_{E_\gamma b}^{-1})\, d\rho'_\gamma, \end{aligned} \tag{101}$$

where $\bar{g}_{p_{ia}}$ is the group element associated with the initial vertex of edge i by $g_\gamma \in \mathcal{G}_\gamma$ and $\bar{g}_{p_{ib}}$ is the group element associated with the final vertex of edge i. Note that, in this scheme, a point may be referred to as the initial and/or final vertex of many edges.

We now perform the change of integration variables,

$$\begin{aligned} g_{ia} &\to \bar{g}_{p_{ia}}^{-1}\, g_{ia}(\bar{g}_{p_{ia}}) \\ g_{ib} &\to \bar{g}_{p_{ib}}^{-1}\, g_{ib}(\bar{g}_{p_{ib}}). \end{aligned} \tag{102}$$

The measure $d\rho'_\gamma$ contains only heat kernel-like measures and delta functions $\delta(g_{vi}, g_{vj})$, where the notation indicates that the arguments of a given delta-function are associated with the same vertex v. Since each such delta-function is unaffected by the above transformation and the heat kernel-like functions ρ_v are conjugation invariant, ρ'_γ is also invariant under (102). Thus,

$$\begin{aligned} R_\gamma(f_\gamma)(g_\gamma[A_\gamma]) = \int_{G^{2E_\gamma}} & (f_\gamma \circ \psi_\gamma^{-1})(\bar{g}_{p_{1a}}\, g_{1a}\bar{A}(e_1)\, g_{1b}^{-1}\bar{g}_{p_{1b}}, ..., \\ & \bar{g}_{p_{E_\gamma a}}\, g_{E_\gamma a}\bar{A}(e_{E_\gamma})\, g_{E_\gamma b}^{-1}(\bar{g}_{p_{E_\gamma b}})^{-1}\, d\rho'_\gamma \\ = {} & R_\gamma(g_\gamma[f_\gamma])(A_\gamma), \end{aligned} \tag{103}$$

verifying gauge covariance for cylindrical f. Since cylindrical functions are dense in $L^2(\bar{A}, \tilde{\mu}_0)$, $L^*_{\bar{g}}$, $L^{\mathbb{C}*}_{\bar{g}}$ are continuous $\forall \bar{g} \in \bar{\mathcal{G}}$ and we have shown that Z is an isometry and thus continuous, it follows that Z commutes with gauge transformations and that Lemma 9 holds. Theorem 2 then follows as a corollary of Lemmas 3–9.

Before concluding, we note that a number of technical issue still remain to be understood. Among these are the exact relationship of $\overline{\mathcal{A}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}}$ to $\mathcal{A}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}$ and a better understanding of the space obtained by completing $L^2(\overline{\mathcal{A}^{\mathbb{C}}}, \mu^{(\nu)}) \cap \mathcal{H}_{\mathscr{C}}(\overline{\mathcal{A}^{\mathbb{C}}})$. It is also not known a *diffeomorphism covariant* coherent state transform can be used to construct a holomorphic representation from $L^2(\bar{\mathscr{A}}, \mu_0)$. While we hope that future investigation will clarify these matters, Theorems 1 and 2 as stated are enough to provide a framework for the construction and analysis of holomorphic representations for theories of connections.

Appendix: The Abelian Case

For compact Abelian G, the transform Z of Section 6 can be expressed in a particularly simple way and it is possible to obtain explicit results. We begin by simply evaluating the transform of the holonomy $T_\alpha: \bar{A} \to \mathbb{C}^N$ associated with an arbitrary piecewise analytic path α. (Note that the results above for $\mathbb{C}$-valued functions on $\bar{\mathscr{A}}$ hold for functions that take values in any Hilbert space.) This holonomy is cylindrical over any graph γ in which the path α may be embedded and may be written as

$$T_\alpha(\bar{A}) = \prod_{i=1}^{E_\gamma} [\bar{A}(e_i)]^{m_i}, \tag{104}$$

where the integer m_i is the (signed) number of times that the path α traces the edge e_i. Thus, the transform is given by the Baez integral over T_α

$$\begin{aligned} R(T_\alpha)(\bar{A}) &= \int_{G_a^{E_\gamma} \times G_b^{E_\gamma}} \prod_{i=1}^{E_\gamma} [g_{ia} \bar{A}(e_i)\, g_{ib}^{-1}]^{m_i}\, d\rho'_\gamma \\ &= T_\alpha(\bar{A}) \int_{G_a^{E_\alpha} \times G_b^{E_\alpha}} \prod_{i=1}^{E_\gamma} [g_{ia}\, g_{ib}^{-1}]^{m_i}\, d\rho'_\gamma \end{aligned} \tag{105}$$

and R is a scaling transformation on T_α. Denote the resulting scaling factor for T_α on the right hand side of (105) by $e^{-l(\alpha)}$, that is, $R[T_\alpha] = e^{-l(\alpha)} T_\alpha$. For the case where ν is a Gaussian measure in standard coordinates, we will show that $l(\alpha)$ is real and positive.

Introduce coordinates $\theta \in [0, 2\pi]$, $r \in (-\infty, \infty)$ on $U(1)^{\mathbb{C}}$ such that $g^{\mathbb{C}} = e^{i\theta} e^r$. We wish to consider a measure $d\nu_\sigma = e^{-r^2/\sigma}(d\theta\, dr/2\pi \sqrt{\pi\sigma})$ and the corresponding heat kernel measure $d\rho_\sigma(\theta) = \sum_{k \in \mathbf{Z}} e^{-[(\theta + 2\pi k)^2]/2\sigma}(d\theta/\sqrt{2\pi\sigma})$. From (105) we find that

$$e^{-l(\alpha)} = \int_{G^{k_\gamma}} \prod_{j=1}^{k_\gamma} e^{iq_j\theta_j}\, d\rho_\sigma(\theta_j) = e^{-(\sigma/2)\sqrt{\sigma/2\pi}\, \sum_j q_j^2} \tag{106}$$

for some $q^j \in \mathbf{Z}$ so that $l(\alpha)$ is real and positive, as claimed. Furthermore, since q_j is a linear function of the m_i, $l(\alpha) = \sum_{i,j} g_\gamma^{ij} m_i m_j$ for some symmetric matrix g_γ^{ij} defined by γ, ψ_γ and λ_γ. The matrix g_γ^{ij} defines a Laplacian operator $\Delta_\gamma = \sum_{i,j} g_\gamma^{ij}(\partial/\partial\theta_i)(\partial/\partial\theta_j)$ on G^{E_γ} and thus a Laplacian on $\mathscr{A}_\gamma$, and our transform is the corresponding coherent state transform on $\mathscr{A}_\gamma$. Consistency of our transform ensures that the Δ_γ are a consistent set of operators and that they define a Laplacian Δ on some dense domain in $L^2(\bar{\mathscr{A}}, \mu_0)$. Our transform is just the coherent state transform on $\bar{\mathscr{A}}$ defined by the heat kernel of the Laplacian Δ.

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