ADE GEOMETRY AND DUALITIES

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Lecture 2
ADE QUIVER REPRESENTATIONS AND BRANES

We return to the situation in the first lecture

$$\begin{array}{ccccccc} Z & \rightarrow & \mathcal{Z} & \rightarrow & \mathcal{X}\times_\rho V & \rightarrow & \mathcal{X} \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ \mathbf{C} & & \rightarrow & & V & \stackrel{\rho}{\rightarrow} & V/W, \end{array}$$

together with

$$\begin{array}{ccc} Z & \rightarrow & X \\ & \searrow & \downarrow \\ & & \mathbf{C} \end{array}$$

where certain curves in Z are contracted to obtain X. In this lecture, we describe how these curves can be identified with representations of certain quivers with relations.

These quivers originate from certain $N=1$ gauge theories arising from wrapped branes on the the cycles $S^2_{i,t}$ in string theory. The relations on the quiver come from the superpotential (F-term constraints).

Rather than explain in detail the physical origins of the particular quivers and relations, we for the most part content ourselves with writing down the quivers and relations, and explaining how this matches the geometry of contractible curves precisely.

A quiver is a directed graph, consisting of a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of directed edges. An edge $e\in\mathcal{E}$ is oriented from its initial vertex $i(e)\in\mathcal{V}$ to its final vertex $f(e)\in V$.

Let's consider a particular ADE singularity S and its associated Dynkin diagram. We construct a quiver from this Dynkin diagram. This is *not* the same as the quivers constructed by picking an orientation of each edge.

The set $\mathcal{V}$ of vertices of the quiver coincides with the set of vertices of the Dynkin diagram.

To each edge e of the Dynkin diagram connecting two vertices v_i, v_j, we associate two edges of the quiver connecting v_i and v_j, one in each direction. That is, an edge $e'\in\mathcal{E}$ with $i(e')=v_i$ and $f(e')=v_j$, together with an edge $e''\in\mathcal{E}$ with $i(e'')=v_i$ and $f(e'')=v_j$.

Finally, for each vertex v of the Dynkin diagram there is an edge e_v with $i(e_v) = f(e_v) = v$.

This quiver will be called an $N = 2$ *ADE quiver* after its physical origin.

Definition. A *representation* of a quiver described by $\mathcal{V}$, $\mathcal{E}$, is the assignment of a vector space W_v to each $v \in \mathcal{V}$ and for each edge $e \in \mathcal{E}$ a linear transformation $Q(e) \in \mathrm{Hom}(W_{i(e)}, W_{j(e)})$.

For example, if a traditional ADE quiver is constructed from the Dynkin diagram by choosing one orientation for each edge, then its simple representations are in 1-1 correspondence with the positive roots of $\tilde{S}$. If $C = \sum n_i C_i$ is such a root, then in the corresponding quiver representation we have $\dim W_i = n_i$. This is part of the content of Gabriel's theorem.

In the $N = 2$ ADE quiver, the vertices will be labelled by $1, \ldots, n$, and if e is an edge with $i(e) = i$ and $f(e) = j$, then we will usually rewrite the linear tranformation $Q(e)$ as Q_{ji}. If e_i is the edge with $i(e_i) = j(e_i) = i$, then we will usually rewrite the linear tranformation $Q(e_i)$ as Φ_i.

In string theory, wrapping N_i branes on $S^2_{i,t}$ gives an $N = 1$ $\prod_i U(N_i)$ gauge theory. This corresponds to a representation of the $N = 2$ ADE quiver with $N_i = \dim W_i$ for each i. The Q_{ji} are bifundamental fields in the $(N_j, \overline{N_i})$ representation, and the Φ_i are $U(N_i)$ adjoints.

We now impose relations on the quiver representations corresponding to the breaking to $N = 1$ supersymmetry in the physical theory. For each $i \neq j \in \mathcal{V}$ we choose s_{ij} such that

$$\begin{aligned} &s_{ij} = 0 \text{ if there is no edge connecting i, j} \\ &s_{ij} = \pm 1 \text{ if there is an edge connecting i, j} \\ &s_{ij} = -s_{ji} \end{aligned}$$

Furthermore, the map $\mathbf{C} \to V$ can be written as $\alpha_i(t) = P_i'(w)$ for $i = 1, \ldots, n$. Then the relations imposed are

$$\sum_i s_{ij} Q_{ji} Q_{ij} + P_j'(\Phi_j), \qquad Q_{ij}\Phi_j = \Phi_i Q_{ij}.$$

These relations in fact are the critical points of the superpotential

$$W(Q_{ij}, \Phi_i) = \sum s_{ij} \mathrm{Tr} Q_{ij} \Phi_j Q_{ji} + \mathrm{Tr} P_i'(\Phi_)$$

For A_n the relations can be written as

$$-Q_{i,i-1}Q_{i-1,i} + Q_{i,i+1}Q_{i+1,i} + P_i'(\Phi_i) = 0, \qquad Q_{ij}\Phi_j = \Phi_i Q_{ij}.$$

A_2 **Example.** We since the map $\mathbf{C} \to V$ is described by $t_i = f_i(w)$ for $i = 1, 2, 3$ with $\sum f_i = 0$, and we have

$$\alpha_1(t) = t_1 - t_2, \qquad \alpha_2(t) = t_2 - t_3$$

we put

$$P_1'(w) = f_1(w) - f_2(w), \qquad P_2'(w) = f_2(w) - f_3(w).$$

The relations are

$$\begin{aligned} Q_{12}Q_{21} + P_1'(\Phi_1) &= 0 \\ -Q_{21}Q_{12} + P_2'(\Phi_2) &= 0 \\ Q_{12}\Phi_2 &= \Phi_1 Q_{12} \\ Q_{21}\Phi_1 &= \Phi_2 Q_{21} \end{aligned}$$

Then we get three classes of simple quiver representations satisfying the required relations.

$$W_1 = \mathbf{C},\ W_2 = 0,\ Q_{12} = Q_{21} = \Phi_2 = 0,\ \Phi_1 = \lambda \cdot I, \qquad (f_1 - f_2)(\lambda) = 0$$

$$W_1 = 0,\ W_2 = \mathbf{C},\ Q_{12} = Q_{21} = \Phi_1 = 0,\ \Phi_2 = \lambda \cdot I, \qquad (f_2 - f_3)(\lambda) = 0$$

$$W_1 = W_2 = \mathbf{C},\ Q_{12} = I,\ Q_{21} = (f_2 - f_3)(\lambda) \cdot I,\ \Phi_1 = \Phi_2 = \lambda \cdot I, \qquad (f_1 - f_3)(\lambda) = 0.$$

To derive these representations, first look for representations where the $Q_{ij} = 0$. This requires only $(f_1 - f_2)(\Phi_1) = 0$, $(f_2 - f_3)(\Phi_2) = 0$. We then see that the only simple representations are those described in the first two classes above. If either $f_1 - f_2$ or $f_2 - f_3$ has multiple roots, then there may be other indecomposible (but not simple) quiver representations satisfying the required relations, e.g. putting $W_1 = \mathbf{C}^2,\ W_2 = 0$ and taking

$$\Phi_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $(f_1 - f_2)(\lambda) = 0$.

To derive the third class of representations, multiply the first relation by Q_{21} on the left and by Q_{12} on the right, and use the third relation to obtain

$$Q_{21}Q_{12}Q_{21}Q_{12} + Q_{21}Q_{12}P_1'(\Phi_2) = 0.$$

Putting $X_{12} = Q_{21}Q_{12}$, this can be written more simply as

$$X_{12}\left(X_{12} + P_1'(\Phi_2)\right) = 0.$$

The second relation now gives $X = P_2'(\Phi_2)$, so we arrive at the identity

$$P_2'(\Phi_2)\,(P_1' + P_2')\,(\Phi_2) = 0.$$

A similar argument gives

$$P_1'(\Phi_1)\,(P_1' + P_2')\,(\Phi_1) = 0.$$

Since $(P_1' + P_2')(w) = (f_1 - f_3)(w)$, the construction is readily completed from here.

Back to the general case, we introduce some more notation. Given a positive root C, we express it as $C = \sum n_i C_i$ with $n_i \geq 0$. Then we define the function $P_C'(w) = \sum n_i P_i'(w)$.

For example, in the A_2 case we have $P_{C_1+C_2}'(w) = (f_1 - f_2)(w) + (f_2 - f_3)(w) = (f_1 - f_3)(w)$.

Now suppose that all roots of $P_R'(w) = 0$ are simple, and that given two roots $R_1 \neq R_2$ in the root system, the set of roots of $P_1'(w) = 0$ is disjoint from the set of roots of $P_2'(w) = 0$.

Assertion. The isomorphism classes of $N = 2$ ADE quiver representations satisfying the given relations together with their morphisms form a semisimple category, and the exceptional curves of $Z \to X$ are in 1-1 correspondence with the simple representations.

This was explained using D-branes in [CKV], with illustrative examples. I expect that a complete proof will follow soon using an extension of the reflection functors associated to elements of the Weyl group as were used to prove Gabriel's theorem. An important point is that like the classical case, reflections construct new $N = 2$ quiver representations from old ones, but the relations are not preserved. Instead a different set of relations is satisfied, arising from the flop of Z along the corresponding curve. This is being completed by my thesis student X. Zhu. In the case of more general P_i', the indecomposable quiver representations correspond to certain sheaves supported on the exceptional curves.

Now we turn to the general case

$$\begin{array}{ccccccc} Y & \to & \mathcal{Y} & \to & \mathcal{X} \times_\rho V/W' & \to & \mathcal{X} \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ \mathbf{C} & & \to & & V/W' & \stackrel{\rho}{\to} & V/W, \end{array}$$

together with the contraction

$$\begin{array}{ccc} Y & \to & X \\ & \searrow & \downarrow \\ & & \mathbf{C} \end{array}$$

We let I index the vertices of the Dynkin diagram corresponding to the curves that are blown up to obtain $\bar{S}$. The complementary set of vertices will be denoted by J. The reflections corresponding to the C_j with $j \in J$ generate W'. A map

$$\mathbf{C} \to V/W'$$

can be given explicitly by choosing generators $\beta_1, \ldots, \beta_n$ for the algebra of W'-invariant functions on V and putting

$$\beta_i = f_i(w), \qquad i = 1, \ldots, n$$

for some holomorphic functions f_i on $\mathbf{C}$.

We now describe a quiver and relations for this situation. The vertices are the set I. We have two edges between i and j whenever i and j were adjacent in the original Dynkin diagram. We still have an edge e_i with $i(e_i) = f(e_i) = i$ for each $i \in I$. Finally, for each $i \in I$ and each edge $j \in J$ adjacent to i in the original Dynkin diagram, we add another edge e_{ji} from i to i.

In a representation, the linear transformations corresponding to the above classes of edges in order will be denoted by

$$Q_{ji} \in \mathrm{Hom}(W_i, W_j),\ \Phi_i \in \mathrm{Hom}(W_i, W_i),\ X_{ji} \in \mathrm{Hom}(W_i, W_i).$$

For the contractions $Z \to X$, the relations can be derived by a simple substitution in terms of a universal relation given by an ideal $\mathcal{R}$ in the algebra

$$\mathbf{C}\left\{Q_{ij}, \Phi_j\right\}[V^*]$$

e.g.

$$\mathcal{R} = (-Q_{i,i-1}Q_{i-1,i} + Q_{i,i+1}Q_{i+1,i} + t_i - t_{i+1}, Q_{ij}\Phi_j - \Phi_i Q_{ij})$$

in the A_n case. We obtain a relation in $\mathrm{Hom}(W_i, W_i)$ from an element of $r \in \mathcal{R}$ beginning and ending at node i by substituting $t_j - t_{j+1} = f_j(\Phi_i)$ for all j.

Given any polynomial functions p_{ij} on V, we can form shifted operators $\tilde{X}_{ij} = X_{ij} + p_{ij}$ and then map

$$\mathbf{C}\{\tilde{X}_{ij}, Q_{ij}\}[V^*] \subset \mathbf{C}\{Q_{ij}, \Phi_i\}[V^*], \qquad \tilde{X}_{ij} \mapsto Q_{ji}Q_{ij} + p_{ij}$$

and put

$$\mathcal{R}' = \mathcal{R} \cap \mathbf{C}\{\tilde{X}_{ij}, Q_{ij}\}[V^*]^{W'}.$$

Assertion. There exist linear functions p_{ij} such that in the generic case, the quiver representations together with their morphisms form a semisimple category, and the exceptional curves of $Y \to X$ are in 1-1 correspondence with the simple representations.

A more general assertion can be made in all cases, involving sheaves supported on the exceptional curves.

Example.

A_2 with only the second vertex blown up. We have one vertex and two edges, whose representations are labeled by Φ_2 and X_{12}. We choose coordinates $t = t_3$ and $\sigma = t_1 t_2$ on V/W', $W' = (1,2) \subset S_3$.

The second relation imposed on the full quiver can be rewritten as $-X_{12} + P_2'(\Phi_2) = 0$, so $-X_{12} + t_2 - t_3 \in \mathcal{R}$. Furthermore, from what we have computed before, $X_{12}\,(X_{12} + t_1 - t_2) \in \mathcal{R}$.

Putting $\tilde{X}_{12} = X_{12} - t_2$, these elements of $\mathcal{R}$ can be rewritten as

$$-\tilde{X}_{12} - t_3 = -\tilde{X}_{12} - t, \qquad \left(\tilde{X}_{12} + t_1\right)\left(\tilde{X}_{12} + t_2\right) = \tilde{X}_{12}^2 - t\tilde{X}_{12} + \sigma$$

which are clearly W'-invariant and so give elements of $\mathcal{R}'$.

A threefold can be constructed by substituting $t = f_1(w),\ \sigma = f_2(w)$ for some functions $f_1,\ f_2$. The equation of X in $\mathbf{C}^4$ is then

$$xy + (z + f_1(w))(z^2 - f_1(w)z + f_2(w)) = 0,$$

and Y is obtained from X by blowing up the ideal $(x, z^2 - f_1(w)z + f_2(w))$. So the only exceptional curves appear over the locus in X given by $x = y = z^2 - f_1(w)z + f_2(w) = z + f_1(w) = 0$. These last two equations imply

$$z = -f_1(w), \qquad f_2(w) + 2\left(f_1(w)\right)^2 = 0.$$

From the elements of $\mathcal{R}'$ given above, the quiver represention has to satisfy

$$-\tilde{X}_{12} - f_1(\Phi_2) = 0,\ \tilde{X}_{12}^2 - f_1(\Phi_2)\tilde{X}_{12} + f_2(\Phi_2) = 0.$$

These coincide with the geometric equations for the exceptional curves after an obvious change of variables. The simple quiver representations clearly arise by choosing λ to be a solution of $(f_2+2f_1^2)(w)=0$ for w, and then taking a representation with $W_2=\mathbf{C}$ and maps $X_{21}=-f_1(\lambda)\cdot 1$ and $\Phi_2=\lambda\cdot 1$.

References

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