ADE GEOMETRY AND DUALITIES

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Abstract: In these lectures, I will explain the geometry of ADE surface singularities, their resolutions, and their deformations, interpret this geometry in gauge theory via quiver representations, and apply these results to N=1 dualities.

1 ADE singularities, deformations, and resolutions

1.1 The A_1 case

Consider an A_1 surface singularity S, analytically isomorphic to the hypersurface in $\mathbf{C}^3$ defined by the equation

$$xy = z^2.$$

This singularity can smoothed in two ways:

- Deforming the equation (complex structure deformation)
- Resolving (blowing up) the singularity (Kähler deformation)

The deformed equation may be taken to be

$$xy = z^2 + s$$

where $s \in \mathbf{C}$. These form a nice family of surfaces X_s over the parameter space $\mathrm{Def}(A_1) = \mathbf{C}$, with $X_0 = S$:

$$\begin{array}{ccc} \mathcal{X} & \subset & \mathrm{Def}(A_1) \times \mathbf{C}^3 \\ & & \downarrow \\ & & \mathrm{Def}(A_1), \end{array}$$

the hypersurface $\mathcal{X} \subset \mathrm{Def}(A_1) \times \mathbf{C}^3 \simeq \mathbf{C}^4$ being defined by the equation $xy = z^2 + s$.

This family admits a nowhere vanishing holomorphic 2-form

$$\omega = \frac{dx \wedge dy}{z} \in \Omega^2_{\mathcal{X}/\mathrm{Def}(A_1)}.$$

The singularity S can be resolved by blowing up the ideal (x,z) (or the maximal ideal (x,y,z)) to obtain a smooth surface $\pi : \tilde{S} \to S$. Explicitly, in one local patch we introduce a new variable $u = z/x$, and in the other patch the variable $v = x/z$. This eliminates z in the first patch and x in the second patch, and the respective equations become

$$y = u^2 x, \qquad vy = z$$

which are clearly smooth. We can take (u,x) (resp. (v,y)) as coordinates on the respective patches. The curve $C = \pi^{-1}(0)$ is isomorphic to $\mathbf{P}^1$ with u and v identified with the standard affine coordinates of $\mathbf{P}^1$. From the gluing maps

$$y = u^2 x, \qquad v = u^{-1}$$

of the two patches it is immediate to compute that $C^2 = -2$.

In the respective patches, we compute

$$\pi^*\omega = 2dx \wedge du, \qquad \pi^*\omega = 2dv \wedge dy,$$

which has no zeros. Thus $\pi^*\omega_S \simeq \omega_{\tilde{S}}$, and S is a canonical surface singularity.

It is not possible to replace S by $\tilde{S}$ and form a family of surfaces over $\mathrm{Def}(A_1)$, but this can be done after a Galois cover of the base, i.e. a change of variables. Substituting $s = -t^2$, we get the equation $xy = z^2 - t^2 = (z-t)(z+t)$. This singular family can be resolved by blowing up the ideal $(x, z-t)$. Explicitly, this gives two patches, one with a new variable $u = (z-t)/x$, the other with a new variable $v = x/(z-t)$. Again, z (resp. x) can be eliminated from the respective patches and we get smooth hypersurfaces with equations

$$y = u(ux + 2t), \qquad vy - t = z.$$

Alternatively, two copies of $\mathbf{C}^2 \times V$ with respective coordinates (u, x, t) and (v, y, t) are glued by the identifications

$$y = u(ux + 2t), \qquad v = u^{-1}, \qquad t = t.$$

In other words, $t \in \mathbf{C}$, not s, is the natural variable parametrizing deformations of $\tilde{S}$. For the purpose of generalizing, put $V = \mathbf{C}$. The map $s = -t^2$ expresses V as a Galois cover of $\mathrm{Def}(A_1)$ with Galois group $W = \mathbf{Z}_2$ generated by $t \mapsto -t$. So we can write $\mathrm{Def}(A_1)$ as V/W.

This construction can be summarized by the diagram

$$\begin{array}{ccccc} \mathcal{Z} & \to & \mathcal{X} \times_\rho V & \to & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \\ & & V & \xrightarrow{\rho} & V/W, \end{array}$$

where the square is a fiber square and $\mathcal{Z} \to \mathcal{X} \times_\rho V$ is the blowup constructed above.

The family $\mathcal{Z} \to V$ is called a *simultaneous resolution* of S. The surface Z_t corresponding to $t \in V$ is isomorphic to X_{-t^2} if $t \neq 0$ while Z_0 is $\tilde{S}$. The exceptional curve $C \subset Z_0$ is given in the respective patches by $x = t = 0$ (resp $y = t = 0$).

The simultaneous resolution can be used to construct noncompact Calabi-Yau threefolds. Let $f : \mathbf{C} \to V$ be holomorphic.[1] A threefold Z is constructed by pulling back $\mathcal{Z}$ via f, i.e. via the fiber square

$$\begin{array}{ccc} Z & \to & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathbf{C} & \stackrel{f}{\to} & V \end{array}$$

Similarly, a possibly singular threefold X can be constructed by pulling back $\mathcal{X} \times_\rho V$ via f:

$$\begin{array}{ccc} X & \to & \mathcal{X} \times_\rho V \\ \downarrow & & \downarrow \\ \mathbf{C} & \stackrel{f}{\to} & V, \end{array}$$

and the map $\mathcal{Z} \to \mathcal{X} \times_\rho V$ induces a map $Z \to X$.

[1]More general and more local constructions can be obtained from holomorphic maps $f : \Delta \to V$ with Δ a complex disk, but we choose $\mathbf{C}$ as the domain of f for application to physics.

Explicitly, we let w be a coordinate on $\mathbf{C}$ and describe f as $t = f(w)$. Then Z is constructed by taking two copies of $\mathbf{C}^3$ with respective coordinates (u, x, w) and (v, y, w) and identifying them via

$$y = u(ux + 2f(w)), \qquad v = u^{-1}, \qquad w = w.$$

Note that Z contains a curve isomorphic to $\mathbf{P}^1$ lying over each solution of $f(w) = 0$. These curves are all contracted by the map $Z \to X$. The explicit equation of X as a hypersurface in $\mathbf{C}^4$ is

$$xy = z^2 - f(w)^2.$$

It can be checked immediately that $du \wedge dx \wedge dw = -dv \wedge dy \wedge dw$, patching to give a global holomorphic 3-form Ω on Z. Thus Z is a (non-compact) Calabi-Yau threefold, which here means a smooth threefold with a nowhere vanishing holomorphic 3 form.

If $f(w) = w$, then X has a node (conifold singularity) at the origin, and $Z \to X$ is a small resolution of X. Note that if instead we take $f(w) = -w$, then we get the same X, but a different small resolution Z'. The birational map $Z - - \to Z'$ is called a *flop*. More generally, given any Z constructed from $f : \mathbf{C} \to V$, we can flop Z by constructing a new Z', using $-f$ in place of f.

Since $\mathcal{Z}$ is a smooth family, all fibers are diffeomorphic. In particular, the exceptional curve $C \simeq \mathbf{P}^1 \simeq S^2$ deforms to a topological 2 sphere S_t^2 in any Z_t. This can be seen explicitly if t is real by changing variables to write Z_t as $x^2 + y^2 + z^2 = t^2$ and then letting x, y, z be real. Put

$$\alpha(t) = \int_{S_t^2} \pi^* \omega.$$

Note that $\alpha(0) = 0$, since

$$\alpha(0) = \int_{S_0^2} \pi^* \omega = \int_C \pi^* \omega = \int_{\pi(C)} \omega = 0$$

since $\pi(C)$ is a point. The quantity $\alpha(t)$ is called the *holomorphic volume* of S_t^2. Its vanishing is necessary and sufficient for the homology class $[S_t^2] \in H^2(Z_t, \mathbf{Z})$ to be representable by a holomorphic cycle. If $t \neq 0$, the cycle $S_t^2 \subset Z_t$ is called a *vanishing cycle*.

1.2 General ADE singularities

Let's generalize the A_1 case. ADE surface singularities are ubiquitous in algebraic geometry and physics, and there are many ways to characterize them. Perhaps the simplest way is to describe these as canonical surface singularities, so that the minimal resolution

$$\pi : \tilde{S} \to S$$

satisfies

$$\pi^* \omega_S \simeq \omega_{\tilde{S}},$$

where ω_X denotes the dualizing sheaf of the variety X.

In particular, if S has trivial canonical bundle, then $\tilde{S}$ is smooth and has trivial canonical bundle.

Each of the ADE surface singularities can be described as a hypersurface in $\mathbf{C}^3$ with a double point. In fact, the ADE singularities are precisely the hypersurfaces with a *rational double points*, and this another way to characterize these singularities.

The equations of the ADE singularities in all cases are

S	Defining Equation
A_n	$xy = z^{n+1}$
D_n $(n \geq 4)$	$x^2 + y^2 z + z^{n-1} = 0$
E_6	$x^2 + y^3 + z^4 = 0$
E_7	$x^2 + y^3 + yz^3 = 0$
E_8	$x^2 + y^3 + z^5 = 0$

In this form, the singularity is at the origin. In general, the singular point will be denoted by $p \in S$.

The ADE singularities can either be resolved or deformed. First let's consider deformations.

For a general $S \subset \mathbf{C}^3$ defined by a polynomial F, choose representative polynomials $G_1, \ldots, G_r$ for a basis of

$$\mathbf{C}[x,y,z]/(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}).$$

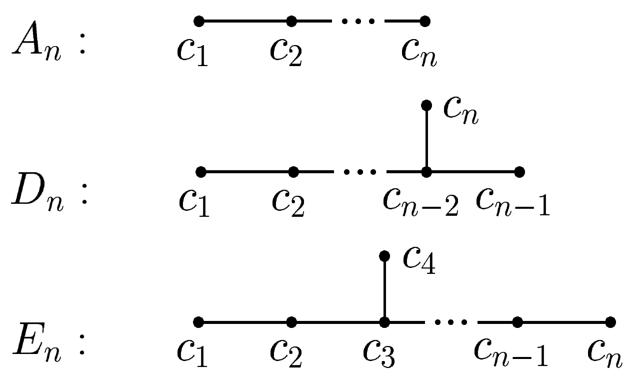
Then $\Phi := F + \mu_1 G_1 + \cdots + \mu_r G_r$ will define a semuniversal deformation of S as a hypersurface in $\mathbf{C}^{3+r}$ with coordinates $(x, y, z, \mu_1, \ldots, \mu_r)$.

Explicitly for the ADE singularities we have the semiuniversal deformations, renaming the deformation parameters μ_i

S	Equation
A_n	$xy = z^{n+1} + \Sigma_{i=2}^{n+1} \alpha_i z^{n-i} = 0$
D_n	$x^2 + y^2 z + z^{n-1} + \Sigma_{i=1}^{n-1} \delta_{2i} z^{n-i-1} + 2\gamma_n y = 0$
E_6	$x^2 + y^3 + z^4 + \epsilon_2 y z^2 + \epsilon_5 yz + \epsilon_6 z^2 + \epsilon_8 y$
	$+\epsilon_9 z + \epsilon_{12} = 0$
E_7	$x^2 + y^3 + yz^3 + \epsilon_2 y^2 z + \epsilon_6 y^2 + \epsilon_8 yz + \epsilon_{10} z^2$
	$+\epsilon_{12} y + \epsilon_{14} z + \epsilon_{18} = 0$
E_8	$x^2 + y^3 + z^5 + \epsilon_2 y z^3 + \epsilon_8 y z^2 + \epsilon_{12} z^3 + \epsilon_{14} yz$
	$+\epsilon_{18} z^2 + \epsilon_{20} y + \epsilon_{24} z + \epsilon_{30} = 0$

In each case, the parameter space is n-dimensional. As we will see, the parameter space can be identified with the quotient of the corresponding root space by the associated Weyl group, and the parameters $\alpha_i, \delta_{2i}, \gamma_n, \epsilon_i$ have been named so that under this identification they are homogenous polynomials on the root space, with degree given by their subscripts. For example, in the motivating case of A_1, we have $\alpha_2 = -t^2$.

Now let S be an ADE singularity. For the minimal resolution $\pi : \tilde{S} \to S$ we have $\pi^{-1}(p) = C_1 \cup \ldots \cup C_n$. The curves C_i are the *exceptional curves*, and each is isomorphic to $\mathbf{P}^1$. Since the canonical bundle of $\tilde{S}$ is trivial, get get $C_i^2 = -2$ by adjunction. Furthermore, the C_i intersect each other transversely in the configuration of the associated Dynkin diagram. A vertex is given for each C_i and an edge is drawn between the vertices corresponding to C_i to C_j if the curves C_i and C_j meet (and intersecting curves intersect at precisely one point).



Let S be one of the ADE singularities and $\pi : \tilde{S} \to S$ its minimal resolution. Put $V = H^2(\tilde{S}, \mathbf{C})$. Then V together with its intersection form is a (complexified) root system of the type that that singularity is named for, with root lattice $H^2(\tilde{S}, \mathbf{Z})$. The curves $C_1, \ldots, C_n$ are identified with a choice of positive simple roots in this root system. In particular, this means that all positive roots C can be expressed as a linear combination $C = \Sigma_i n_i C_i$ with $n_i \in \mathbf{Z}_{\geq 0}$. Geometrically, this means that the cycle C satisfies $C^2 = -2$, so that with the natural scheme structure defined by $\Pi_i \mathcal{I}_{C_i}^{n_i}$, the subscheme $C \subset \tilde{S}$ has arithmetic genus 0 by adjunction. Here $\mathcal{I}_{C_i}$ is the ideal sheaf of C_i in $\tilde{S}$.

Each root C generates a reflection r_C in the hyperplane in V orthogonal to C

$$r_C : V \to V, \qquad r_C(D) = D + (D \cdot C)C.$$

The *Weyl group* $W \subset \mathrm{Aut}(V)$ is the group generated by the r_{C_i}. It is a finite group.

The Weyl group of A_n is the symmetric group S_{n+1}. This can be seen explicitly as follows. Embed $V \subset \mathbf{C}^{n+1}$ by mapping C_i to $e_i - e_{i+1}$, where the $\{e_i\}$ are the standard basis for $\mathbf{C}^{n+1}$. Let $\sigma \in S_{n+1}$ denote the transposition $(i, i+1)$. Then we compute

$$r_{C_i}(e_j - e_{j+1}) = e_{\sigma(j)} - e_{\sigma(j+1)}.$$

Thus r_{C_i} acts as the transposition $(i, i+1)$, and the assertion follows since these tranpositions generate S_{n+1}.

We can now give the main geometric assertion.

Given the semi-universal deformation $\mathcal{X}$ of S parametrized by V/W as described explicitly above, a simultaneous resolution $\mathcal{Z}$ exists over V, so that the family $\mathcal{Z}$ is the semi-universal deformation of $\tilde{S}$.

The situation can be summarized by the diagram

$$\begin{array}{ccccccc} \mathcal{Z} & \rightarrow & \mathcal{X} \times_\rho V & \rightarrow & \mathcal{X} & \subset V/W \times \mathbf{C}^3 \\ & \searrow & \downarrow & & \downarrow & \\ & & V & \stackrel{\rho}{\rightarrow} & V/W & \end{array}$$

As in the A_1 case, we can now pull back the simultaneous resolution via a holomorphic map $f : \mathbf{C} \to V$, obtaining a smooth threefold Z over $\mathbf{C}$ which maps to a possibly singular threefold X. For any w with $f(w) = 0$, there will be an ADE configuration of curves $C_1 \cup \ldots \cup C_n$ lying over w.

We illustrate by constructing the simultaneous resolution for A_n. We have identified

$$V = \left\{(t_1, \ldots, t_{n+1}) \in \mathbf{R}^{n+1} \mid \sum t_i = 0\right\}$$

Recall the equation for $\mathcal{X}$

$$xy = z^{n+1} + \sum_{i=2}^{n+1} \alpha_i z^{n-i}.$$

We make the substitution $\alpha_i = \sigma_i(t_1, \ldots, t_{n+1})$, where the σ_i are the usual elementary symmetric functions. Note that this generalizes the A_1 case. The t_i are naturally functions on V, and the α_i are functions on V/W, which are homogeneous of degree i when thought of as functions on V.

We construct $\mathcal{Z}$ from $\mathcal{X} \times_\rho V$ as the closure of the graph of the rational map

$$\begin{array}{rcl} \mathcal{X} \times_\rho V & \to & (\mathbf{P}^1)^n \\ (x, y, z, t_1, \ldots, t_{n+1}) & \mapsto & [x, \Pi_{\nu=1}^{i}(z + t_\nu)]_{i=1,\ldots,n} \end{array}$$

Let us now consider in the general case a threefold Z constructed from a map $f : \mathbf{C} \to V$

$$\begin{array}{ccc} Z & \to & X \\ & \searrow & \downarrow \\ & & \mathbf{C} \end{array}$$

Let C be a root. A fundamental result is that the cycle C (with multiplicities) appears in the fiber of Z over $w \in \mathbf{C}$ if and only if $f(w) \cdot C = 0$, i.e. if and only if $f(w)$ lies in the hyperplane orthogonal to C. Furthermore, in such a case, inside the fiber of Z over w (which by construction is diffeomorphic to $\tilde{S}$), the fiber of $Z \to X$ contains a holomorphic cycle in the homology class $[C]$.

Let's say a few more words for A_2. We write $f : \mathbf{C} \to V \subset \mathbf{C}^3$ as

$$f = (f_1, f_2, f_3) : \mathbf{C} \to \mathbf{C}^3, \qquad f_1 + f_2 + f_3 = 0.$$

Then $X \subset \mathbf{C} \times \mathbf{C}^3$ is defined by the equation

$$xy = (z + f_1(w))(z + f_2(w))(z + f_3(w)).$$

We obtain Z as the closure of the graph of the rational map

$$X \to \left(\mathbf{P}^1\right)^2,$$

$$(x, y, z, w) \mapsto ([x, z+f_1(w)], [x, (z+f_1(w))(z+f_2(w))]).$$

Using the identities

$$[x, z + f_1(w)] = [(z + f_2(w)(z + f_3(w), -y],$$

$$[x, (z + f_1(w))(z + f_2(w))] = [z + f_3(w), -y],$$

we see that the indeterminacy locus of this map is given by

$$x = y = 0,\ z = -f_i(w),\ f_i(w) = f_j(w)$$

for any $i \neq j$. So any of these cases gives an exceptional curve in Z.

If $f_1(w) = f_2(w)$, then $f(w)$ is orthogonal to C_1 and we get an exceptional curve homologous to C_1 in its fiber.
If $f_2(w) = f_3(w)$, then $f(w)$ is orthogonal to C_2 and we get an exceptional curve homologous to C_2 in its fiber.
If $f_1(w) = f_3(w)$, then $f(w)$ is orthogonal to the root $C_1 + C_2$ and we get an exceptional curve homologous to $C_1 + C_2$ in its fiber.

Since $H^2(\tilde{S}, \mathbf{Z}) \simeq \mathbf{Z}^n$, we have n vanishing cycles $S^2_{i,t} \subset Z_t$, $i = 1, \ldots, n$. For an appropriate family of 2-forms $\omega^2_{\mathcal{Z}/V}$ we put

$$\alpha_i(t) = \int_{S^2_{i,t}} \omega.$$

Here $t \in V$. Note that $\alpha_i(t)$ vanishes when the cycle $S^2_{i,t}$ is holomorphic. As remarked above, this happens when t is in the hyperplane orthogonal to the root C_i. We can choose ω such that in the A_n case we have

$$\alpha_i = t_i - t_{i+1}$$

with the conventions above. In the D_n case we have

$$\alpha_i = t_i - t_{i+1},\ i = 1, \ldots, n-1, \qquad \alpha_n = t_{n-1} + t_n,$$

where $\mathcal{X} \times_\rho V$ is given by the equation

$$x^2 + y^2 z + \frac{\Pi_{i=1}^n (z + t_i^2) - \Pi_{i=1}^n t_i^2}{z} + 2 \prod_{i=1}^n t_i y,$$

and the blowup $\mathcal{Z} \to \mathcal{X} \times_\rho V$ can be given explicitly.

The E_n descriptions can also be given explicitly by explicit formulas for the $\alpha_i(t)$ and an explicit geometric construction in lieu of an explicit equation.

There is a more general construction obtained from *partial simultaneous resolutions* $\bar{S}$ of S. Instead of blowing up all of $C_1, \ldots, C_n$, we pick a subset $\{C_i \mid i \in I\}$ of the C_i, and contract the curves $\{C_j \mid j \notin I\}$ in $\tilde{S}$ to obtain a surface $\bar{S}$ (with ADE singularities) and a birational morphism $\bar{S} \to S$. Let $W' \subset W$ be the subset of W generated by the reflections $\{r_{C_i} \mid i \in I\}$. It can be shown that V/W' is smooth (and is in fact isomorphic to $\mathbf{C}^n$), and that there is a diagram

$$\begin{array}{ccccccc} \mathcal{Y} & \to & \mathcal{X} \times_\rho V/W' & \to & \mathcal{X} & \subset V/W \times \mathbf{C}^3 \\ & \searrow & \downarrow & & \downarrow & \\ & & V/W' & \stackrel{\rho}{\to} & V/W & \end{array}$$

where the fibers of $\mathcal{Y}$ are partial resolutions of the fibers of $\mathcal{X}$, the fiber over the origin being the partial resolution $\bar{S}$. Furthermore, $\mathcal{Y}$ is a semi-universal deformation of $\bar{S}$.

A threefold Y can be constructed from a holomorphic map $f : \mathbf{C} \to V/W'$ by pulling back $\mathcal{Y}$. Pulling back $\mathcal{X} \times_\rho V/W'$ gives a threefold X, and there is an induced map $Y \to X$ contracting some curves. The threefold Y may be singular, but will be smooth if f is carefully chosen.

References

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