NOTES ON LAGRANGIAN FIBRATIONS

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Abstract. These notes are written to complement a series of lectures on Lagrangian fibrations that will take place at IST from February 28 to April 3, 2012. As such, they only cover a few topics viewed from a very specific point of view (*i.e.* understanding the symplectic topology of completely integrable Hamiltonian systems). At the moment they are work in progress, so please feel free to comment on versions of this draft.

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1. Motivation

2. Definition and examples

Proviso. Throughout these notes we work in the C^∞ category unless otherwise stated. Moreover, all manifolds are assumed to be Hausdorff.

Definition 2.1. Let (M, ω) be a symplectic manifold. A *Lagrangian fibration* is a surjective map $\pi : (M, \omega) \to B$ whose regular fibres are Lagrangian submanifolds of (M, ω).

Remark 2.2. Definition 2.1 implies that $\dim M = 2 \dim B$.

2.1. **Examples.**

2.1.1. *The cotangent bundle.* Let B be any manifold and consider its cotangent bundle T^*B endowed with the canonical symplectic form Ω_{can} (cf. [MS95]). The footpoint projection

$$\text{pr} : (T^*B, \Omega_{\text{can}}) \to B$$

is a Lagrangian fibration (in fact, a fibre bundle). Note that the zero section is Lagrangian: in fact, all sections $\alpha : B \to T^*B$ which are closed 1-forms are Lagrangian. While this is a very simple example, it is central to understanding the topology and symplectic geometry of Lagrangian fibrations (cf. Section 2.2).

This example can be generalised slightly by considering *magnetic terms.* Fix $[\kappa] \in H^2(B; \mathbb{R})$ and let κ be a closed 2-form on B whose cohomology class is $[\kappa]$. Then the 2-form $\Omega_{\text{can}} + \text{pr}^*\kappa$ is closed and non-degenerate on T^*B and it makes the footpoint projection into a Lagrangian fibration.

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Exercise 1. The Lagrangian fibration

$$\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}} + \mathrm{pr}^*\kappa) \to B$$

admits a Lagrangian section $s : B \to \mathrm{T}^*B$ if and only if $[\kappa] = 0$.

Notation. A Lagrangian fibration which is a fibre bundle is called a Lagrangian *bundle.*

2.1.2. *Local model of Lagrangian fibre bundles with compact fibres.* Consider the Lagrangian bundle $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$ and let $a^1, \ldots, a^n$ be the standard coordinates on $\mathbb{R}^n$. Fix a trivialisation $\mathrm{T}^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, so that there exist coordinates $a^1, \ldots, a^n, p^1, \ldots, p^n$ on $\mathbb{R}^n \times \mathbb{R}^n$ with

$$\Omega_{\mathrm{can}} = \sum_{i=1}^{n} \mathrm{d}a^i \wedge \mathrm{d}p^i.$$

Define a $\mathbb{Z}^n$-action on $\mathrm{T}^*\mathbb{R}^n$

$$\begin{aligned} \mathbb{Z}^n \times \mathrm{T}^*\mathbb{R}^n &\to \mathrm{T}^*\mathbb{R}^n \\ (\mathbf{k}, (\mathbf{a}, \mathbf{p})) &\mapsto (\mathbf{a}, \mathbf{p} + \mathbf{k}). \end{aligned} \tag{1}$$

The action of equation (1) is free and properly discontinuous, so that the quotient $\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n$ is a smooth manifold. Moreover, it preserves the canonical symplectic form Ω_{can}; thus $\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n$ inherits a symplectic form ω_0. Finally, the above action preserves the footpoint projection pr; this map descends to a well-defined map

$$(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n, \omega_0) \to \mathbb{R}^n$$

whose fibres are Lagrangian tori.

Exercise 2. Check the above claims.

This example provides the local model for Lagrangian bundles with compact and connected fibres. It turns out that if B is the base of such a bundle, then B inherits the structure of an *integral affine manifold*, *i.e.* a (maximal) atlas $\mathcal{A}$ whose changes of coordinates, on each connected component, are restrictions of elements of the group

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n$$

of integral affine transformations of $\mathbb{R}^n$ (cf. Section 4).

2.1.3. *Completely integrable Hamiltonian systems.*

Definition 2.3. Let (M, ω) be a $2n$-dimensional symplectic manifold. A *completely integrable Hamiltonian system* (CIHS for short) on (M, ω) is a map (called *momentum map*

$$(f_1, \ldots, f_n) : (M, \omega) \to \mathbb{R}^n$$

whose components satisfy

- (involutivity) $\{f_i, f_j\} = 0$ for all i, j;
- (functional independence) $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n \neq 0$ almost everywhere on M[1].

There are several reasons to study CIHS, coming from within and outwith mathematics. For instance, CIHS can be used to model interesting physical and chemical problems which have 'real life' applications (cf. [ELS09]). In particular, it is important to provide scientists with appropriate tools to study the dynamical properties of these systems, to determine whether two such systems are equivalent (up to some specified notion) and to deal with motion around *singularities*, *i.e.* fibres of the momentum map which contain critical points. On the other hand, from a mathematical point of view, a CIHS is a Hamiltonian $\mathbb{R}^n$-action on (M, ω); as such, they generalise toric manifolds and present a much richer (but harder to understand) structure due to the non-compactness of the group acting. For instance, in general it is not true that the fibres of the momentum map are connected, unlike the toric case. Moreover, the types of singularities that can arise are much more complicated than in the toric case, as illustrated in some examples below.

[1] Note that this condition is loosely stated here as the conventions vary depending on the point of view taken to study integrable systems. For the purposes of these notes, 'almost everywhere on M' means 'on a dense open subset of M'.

Seeing as singularities of CIHS play an important role in the study of Lagrangian fibrations, it may be helpful to present a few of the simplest examples which are discussed in further detail in 5.

i) *Elliptic*: Consider the Lagrangian fibration given by

$$(2) \qquad \begin{aligned} (\mathrm{T}^*\mathbb{R}^2, \Omega_{\mathrm{can}}) &\to \mathbb{R} \\ (a,p) &\mapsto a^2 + p^2. \end{aligned}$$

The only critical point is the origin, which is, in fact, a fibre, while all other fibres are diffeomorphic to S^1. By taking products of the fibration of equation (2), it is possible to construct all local models for the singularities of toric manifolds (cf. and [Sym03]).

ii) *Hyperbolic*: Consider the Lagrangian fibration

$$(3) \qquad \begin{aligned} (\mathrm{T}^*\mathbb{R}^2, \Omega_{\mathrm{can}}) &\to \mathbb{R} \\ (a,p) &\mapsto ap. \end{aligned}$$

Like the elliptic case above, there is only one critical point at the origin, but the fibre on which it lies is not compact. Moreover, all fibres have two connected components, while the critical fibre (*i.e.* the one containing the origin) with the origin removed has four connected components. The topology of this singularity is more complicated than in the elliptic case and, as such, it makes the construction of Lagrangian fibrations with this type of singularity harder. However, it is important to remark that the vast majority of CIHS have both elliptic and hyperbolic singularities.

iii) *Focus-focus*: Consider the following (holomorphic) Lagrangian fibration

$$(4) \qquad \begin{aligned} (\mathbb{C}^2, \mathrm{Re}(\mathrm{d}z \wedge \mathrm{d}w)) &\to \mathbb{C} \\ (z,w) &\mapsto zw. \end{aligned}$$

In this example the momentum map is given by the real and imaginary parts of the above complex function. In some sense this singularity is the complex analogue of the hyperbolic case presented above; as such, all fibres are connected and the fibre containing the only critical point consists of two Lagrangian discs in $\mathbb{C}^2$ transversely intersecting. Focus-focus singularities have been central in the development of the theory of Lagrangian fibrations as their presence indicates non-trivial dynamical phenomena (cf. [DDSZ09]). In particular, Duistermaat's paper [Dui80] on the local structure of Lagrangian fibrations was motivated by the presence of such a singularity in the case of the integrable system given by a spherical pendulum. On the other hand, these singular points occur naturally in the study of Lagrangian fibrations from the point of view of mirror symmetry (cf. [CB04]) and exhibit interesting symplectic invariants (cf. Section 5 and [Ngo03]).

2.2. **A simple example.** In this section the cotangent bundle

$$\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$$

is studied in further details, in order to present some important properties enjoyed by a large family of Lagrangian fibrations which allow to develop a classification theory (cf. Section 3).

2.2.1. *Infinitesimal fibrewise action.* Let $U \subset B$ be an open set and consider $\alpha : U \to \mathrm{T}^*U \subset \mathrm{T}^*B$ be a locally defined 1-form. By definition α is also a locally defined section of pr. The equation

$$(5) \qquad \iota(X_{\mathrm{pr}^*\alpha})\Omega_{\mathrm{can}} = \mathrm{pr}^*\alpha,$$

where ι denotes interior product, defines a local vector field on T^*U.

Claim 2.4. $X_{\mathrm{pr}^*\alpha} \in \ker \mathrm{pr}_*$.

Proof. Let $q^1, \ldots, q^n$ be local coordinates on U and set

$$\alpha = \sum_{i=1}^n \alpha_i \mathrm{d}q^i$$

for some smooth functions $\alpha_i : U \to \mathbb{R}$, $i = 1, \ldots, n$. Then, by definition,

$$(6) \qquad X_{\mathrm{pr}^*\alpha} = \sum_{i=1}^n (\mathrm{pr}^*\alpha_i) X_{\mathrm{pr}^*\mathrm{d}q^i}.$$

Seeing as the condition that needs checking is pointwise, it suffices to prove that for all locally defined smooth functions $f \in \mathrm{C}^\infty(U)$, $X_{\mathrm{pr}^*\mathrm{d}f} \in \ker \mathrm{pr}_*$. Fix such a function and let $Y \in \ker \mathrm{pr}_*$. Since $\mathrm{pr}^* f$ is constant along the fibres of pr, it follows that $Y(\mathrm{pr}^* f) = 0$. On the other hand,

$$Y(\mathrm{pr}^* f) = (\mathrm{pr}^*(\mathrm{d}f))(Y) = \Omega_{\mathrm{can}}(X_{\mathrm{pr}^*\mathrm{d}f}, Y)$$

by definition of the vector field $X_{\mathrm{pr}^*\mathrm{d}f}$. Therefore

$$\Omega_{\mathrm{can}}(X_{\mathrm{pr}^*\mathrm{d}f}, Y) = 0; \tag{7}$$

since equation (7) holds for all $Y \in \ker \mathrm{pr}_*$, it follows that $X_{\mathrm{pr}^*\mathrm{d}f} \in (\ker \mathrm{pr}_*)^{\Omega_{\mathrm{can}}}$, where the upper script denotes the symplectic orthogonal with respect to the canonical form. The map pr defines a Lagrangian fibration, which means that $(\ker \mathrm{pr}_*)^{\Omega_{\mathrm{can}}} = \ker \mathrm{pr}_*$ and the result follows. □

Exercise 3.

i) If $(\mathbf{q}, \mathbf{p})$ are local canonical coordinates on T^*U, so that $q^1, \ldots, q^n$ are coordinates on U and

$$\Omega_{\mathrm{can}} = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p^i,$$

find a coordinate expression for $X_{\mathrm{pr}^*\alpha}$ for any $\alpha : U \to \mathrm{T}^*U$;

ii) Using the result of part (i), prove that for all functions $f, g \in \mathrm{C}^\infty(U)$,

$$[X_{\mathrm{pr}^*\mathrm{d}f}, X_{\mathrm{pr}^*\mathrm{d}g}] = 0,$$

where $[.,.]$ denotes the standard Lie bracket of vector fields. Hence, or otherwise, prove that for all $\alpha, \beta : U \to \mathrm{T}^*U$

$$[X_{\mathrm{pr}^*\alpha}, X_{\mathrm{pr}^*\beta}] = 0.$$

Using Claim 2.4 and the results of Exercise 3 it is possible to define a smooth fibrewise infinitesimal action of $\mathrm{T}^*B \to B$ on itself, defined by

$$\begin{aligned} \Gamma(\mathrm{T}^*B) &\to \Gamma(\mathrm{T}\mathrm{T}^*B) \\ \alpha &\mapsto X_{\mathrm{pr}^*\alpha}; \end{aligned} \tag{8}$$

note that, in fact, the image of the action is tangent to the fibres of the footpoint projection. The most appropriate framework to define this action is that of Lie algebroids (cf. Section 3), as each cotangent space (considered as an abelian Lie algebra) acts infinitesimally on itself by the above map. Moreover, the image of a local frame $\alpha_1, \ldots, \alpha_n$ of T^*B gives n linearly independent vector fields tangent to the fibres of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$ by non-degeneracy of Ω_{can}.

2.2.2. *Smooth fibrewise action.* Given an action of a finite dimensional Lie algebra $\mathfrak{g}$ on a manifold M it is natural to ask whether it can be integrated to an action of the corresponding Lie group G on M. This section shows that the above fibrewise infinitesimal action can, in fact, be integrated to a smooth action; this follows immediately once it is observed that, for any $\alpha : U \to \mathrm{T}^*U$, its flow ϕ^t_α is defined for all $t \in \mathbb{R}$, *i.e.* the image of the infinitesimal action of equation (8) consists of *complete* vector fields.

Exercise 4. Let $\alpha : U \to \mathrm{T}^*U$. Prove that the flow of $X_{\mathrm{pr}^*\alpha}$ is given by

$$\begin{aligned} \phi^t_\alpha : \mathrm{T}^*U &\to \mathrm{T}^*U \\ \beta &\mapsto \beta + t\alpha. \end{aligned} \tag{9}$$

Consider the *fibre product*

$$\mathrm{T}^*B \;{}_{\mathrm{pr}}\!\times_{\mathrm{pr}} \mathrm{T}^*B := \{(\alpha, \beta) \in \mathrm{T}^*B \times \mathrm{T}^*B \;:\; \mathrm{pr}(\alpha) = \mathrm{pr}(\beta)\};$$

it can be checked that this is a smooth manifold (since pr is a surjective submersion). The map

$$\begin{aligned} \mu : \mathrm{T}^*B \;{}_{\mathrm{pr}}\!\times_{\mathrm{pr}} \mathrm{T}^*B &\to \mathrm{T}^*B \\ (\alpha, \beta) &\mapsto \phi^1_\alpha(\beta) = \beta + \alpha \end{aligned} \tag{10}$$

defines a smooth fibrewise action of T^*B on the fibres of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$. Note that since the infinitesimal action of equation (8) sends a local frame of T^*B to a local frame of vector fields tangent to the fibres of pr, it follows that the fibrewise action defined by μ is transitive. As in the infinitesimal case, the best framework to describe μ as an action is using the language of Lie groupoids (cf. Section 3).

2.2.3. *Smooth trivialisations.* Since $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$ is a submersion, there exist local sections. Upon a choice of one such $\sigma : U \to \mathrm{pr}^{-1}(U)$, the map μ of equation (10) allows to define a smooth diffeomorphism between $\mathrm{pr}^{-1}(U)(= \mathrm{T}^*U)$ and T^*U which identifies σ with the zero section in T^*U. While this is almost tautological in this simple example, the underlying principle is the key to understanding the smooth classification of a large class of Lagrangian fibrations.

Pick a section σ as above and define a smooth map

$$\begin{aligned} \psi_\sigma : \mathrm{T}^*U &\to \mathrm{pr}^{-1}(U)(= \mathrm{T}^*U) \\ \beta &\mapsto \phi^1_\beta(\sigma \circ \mathrm{pr}(\beta))(= \beta + \sigma \circ \mathrm{pr}(\beta)); \end{aligned} \tag{11}$$

it is evident that this is a diffeomorphism which maps the zero section of $\mathrm{T}^*U \to U$ to the chosen section σ. Moreover, note that $\mathrm{pr} \circ \psi_\sigma = \mathrm{pr}$, *i.e.* the above diffeomorphism preserves the fibration defined by pr.

Remark 2.5. In general, the diffeomorphism ψ_σ does *not* yield a symplectomorphism. In fact, it can be checked that

$$\psi^*_\sigma \Omega_{\mathrm{can}} - \Omega_{\mathrm{can}} = \mathrm{pr}^* \mathrm{d}\sigma.$$

2.2.4. *Symplectic trivialisations.* In light of Remark 2.5, the diffeomorphism ψ_σ is a symplectomorphism if and only if $\mathrm{pr}^*\mathrm{d}\sigma = 0$. Since pr is a submersion, it follows that $\mathrm{pr}^*\mathrm{d}\sigma = 0$ if and only if $\mathrm{d}\sigma = 0$. However, this condition can be rephrased as stating that $\sigma : U \to \mathrm{pr}^{-1}(U)$ is a *Lagrangian section* of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$.

Exercise 5. Check that a 1-form $\sigma : U \to \mathrm{T}^*U$ is closed if and only if $\sigma^*\Omega_{\mathrm{can}} = 0$, *i.e.* if and only if σ is a Lagrangian section of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$.

Note that, locally, it is possible to pick a Lagrangian section of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$, as any locally defined closed section works. Fix such a choice $\sigma : U \to \mathrm{pr}^{-1}(U)$ and construct ψ_σ as in equation (11). By construction this map is a symplectomorphism between $(\mathrm{T}^*U, \Omega_{\mathrm{can}})$ and $(\mathrm{pr}^{-1}(U), \Omega_{\mathrm{can}})$ which maps the zero section to σ and satisfies $\mathrm{pr} \circ \psi_\sigma = \mathrm{pr}$.

3. Global topological and symplectic classification

In this section the topology and symplectic geometry of an important family of Lagrangian fibrations is studied. The aim is to generalise the Liouville-Mineur-Arnol'd theorem concerning the existence of action-angle coordinates in a neighbourhood of a compact and connected fibre of a Lagrangian fibration (cf. Theorem 3.1 below) and to study, in the spirit of Duistermaat's seminal work [Dui80], global consequences of this generalised theorem.

Assumption 1. Throughout this section, any Lagrangian fibration $\pi : (M, \omega) \to B$ has connected fibres and is a surjective submersion unless otherwise stated.

Suppose, in addition to the above assumption, that the fibration is proper. It follows from a theorem of Ehresmann (cf. [CB97]) that $\pi : (M, \omega) \to B$ is a fibre bundle with compact fibres. Then the following theorem holds.

Theorem 3.1 (Liouville-Mineur-Arnol'd, Duistermaat [Dui80])**.** *There exists an atlas $\mathcal{A} = \{U_i, \chi_i\}$ of B such that π can be trivialised over U_i via symplectomorphisms*

$$\varphi_i : (\pi^{-1}(U_i), \omega|_{\pi^{-1}(U_i)}) \to (\chi^*_i(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n), \chi^*_i\omega_0),$$

where $\mathrm{pr} : (\chi^*_i(\mathrm{T}^*\mathbb{R}^n/\mathbb{Z}^n), \chi^*_i\omega_0) \to U_i$ is the Lagrangian fibration obtained by pulling back Example 2.1.2 along the coordinate map $\chi_i : U_i \to \mathbb{R}^n$.

Remark 3.1. The Liouville-Mineur-Arnol'd theorem also describes the dynamics of a CIHS in a neighbourhood of a compact and connected fibre of the momentum map (cf. [Arn78]); while this result is important for mechanical purposes, it shall not be considered further in these notes.

As a corollary to the above theorem, obtain the following.

Corollary 3.2. *Under the above assumptions, the fibres of* $\pi : (M, \omega) \to B$ *are* n*-dimensional tori.*

Note that the trivialisations of Theorem 3.1 are given by quotients of the cotangent bundle by some smoothly varying lattice (which is also a Lagrangian submanifold of the cotangent space). In what follows the aim is to try and mimick this construction in a greater degree of generality. Furthermore, Theorem 3.1 also allows to describe the topological and symplectic classification of Lagrangian fibre bundles with compact and connected fibres (cf. [DD87, Dui80]).

3.1. **Infinitesimal action of** $\mathrm{T}^*B \to B$ **along** $\pi : (M, \omega) \to B$**.** Henceforth, fix a Lagrangian fibration $\pi : (M, \omega) \to B$. Recall that if (M, ω) is a symplectic manifold, ω induces a *Poisson bracket* $\{.,.\}_\omega$ on the space of (locally defined) smooth functions $\mathrm{C}^\infty(M)$ by setting, for all $f, q \in \mathrm{C}^\infty(M)$

$$\{f, g\} := \omega(X_{\mathrm{d}f}, X_{\mathrm{d}g}),$$

where $X_{def}, X_{\mathrm{d}g}$ are the Hamiltonian vector fields of the functions f, g, defined as in equation (5).

Definition 3.2. Let $(P_i, \{.,.\}_i)$ be Poisson manifolds for $i = 1, 2$ and let $F : P_1 \to P_2$ be a smooth map. Say that F is a *Poisson morphism* if for all $f, g \in \mathrm{C}^\infty(P_2)$

$$F^*\{f, g\}_2 = \{F^*f, F^*g\}_1.$$

The following proposition explains the importance of the Poisson structure induced by ω on the total space of $\pi : (M, \omega) \to B$; it is stated without proof (cf. [Vai94]).

Proposition 3.3. *The Lagrangian fibration* $\pi : (M, \omega) \to B$ *induces the zero Poisson structure on* B *so that the map*

$$\pi : (M, \{.,.\}_\omega) \to (B, 0)$$

is a Poisson morphism.

Thus B is henceforth considered as a Poisson manifold with trivial Poisson structure. Recall that any Poisson structure .,. on a manifold P defines a *Lie algebroid* structure on $\mathrm{pr} : \mathrm{T}^*P \to P$ (cf. [CF04]). In particular, associated to the zero Poisson structure on B there is a Lie algebroid structure on $\mathrm{pr} : \mathrm{T}^*B \to B$, which makes it into a bundle of abelian Lie algebras. The advantage of using the formalism of Lie algebroids is that there is a natural notion of action for such objects.

Let $U \subset B$ be an open set and consider a locally defined 1-form $\alpha : U \to \mathrm{T}^*U$. Define a vector field $X_{\pi^*\alpha} \in \Gamma(\mathrm{T}M)$ by setting

$$\iota(X_{\pi^*\alpha})\omega = \pi^*\alpha, \tag{12}$$

(cf. equation (5)). Note that, for all $b \in B$ and $m \in \pi^{-1}(b)$, equation (12) can be used to define a map

$$\mathrm{T}_b^*B \to \mathrm{T}_m M;$$

viewing T_b^*B as an abelian Lie algebra, the above map can be seen as an infinitesimal action of T_b^*B. The following proposition states that this pointwise infinitesimal action is in fact a smooth action of $\mathrm{T}^*B \to B$ on M.

Proposition 3.4. *The map*

$$\begin{aligned} A : \Gamma(\mathrm{T}^*B) &\to \Gamma(\mathrm{T}M) \\ \alpha &\mapsto X_{\pi^*\alpha} \end{aligned} \tag{13}$$

defines a Lie algebroid action of $\mathrm{T}^*B \to B$ *with structure induced by the zero Poisson structure on* B *along the Lagrangian fibration* $\pi : (M, \omega) \to B$.

Proof. It can be checked that the above statement is equivalent to proving the following four properties (cf. [MM03]): –

(i) for all $\alpha, \beta \in \Gamma(\mathrm{T}^*B)$, $X_{\pi^*(\alpha+\beta)} = X_{\pi^*\alpha} + X_{\pi^*\beta)}$;
(ii) for all $\alpha \in \Gamma(\mathrm{T}^*B)$, for all $f \in \mathrm{C}^\infty(B)$, $X_{\pi^*(f\alpha)} = (\pi^*f)X_{\pi^*\alpha}$;
(iii) for all $\alpha \in \Gamma(\mathrm{T}^*B)$, $X_{\pi^*\alpha} \in \ker \pi_*$;
(iv) for all $\alpha, \beta \in \Gamma(\mathrm{T}^*B)$, $[X_{\pi^*\alpha}, X_{\pi^*\beta}] = 0$.

Properties (i) and (ii) follow immediately from the definition of the map A. The proof of property (iii) is identical to the proof of Claim 2.4 and is thus omitted. It remains to prove property (iv). Choose local coordinates $q^1, \ldots, q^n$ on B and set

$$\alpha = \sum_{i=1}^{n} \alpha_i \mathrm{d}q^i \qquad \beta = \sum_{j=1}^{n} \beta_j \mathrm{d}q^j$$

for smooth functions α_i, β_j. Then

$$\begin{aligned}[X_{\pi^*\alpha}, X_{\pi^*\beta}] = \sum_{i,j=1}^{n} [(\pi^*\alpha_i)X_{\pi^*\mathrm{d}q^i}, (\pi^*\beta_j)X_{\pi^*\mathrm{d}q^j}] = \sum_{i,j=1}^{n} \Big((\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^*\mathrm{d}q^i}, X_{\pi^*\mathrm{d}q^j}] \\ + (\pi^*\alpha_i)(X_{\pi^*\mathrm{d}q^i}(\pi^*\beta_j))X_{\pi^*\mathrm{d}q^j} - (\pi^*\beta_j)(X_{\pi^*\mathrm{d}q^j}(\pi^*\alpha_i))X_{\pi^*\mathrm{d}q^i} \Big) \\ = \sum_{i,j=1}^{n} (\pi^*\alpha_i)(\pi^*\beta_j)[X_{\pi^*\mathrm{d}q^i}, X_{\pi^*\mathrm{d}q^j}]\end{aligned}$$

where the first equality follows from properties (i) and (ii) and the last from property (iii). Thus it suffices to show that for any $f, g \in \mathrm{C}^\infty(B)$, $[X_{\pi^*\mathrm{d}f}, X_{\pi^*\mathrm{d}g}] = 0$. Note that the homomorphism

$$\begin{aligned}\mathrm{C}^\infty(M) &\to \Gamma(\mathrm{T}M) \\ h &\mapsto X_{\mathrm{d}h}\end{aligned}$$

is a Lie algebra homomorphism with respect to the Poisson bracket $\{.,.\}_\omega$ and the Lie bracket $[.,.]$. In particular, this implies that for $f, g \in \mathrm{C}^\infty(B)$,

$$[X_{\pi^*\mathrm{d}f}, X_{\pi^*\mathrm{d}g}] = X_{\{\pi^* f, \pi^* g\}_\omega} = 0,$$

where the second equality follows from the fact that $\pi : (M, \{.,.\}_\omega) \to (B, 0)$ is a Poisson morphism. This completes the proof of property (iv) and of the proposition. □

Remark 3.3.

i) Note that what really lies at the heart of the proof of Proposition 3.4 is the Poisson structure on M rather than its symplectic structure;
ii) The action of equation (13) is *not* symplectic, *i.e.* the vector fields $X_{\pi^*\alpha}$ do not, in general satisfy $\mathcal{L}_{X_{\pi^*\alpha}}\omega = 0$;
iii) For each $b \in B$ and each $m \in \pi^{-1}(b)$, $A(\mathrm{T}^*_b B) = \ker \pi_*(m)$, since by Proposition 3.4, $A(\mathrm{T}^*_b B) \subset \ker \pi_*(m)$ and $\dim A(\mathrm{T}^*_b B) = \dim \mathrm{T}^*_b B$ as the symplectic form is non-degenerate. Since
$$\dim \ker \pi_*(m) = n = \dim \mathrm{T}^*_b B,$$
the claim follows.

Exercise 6. Find necessary and sufficient conditions on $\alpha \in \Gamma(\mathrm{T}^*B)$ so that $\mathcal{L}_{X_{\pi^*\alpha}} = 0$.

3.2. **Groupoid action of the cotangent bundle.** In this section, the Lie algebroid action of Proposition 3.4 is integrated to a *Lie groupoid* $\mathrm{T}^*B \rightrightarrows B$ action on M along $\pi : (M, \omega) \to B$. However, just as in the case of integration of infinitesimal Lie algebra actions on manifolds, some restriction on the topology of M is required so as to be able to integrate the vector fields of equation (13).

Assumption 2. Henceforth, consider only *complete* Lagrangian fibrations, *i.e.* for every compactly supported 1-form α on the base space the vector field $X_{\pi^*\alpha}$ constructed via equation (12) is complete.

Remark 3.4. The above assumption is reasonable, not only because it allows to integrate the Lie algebroid action of equation (13), but also because almost all Lagrangian fibrations arising from CIHS and in mirror symmetry are complete away from the singular points.

In light of the above assumption, for each compactly supported locally defined 1-form $\alpha : U \to \mathrm{T}^*U$, let $\phi^t_\alpha : \pi^{-1}(U) \to \pi^{-1}(U)$ be the flow of the vector field $X_{\pi^*\alpha}$. Note that for each covector $\alpha_b \in \mathrm{T}^*_b B$ there exists a compactly supported locally defined 1-form $\alpha : U \to \mathrm{T}^*U$ such that $\alpha(b) = \alpha_b$. The vector field

$X_{\pi^*\alpha}$ is tangent to the fibres of $\pi : (M, \omega) \to B$ by Proposition 3.4; therefore, its flow ϕ^t_α lies along the fibres of $\pi : (M, \omega) \to B$ for all $t \in \mathbb{R}$. Note that for each $m \in \pi^{-1}(b)$ the value of $X_{\pi^*\alpha}(m)$ only depends on α_b and not on the choice of compactly supported 1-form $\alpha : U \to \mathrm{T}^*U$ satisfying $\alpha(b) = \alpha_b$. Therefore, for each $\alpha_b \in \mathrm{T}^*_b B$, there is a well-defined diffeomorphism

$$\phi^1_{\alpha_b} := \phi^1_\alpha|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(b), \tag{14}$$

where $\alpha : U \to \mathrm{T}^*U$ is any compactly supported form such that $\alpha(b) = \alpha_b$. Note that, in fact, the assignment

$$\begin{aligned} \mathrm{T}^*_b B &\to \mathrm{Diff}(\pi^{-1}(b)) \\ \alpha_b &\mapsto \phi^1_{\alpha_b} \end{aligned} \tag{15}$$

is a Lie group homomorphism, where $\mathrm{T}^*_b B$ has the structure of an abelian Lie group.

The fibrewise Lie group structure of $\mathrm{T}^*B \to B$ yields a structure of Lie groupoid (in fact, simply a bundle of abelian Lie groups), with source and target maps being the footpoint projection pr, multiplication given by fibrewise addition of covectors, unit being the zero section and inversion given by taking negatives of covectors. Denote this Lie groupoid by $\mathrm{T}^*B \rightrightarrows B$. The following proposition merely states that the fibrewise actions defined by equation (15) for each $b \in B$ vary smoothly as $b \in B$ varies and is a simple consequence of Proposition 3.4 (for a definition of Lie groupoid action, see [MM03]).

Proposition 3.5. *Let $\pi : (M, \omega) \to B$ be a complete Lagrangian fibration. The following smooth map*

$$\begin{aligned} \mu : \mathrm{T}^*B \ {}_{\mathrm{pr}}\!\times_\pi M &\to M \\ (\alpha, m) &\mapsto \phi^1_\alpha(m) \end{aligned} \tag{16}$$

defines a smooth left-action of $\mathrm{T}^*B \rightrightarrows B$ on M along $\pi : (M, \omega) \to B$.

Remark 3.5.

i) The fibrewise action defined by equation (15) is, in fact, transitive, since the infinitesimal fibrewise action mapped $\mathrm{T}^*_b B$ onto $\mathrm{T}_m \pi^{-1}(b)$ for all $m \in \pi^{-1}(b)$ (cf. Remark 3.3.iii);

ii) Note further that since for all $b \in B$ the fibrewise action of equation (15) is by abelian Lie groups, the isotropy subgroups at points $m, m' \in \pi^{-1}(b)$ can be identified *canonically.* Therefore, it makes sense to consider the subgroup of *periods* $\Lambda_b \subset \mathrm{T}^*_b B$, defined by

$$\Lambda_b = \{\alpha_b \in \mathrm{T}^*_b B : \exists m \in \pi^{-1}(b) \text{ s.t. } \phi^1_{\alpha_b}(m) = m\}, \tag{17}$$

since if $\phi^1_{\alpha_b}(m) = m$ for some $m \in \pi^{-1}(b)$, then $\phi^{-1}_{\alpha_b} = \mathrm{id}$. Moreover, since $\dim \mathrm{T}^*_b B = \dim \pi^{-1}(b)$, for each $b \in B$ the subgroup Λ_b is discrete, *i.e.* isomorphic to $\mathbb{Z}^k$ for $k \leq n$.

Definition 3.6 ([DD87, Dui80, Vai94])**.** The subset

$$\Lambda := \coprod_{b \in B} \Lambda_b \subset \mathrm{T}^*B \tag{18}$$

is called the *period net* associated to the Lagrangian fibration $\pi : (M, \omega) \to B$.

Remark 3.7. When dealing with CIHS, the period net is also known as *period lattice* or *period lattice bundle*, since, in this case, the dimension of each period subgroup Λ_b is maximal and thus it forms a lattice inside $\mathrm{T}^*_b B$. The period lattice associated to a CIHS is one of its most important topological and symplectic invariants (cf. Sections 4, 5 and [Dui80, Ngo03]). The terminology used throughout these notes comes from the theory of symplectic realisations of regular Poisson manifolds (cf. [DD87, Vai94]).

The period net Λ associated to a Lagrangian fibration $\pi : (M, \omega) \to B$ plays a fundamental role in the topological and symplectic classification problems. *A priori*, it is not clear that Λ is a smooth submanifold of T^*B and, in fact, its smoothness follows from the fact that there exists of a Lagrangian fibration $\pi : (M, \omega) \to B$ associated to it (cf. [Zun03]).

Theorem 3.6. *Fix a complete Lagrangian fibration $\pi : (M, \omega) \to B$ and let Λ be the associated period net. Then*

i) Λ is a closed Lagrangian submanifold of T^*B;

ii) The quotient T^*B/Λ is a smooth manifold.

Proof. The idea of the proof is to show that Λ is given, locally, as the graph of some sections $\alpha_\sigma : U \to \mathrm{T}^*U$, where $\sigma : U \to \pi^{-1}(U)$ denotes a local section of $\pi : (M,\omega) \to B$.

Firstly, note that since $\pi : (M,\omega) \to B$ is a surjective submersion, then for each $b \in B$ there exists a local section $\sigma : U \to \pi^{-1}(U)$ defined in an open neighbourhood U of b. Fix such a section σ and consider the smooth map

$$\begin{aligned} \psi_\sigma : \mathrm{T}^*U &\to \pi^{-1}(U) \\ \alpha &\mapsto \phi^1_\alpha(\sigma \circ \mathrm{pr}(\alpha)) \end{aligned} \tag{19}$$

(cf. equation (11)). The claim is that ψ_σ is a local diffeomorphism at all points of T^*U. Note that since $\dim \mathrm{T}^*U = \dim \pi^{-1}(U)$, it suffices to prove that $\ker D\psi_\sigma(\alpha) = 0$ for all $\alpha \in \mathrm{T}^*U$. Fix a point $\alpha_0 \in \mathrm{T}^*U$ and begin by observing that if $X \in \mathrm{T}_{\alpha_0}\mathrm{T}^*U$ is tangent to the fibres of pr, then $D\psi_\sigma(\alpha_0)(X) = 0$ if and only if $X = 0$ by Remark 3.3.iii. Therefore, if $\ker D\psi_\sigma(\alpha_0)(Y) = 0$ and $Y \neq 0$, then $D\mathrm{pr}(\alpha_0)(Y) \neq 0$. Any such vector $Y \in \mathrm{T}_{\alpha_0}\mathrm{T}^*U$ is mapped to a non-zero vector $Y' \in \mathrm{T}_{\psi_\sigma(\alpha_0)}\pi^{-1}(U)$ such that $D\pi(\psi_\sigma(\alpha)_0) \neq 0$, since σ is an immersion and the action of equation (16) preserves the Lagrangian fibration $\pi : (M,\omega) \to B$. Thus $\ker D\psi_\sigma(\alpha_0) = 0$ as claimed.

Let $b_0 \in U$ and $\alpha_0 \in \Lambda_{b_0}$. By definition

$$\psi_\sigma(\alpha_0) = \sigma \circ \mathrm{pr}(\alpha_0).$$

Since ψ_σ is a local diffeomorphism, there exists an inverse ψ_σ^{-1} defined on an open neighbourhood $V \subset \pi^{-1}(U)$ of $\sigma \circ \mathrm{pr}(\alpha_0)$. By shrinking U if needed, may assume that $U = \pi(V)$. The composite

$$\alpha_\sigma = \psi_\sigma^{-1} \circ \sigma : U \to \mathrm{T}^*U \tag{20}$$

is a locally defined 1-form, since $\mathrm{pr} = \pi \circ \psi_\sigma$. This local section of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$ is what yields the smooth structure on Λ. By definition, for all $b \in U$

$$\sigma(b) = \psi_\sigma \circ \alpha_\sigma(b) = \phi^1_{\alpha_\sigma(b)}(\sigma(b));$$

therefore, for all $b \in U$, $\alpha_\sigma(b) \in \Lambda_b$. Set $W = \psi_\sigma^{-1}(V)$; since ψ_σ^{-1} is an open map, W is an open neighbourhood (diffeomorphic to V). The above argument shows that $\alpha_\sigma(U) \subset W \cap \Lambda$. If the reverse inclusion also holds, then Λ is a smooth submanifold of T^*B.

Suppose that $\beta \in W \cap \Lambda$; the aim is to show that $\beta = \alpha_\sigma(\mathrm{pr}(\beta))$. Since $\beta \in W$, then there exists $m \in V = \psi_\sigma(W)$ such that

$$m = \psi_\sigma(\beta) = \phi^1_\beta(\sigma \circ \mathrm{pr}(\beta)). \tag{21}$$

On the other hand, $\beta \in \Lambda_{\mathrm{pr}(\beta)}$ implies that for all $m' \in \pi^{-1}(\mathrm{pr}(\beta))$, $\phi^1_\beta(m') = m'$. Therefore

$$\phi^1_\beta(\sigma \circ \mathrm{pr}(\beta)) = \sigma \circ \mathrm{pr}(\beta); \tag{22}$$

in light of equation (22), it is possible to rewrite equation (21) as

$$\psi_\sigma(\beta) = \sigma \circ \mathrm{pr}(\beta).$$

Applying ψ_σ^{-1} to both sides of the above equality, obtain that

$$\beta = \psi_\sigma^{-1} \circ \sigma \circ \mathrm{pr}(\beta) = \alpha_\sigma \circ \mathrm{pr}(\beta),$$

thus proving that $\beta \in \alpha_\sigma(U)$. This completes the proof that Λ is a smooth submanifold of T^*B.

In order to show that Λ is closed, let $(\beta_n) \subset \Lambda$ be a sequence that converges to $\beta \in \mathrm{T}^*B$. The aim is to prove that $\beta \in \Lambda$. By taking a small enough neighbourhood W' of β in T^*B, it is possible to ensure that all but finitely many β_n lie in W' and that there exists a locally defined section $\sigma : U' = \mathrm{pr}(W') \subset B \to M$. Consider the diffeomorphism ψ_σ constructed as in equation (19). For all but finitely many n, have that

$$\psi_\sigma(\beta_n) = \sigma \circ \mathrm{pr}(\beta_n),$$

since, for all n, $\beta_n \in \Lambda_{\mathrm{pr}(\beta_n)}$. By continuity of σ and pr, the left hand side converges to $\sigma \circ \mathrm{pr}(\beta)$. On the other hand, the right hand side converges to $\psi_\sigma(\beta)$, again by continuity of ψ_σ. Since M is assumed to be Hausdorff, limits in M are unique and, therefore, $\psi_\sigma(\beta) = \sigma \circ \mathrm{pr}(\beta)$. By definition of ψ_σ, it follows that $\beta \in \Lambda$ and the proof of (i) is complete.

The proof of (ii) hinges upon the following important property (cf. [Vai94]). If M' is a smooth manifold and Q is an equivalence relation on M' whose graph in $M' \times M'$ is a closed submanifold, then the quotient M'/Q is a smooth manifold[2]. In this case, two elements $\alpha, \beta \in \mathrm{T}^*B$ are equivalent if and only if $\alpha - \beta \in \Lambda$. The proof that

$$\{(\alpha, \beta) \in \mathrm{T}^*B \times \mathrm{T}^*B \ : \ \alpha - \beta \in \Lambda\} \subset \mathrm{T}^*B \times \mathrm{T}^*B$$

is a closed submanifold of $\mathrm{T}^*B \times \mathrm{T}^*B$ is left as an exercise to the reader, as it can be proved using existence of the sections α_σ constructed in the proof of (i) and by following the above arguments. □

Exercise 7. Complete the proof of part (ii) of Theorem 3.6.

Remark 3.8. Existence of the open sets $W \subset \mathrm{T}^*B$ in the proof of part (i) of Theorem 3.6 uses crucially the fact that $\Lambda \subset \mathrm{T}^*B$ is defined using a smooth Lagrangian fibration $\pi : (M, \omega) \to B$, as these open sets are constructed from open sets of M.

Theorem 3.6 implies the following corollary.

Corollary 3.7.

i) The inclusion $\Lambda \hookrightarrow \mathrm{T}^*B$ makes Λ the space of arrows of a wide, étale, Lie subgroupoid of $\mathrm{T}^*B \rightrightarrows B$;

ii) The quotient T^*B/Λ inherits the structure of Lie groupoid over B. The left Lie groupoid action of $\mathrm{T}^*B \rightrightarrows B$ on M along $\pi : (M, \omega) \to B$ of equation (16) descends to a a free left Lie groupoid action of $\mathrm{T}^*B/\Lambda \rightrightarrows B$ on M along $\pi : (M, \omega) \to B$ denoted by

$$\hat{\mu} : \mathrm{T}^*B/\Lambda \ {}_s\!\times_\pi M \to M$$

where $s : \mathrm{T}^*B/\Lambda \to B$ denotes the source map of $\mathrm{T}^*B/\Lambda \rightrightarrows B$;

iii) A choice of locally defined section $\sigma : U \subset B \to \pi^{-1}(U)$ induces a diffeomorphism

$$\hat{\psi}_\sigma : \mathrm{T}^*U/\Lambda|_U \to \pi^{-1}(U)$$

which commutes with the projections onto U.

Proof. The proofs of (i) and (ii) are left as exercises to the reader. The proof of (iii) follows from the fact that $\hat{\psi}_\sigma$ is a bijective local diffeomorphism; this, in turn, is implied by the fact that ψ_σ constructed in the proof of Theorem 3.6 is a local diffeomorphism. □

Exercise 8. Prove parts (i) and (ii) of Corollary 3.7.

Remark 3.9. The diffeomorphism $\hat{\psi}_\sigma$ of Corollary 3.7 should be thought of as a local trivialisation of $\pi : (M, \omega) \to B$. Fix a local section $\sigma : U \to \pi^{-1}(U)$ and, thus, a local trivialisation $\hat{\psi}_\sigma$. It is immediate to check that the zero section of $\mathrm{T}^*U \to U$ is mapped to the image of σ under $\hat{\psi}_\sigma$.

3.3. **Smooth classification of Lagrangian fibrations.** The diffeomorphisms $\hat{\psi}_\sigma$ constructed in Corollary 3.7 can be used to develop a topological (in fact smooth) classification theory of complete Lagrangian fibrations. The key issue is the fact that a Lagrangian fibration $\pi : (M, \omega) \to B$ may not admit a globally defined section and, therefore, there is no natural choice of locally defined sections $\sigma : U \to \pi^{-1}(U)$ to construct the trivialisations $\hat{\psi}_\sigma$. This is analogous to the classification of principal G-bundles for a Lie group G.

Let $U_i, U_j \subset B$ be open sets such that $U_i \cap U_j \neq \emptyset$. Pick sections $\sigma_i : U_i \to \pi^{-1}(U_i)$, $\sigma_j : U_j \to \pi^{-1}(U_j)$ and construct local trivialisations $\hat{\psi}_{\sigma_i}$, $\hat{\psi}_{\sigma_j}$ as in Corollary 3.7. Consider the diffeomorphism

$$\hat{\psi}_{\sigma_j}^{-1} \circ \hat{\psi}_{\sigma_i} : \mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j} \to \mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j},$$

[2]If the graph is not closed, the quotient M'/Q is a non-Hausdorff manifold.

which, by definition, leaves the projection onto B invariant. Moreover, notice that $\hat{\psi}_{\sigma_j}^{-1} \circ \hat{\psi}_{\sigma_i}$ sends the zero section to $\hat{\psi}_{\sigma_j}^{-1}(\sigma_i)$. By part (ii) of Corollary 3.7, this suffices to show that

$$\hat{\psi}_{\sigma_j}^{-1} \circ \hat{\psi}_{\sigma_i}(\alpha) = \alpha + \hat{\psi}_{\sigma_j}^{-1}(\sigma_i \circ \mathrm{pr}(\alpha)) \tag{23}$$

for all $\alpha \in \mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j}$. If each diffeomorphism $\hat{\psi}_{\sigma_i}$ is thought of as a trivialisation of $\pi : (M, \omega) \to B$, then the diffeomorphisms $\hat{\psi}_{\sigma_j}^{-1} \circ \hat{\psi}_{\sigma_i}$ are transition functions for the above trivialisations. What equation (23) shows is that these transition functions are given by smooth sections of $\mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j} \to U_i \cap U_j$. It is important to notice that $\hat{\psi}_{\sigma_j}^{-1}(\sigma_i)$ can be intrinsically defined using the groupoid action of $\mathrm{T}^*B/\Lambda \rightrightarrows B$ on M along $\pi : (M, \omega) \to B$. In fact, $\hat{\psi}_{\sigma_j}^{-1}(\sigma_i)$ is the unique section s_{ji} of $\mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j} \to U_i \cap U_j$ satisfying

$$\phi^1_{s_{ji}}(\sigma_j) = \sigma_i. \tag{24}$$

Fix a good open cover $\mathcal{U} = \{U_i\}$ in the sense of Leray, *i.e.* all subsets U_i and all finite intersections of these subsets are contractible. The above construction yields locally defined smooth sections $s_{ji} : U_i \cap U_j \to \mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i \cap U_j}$ for each pair i, j whose respective open sets in $\mathcal{U}$ intersect non-trivially. Let $\mathcal{C}^\infty(\mathrm{T}^*B/\Lambda)$ denote the sheaf of smooth sections of $\mathrm{T}^*B/\Lambda \to B$. By definition, the family s_{ji} defines a Čech 1-cocycle for the cohomology of B with coefficients in $\mathcal{C}^\infty(\mathrm{T}^*B/\Lambda)$ and, therefore, a cohomology class in $\eta \in \mathrm{H}^1(B; \mathcal{C}^\infty(\mathrm{T}^*B/\Lambda))$. The cohomology class η classifies the Lagrangian fibration $\pi : (M, \omega) \to B$ up to fibrewise diffeomorphism (cf. [DD87, Gro58]).

Remark 3.10. Another way to see that the class η classifies $\pi : (M, \omega) \to B$ up to fibrewise diffeomorphism is to notice that the action $\hat{\mu}$ of part (ii) of Corollary 3.7 endows $\pi : (M, \omega) \to B$ with the structure of a principal $\mathrm{T}^*B/\Lambda \rightrightarrows B$-bundle (cf. [Ros04]). With this interpretation, the cohomology class η should be seen as the obstruction for $\pi : (M, \omega) \to B$ to be globally diffeomorphic to $\mathrm{T}^*B/\Lambda \to B$, which is equivalent to asking that $\pi : (M, \omega) \to B$ admits a globally defined section.

Let $\mathcal{P}_\Lambda$ be the sheaf of smooth sections of $\Lambda \to B$. There is a short exact sequence of sheaves (cf. [DD87, Dui80])

$$0 \to \mathcal{P}_\Lambda \to \mathcal{C}^\infty(\mathrm{T}^*B) \to \mathcal{C}^\infty(\mathrm{T}^*B/\Lambda) \to 0 \tag{25}$$

where the first non-trivial map is given by the inclusion $\Lambda \hookrightarrow \mathrm{T}^*B$ and $\mathcal{C}^\infty(\mathrm{T}^*B)$ is the sheaf of 1-forms of B (equivalently, the sheaf of sections of $\mathrm{T}^*B \to B$). Recall that $\mathcal{C}^\infty(\mathrm{T}^*B)$ is a *fine* sheaf (cf. [GH94]), which, for the purposes of these notes, can be taken to mean that the sheaf satisfies the following property: for any locally defined smooth function $f \in \mathrm{C}^\infty(B)$ and any locally defined smooth section $s : U \to \mathrm{T}^*U$, the product $fs : U \to \mathrm{T}^*U$ is also a smooth section wherever defined. In particular, this property implies that for all $kgeq1$

$$\mathrm{H}^1(B; \mathcal{C}^\infty(\mathrm{T}^*B)) = 0.$$

This, in turn, means that the long exact sequence in cohomology induced by equation (25) induces an isomorphism

$$\delta : \mathrm{H}^1(B; \mathcal{C}^\infty(\mathrm{T}^*B/\Lambda)) \to \mathrm{H}^2(B; \mathcal{P}_\Lambda). \tag{26}$$

Definition 3.11 ([DD87, Dui80])**.** The image $\delta\eta = c \in \mathrm{H}^2(B; \mathcal{P}_\Lambda)$ is the *Chern class* associated to the Lagrangian fibration $\pi : (M, \omega) \to B$.

In light of the above discussion, the Chern class of a Lagrangian fibration is the obstruction to the existence of a globally defined section. This completes the topological (smooth) classification of complete Lagrangian fibrations. Given $\pi : (M, \omega) \to B$, there are two topological (smooth) invariants which completely characterise the fibration, namely

i) its period net $\Lambda \subset \mathrm{T}^*B$;
ii) its Chern class $c \in \mathrm{H}^2(B; \mathcal{P}_\Lambda)$.

What the above statement means is that two Lagrangian fibrations are fibrewise diffeomorphic if and only if they have diffeomorphic period nets and equal (up to the diffeomorphism relating the period nets) Chern classes.

3.4. **Symplectic classification of Lagrangian fibrations.** The results of Section 3.3 leave unanswered the question of classifying complete Lagrangian fibrations up to fibrewise symplectomorphism. It is natural to ask whether the diffeomorphisms $\hat{\psi}_\sigma$ of Corollary 3.7 can be chosen so that they are symplectomorphisms; however, in order for this question to make sense, the space T^*B/Λ must be endowed with a *reference* symplectic structure. This is precisely the analogue of constructing action-angle coordinates in the Liouville-Mineur-Arnol'd theorem (cf. Theorem 3.1).

As a first step, it must be noticed that the groupoid action μ of $\mathrm{T}^*B \rightrightarrows B$ on M along $\pi : (M, \omega) \to B$ given by Proposition 3.5 is not by symplectomorphisms. In fact, the following proposition holds.

Proposition 3.8. *Fix a locally defined 1-form $\alpha : U \to \mathrm{T}^*U$. Then*

$$(\phi^1_\alpha)^*\omega - \omega = \pi^*\mathrm{d}\alpha.$$

Proof. Note that

$$\begin{aligned}(\phi^1_\alpha)^*\omega - \omega &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\phi^t_\alpha)^*\omega \mathrm{d}t = \int_0^1 (\phi^t_\alpha)^*(\mathcal{L}_{X_{\pi^*\alpha}}\omega)\mathrm{d}t \\ &= \int_0^1 (\phi^t_\alpha)^*\mathrm{d}(\iota(X_{\pi^*\alpha})\omega)\mathrm{d}t = \int_0^1 (\phi^t_\alpha)^*\mathrm{d}(\pi^*\alpha)\mathrm{d}t \\ &\int_0^1 (\pi \circ \phi^t_\alpha)^*\mathrm{d}\alpha\mathrm{d}t = \int_0^1 \pi^*\mathrm{d}\alpha\mathrm{d}t \\ &= \pi^*\mathrm{d}\alpha,\end{aligned}$$

since $\pi \circ \phi^t_\alpha = \pi$ for all t. The above calculation finishes the proof of the proposition. ☐

Corollary 3.9. *Any section $\alpha : U \to \Lambda|_U$ is a closed 1-form.*

Proof. Note that Λ is defined as the isotropy subgroupoid of μ. In patricular, for any section $\alpha : U \to \Lambda|_U$, $\phi^1_\alpha = \mathrm{id}$, which implies that $(\phi^1_\alpha)^*\omega = \omega$. Therefore $\pi^*\mathrm{d}\alpha = 0$ by Proposition 3.8. Since π is a submersion, it follows that $\mathrm{d}\alpha = 0$ as required. ☐

Remark 3.12.

i) Corollary 3.9 implies that Λ is a Lagrangian submanifold of $(\mathrm{T}^*B, \Omega_{\mathrm{can}})$, since it is given locally by the image of closed 1-forms;
ii) There is a natural groupoid action of $\Lambda \rightrightarrows B$ on T^*B along $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$. As a consequence of Corollary 3.9, this action is by symplectomorphisms. In particular, this implies that the quotient space T^*B/Λ inherits a symplectic form ω_0 which makes the induced projection

$$(\mathrm{T}^*B/\Lambda, \omega_0) \to B$$

into a complete Lagrangian fibration.

Definition 3.13. Given a complete $\pi : (M, \omega) \to B$ with period net $\Lambda \subset \mathrm{T}^*B$, the complete Lagrangian fibration given by

$$\mathrm{pr} : (\mathrm{T}^*B/\Lambda, \omega_0) \to B \tag{27}$$

is called the *symplectic reference* or *Jacobian* Lagrangian fibration associated to $\pi : (M, \omega) \to B$.

Remark 3.14. Note that any symplectic reference Lagrangian fibration admits a globally defined Lagrangian section.

Exercise 9. Prove the statement of Remark 3.14. [Hint: consider what happens to the zero section $B \to \mathrm{T}^*B$ under the action of Λ on T^*B.]

Part (iii) of Corollary 3.7 can be interpreted as saying that a choice of locally defined section $\sigma : U \to \pi^{-1}(U)$ of a Lagrangian fibration $\pi : (M, \omega) \to B$ yields a local fibrewise diffeomorphism between the given Lagrangian fibration and its associated symplectic reference fibration. The result of the proposition below shows that not all such local trivialisations are symplectomorphisms with respect to the symplectic structure ω_0 on T^*B/Λ. In fact, what goes wrong is that the section σ may not be *Lagrangian*.

Proposition 3.10. *For any locally defined section $\sigma : U \to \pi^{-1}(U)$,*

$$\hat{\psi}_\sigma^* \omega = \omega_0 + \mathrm{pr}^* \circ \sigma^* \omega.$$

Proof. Let $\alpha : U \to \mathrm{T}^*U$ be a locally defined form and set $q : \mathrm{T}^*U \to \mathrm{T}^*U/\Lambda|_U$ be the quotient map. Then $q \circ \alpha : U \to \mathrm{T}^*U/\Lambda|_U$ is a locally defined section of the symplectic reference Lagrangian fibration associated to $\pi : (M, \omega) \to B$. By construction, $q^*\omega_0 = \Omega_{\mathrm{can}}$. Proposition 3.8 gives that

$$(\phi_\alpha^1)^*\omega = \omega + \pi^*\alpha^*\Omega_{\mathrm{can}} = \omega + \pi^*(q \circ \alpha)^*\omega_0, \tag{28}$$

where the identity $\mathrm{d}\alpha = \alpha^*\Omega_{\mathrm{can}}$ is used (cf. [MS95]). Applying σ^* to both sides of equation (28) and recalling that $\pi \circ \sigma = \mathrm{id}_U$, obtain that

$$(\phi_\sigma^1 \circ \sigma)^*\omega = \sigma^*\omega + (q \circ \alpha)^*\omega_0.$$

By definition, $\phi_\sigma^1 \circ \sigma = \hat{\psi}_\sigma \circ q \circ \alpha$ and $\mathrm{pr} \circ q \circ \alpha = \mathrm{id}_U$, so that the above equation can be written as

$$(q \circ \alpha)^*((\hat{\psi}_\sigma)^*\omega - \mathrm{pr}^*\sigma^*\omega - \omega_0) = 0. \tag{29}$$

The result follows if $(\hat{\psi}_\sigma)^*\omega - \omega_0 = \mathrm{pr}^*(\beta)$ for some β locally defined 2-form on U. This is a consequence of the fact that, for all locally defined $\alpha : U \to \mathrm{T}^*U$, the following equation holds

$$D\hat{\psi}_\sigma(X_{\mathrm{pr}^*\alpha}) = X_{\pi^*\alpha} \tag{30}$$

(cf. [Gro01]). This completes the proof of the proposition. □

Exercise 10. Prove the formula of equation (30).

The following corollary is immediate from Proposition 3.10.

Corollary 3.11. *$\hat{\psi}_\sigma^*\omega = \omega_0$ if and only if $\sigma^*\omega = 0$, i.e. σ is a Lagrangian section.*

Corollary 3.11 implies that if $\sigma : U \to \pi^{-1}(U)$ is a Lagrangian section, then the local trivialisation $\hat{\psi}_\sigma$ constructed as in Corollary 3.7 is a symplectomorphism

$$(\mathrm{T}^*U/\Lambda|_U, \omega_0) \to (\pi^{-1}(U), \omega|_U)$$

which satisfies $\pi \circ \hat{\psi}_\sigma = \mathrm{pr}$, *i.e.* it preserves the fibrations pr and π. The next lemma shows that near each point $b \in B$ there exists a Lagrangian section σ; as a consequence, it follows that B can be covered by open sets over which there exist local symplectic trivialisations.

Lemma 3.12. *For each $b \in B$ there exists an open neighbourhood $U \subset B$ of b and a Lagrangian section $\sigma : U \to \pi^{-1}(U)$.*

Proof. Fix $b \in B$ and let U be an open neighbourhood of b over which there exists a section $\bar{\sigma} : U \to \pi^{-1}(U)$ and small enough that $\mathrm{H}^2(U; \mathbb{R}) = 0$. Proposition 3.10 implies that the local smooth trivialisation $\hat{\psi}_{\bar{\sigma}}$ yields a symplectomorphism

$$(\mathrm{T}^*U/\Lambda|_U, \mathrm{pr}^* \circ \bar{\sigma}^*\omega + \omega_0) \to (\pi^{-1}(U), \omega).$$

Therefore it suffices to prove that

$$\mathrm{pr} : (\mathrm{T}^*U/\Lambda|_U, \mathrm{pr}^* \circ \bar{\sigma}^*\omega + \omega_0) \to U \tag{31}$$

admits a Lagrangian section. Note that, since $\mathrm{H}^2(U; \mathbb{R})$, $\bar{\sigma}^*\omega = \mathrm{d}\beta$ for some locally defined 1-form $\beta : U \to \mathrm{T}^*U$. Set $s := q \circ (-\beta) : U \to \mathrm{T}^*U/\Lambda|_U$ be the composite of the negative of β and the quotient map $q : \mathrm{T}^*U \to \mathrm{T}^*U/\Lambda|_U$. Then

$$s^*(\mathrm{pr}^* \circ \bar{\sigma}^*\omega + \omega_0) = \bar{\sigma}^*\omega + (-\beta)^*\Omega_{\mathrm{can}} = \mathrm{d}\beta - \mathrm{d}\beta = 0,$$

since $q^*\omega_0 = \Omega_{\mathrm{can}}$. This proves that s is a Lagrangian section of the Lagrangian fibration of equation (31), thus finishing the proof of the lemma. □

Lemma 3.12 implies that it is possible to construct a *symplectic* classification theory for Lagrangian fibrations. Fix such a fibration $\pi : (M,\omega) \to B$ and choose a good open cover $\mathcal{U} = \{U_i\}$ of B over which there exist Lagrangian sections $\sigma_i : U_i \to \pi^{-1}(U_i)$. This can always be achieved, since any open cover of a manifold admits a good refinement (cf. [BT99]). Construct local symplectomorphisms

$$\hat{\psi}_{\sigma_i} : (\mathrm{T}^*U/\Lambda|_{U_i}, \omega_0) \to (\pi^{-1}(U_i), \omega|_{\pi^{-1}(U_i)}),$$

and consider, for each pair of indices i, j such that $U_i \cap U_j \neq \emptyset$, the symplectomorphisms

$$\begin{aligned} \hat{\psi}^{-1}_{\sigma_j} \circ \hat{\psi}_{\sigma_i} : (\mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i\cap U_j}, \omega_0) &\to (\mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i\cap U_j}, \omega_0) \\ \alpha &\mapsto \alpha + \hat{\psi}^{-1}_{\sigma_j}(\sigma_i \circ \mathrm{pr}(\alpha)) \end{aligned} \tag{32}$$

Since, for each pair i, j, the map of equation (32) is a symplectomorphism, the sections $s_{ji} := \hat{\psi}^{-1}_{\sigma_j}(\sigma_i)$ are Lagrangian sections of $\mathrm{pr} : (\mathrm{T}^*(U_i \cap U_j)/\Lambda|_{U_i\cap U_j}, \omega_0) \to U_i \cap U_j$. Denote this sheaf of sections by $\mathcal{Z}^1(\mathrm{T}^*B/\Lambda)$. As in Section 3.3, these locally defined Lagrangian sections s_{ji} define a cohomology class $l \in \mathrm{H}^1(B; \mathcal{Z}^1(\mathrm{T}^*B/\Lambda)$.

Definition 3.15 ([DD87, Dui80, Zun03])**.** l is called the *Lagrangian Chern class* associated to the complete Lagrangian fibration $\pi : (M,\omega) \to B$.

As in Section 3.3, it can be shown that two Lagrangian fibrations $\pi : (M,\omega) \to B$ and $\pi' : (M',\omega') \to B$ (with associated period nets Λ, Λ') are fibrewise symplectomorphic if and only if the following two conditions hold

i) there exists a diffeomorphism $f : B \to B'$ whose induced symplectomorphism of $(\mathrm{T}^*B, \Omega_{\mathrm{can}})$ maps Λ diffeomorphically to Λ';
ii) their Lagrangian Chern classes l, l' coincide (up to the above symplectomorphism of $(\mathrm{T}^*B, \Omega_{\mathrm{can}})$.

Remark 3.16. Fix a complete Lagrangian fibration $\pi : (M,\omega) \to B$ with period net Λ. There exists a short exact sequence of sheaves

$$0 \to \mathcal{P}_\Lambda \to \mathcal{Z}^1(\mathrm{T}^*B) \to \mathcal{Z}^1(\mathrm{T}^*B/\Lambda) \to 0, \tag{33}$$

where $\mathcal{Z}^1(\mathrm{T}^*B)$ denotes the sheaf of closed 1-forms of B, which can also be equivalently described as the sheaf of *Lagrangian* sections of $\mathrm{pr} : (\mathrm{T}^*B, \Omega_{\mathrm{can}}) \to B$ (Cf. [DD87]). The long exact sequence in cohomology induced by equation (33) gives homomorphisms

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{H}^1(B; \mathcal{Z}^1(\mathrm{T}^*B/\Lambda)) & \longrightarrow & \mathrm{H}^2(B; \mathcal{P}_\Lambda) & \xrightarrow{\mathcal{D}_\Lambda} & \mathrm{H}^2(B; \mathcal{Z}^1(\mathrm{T}^*B)) & \longrightarrow \cdots \\ & l & \longmapsto & c & \longmapsto & 0, & \end{array} \tag{34}$$

where $c \in \mathrm{H}^2(B; \mathcal{P}_\Lambda)$ is the Chern class associated to $\pi : (M,\omega) \to B$ (cf. Definition 3.11). In particular, c lies in the kernel of the *Dazord-Delzant* homomorphism

$$\mathcal{D}_\Lambda : \mathrm{H}^2(B; \mathcal{P}_\Lambda) \to \mathrm{H}^2(B; \mathcal{Z}^1(\mathrm{T}^*B)) \cong \mathrm{H}^3(B; \mathbb{R}), \tag{35}$$

where the isomorphism follows, say, from the structure of the Čech-de Rham double complex on B (cf. [BT99]). It is importante to notice that the Dazord-Delzant homomorphism depends on the period net Λ (hence, the subscript in the notation above).

In fact, the following theorem (stated below without proof) can be proved.

Theorem 3.13 ([DD87, Vai94])**.** *For a fixed period net $\Lambda \subset (\mathrm{T}^*B, \Omega_{\mathrm{can}})$, all elements lying in the kernel of the Dazord-Delzant homomorphism $\mathcal{D}_\Lambda$ arise as the Chern class of some complete Lagrangian fibration $\pi : (M,\omega) \to B$ whose associated period net is diffeomorphic to Λ.*

4. Integral affine geometry and Lagrangian fibrations

This section studies the relation between the geometry of *affine* manifolds and Lagrangian fibrations. On the one hand, it is known that the leaves of any Lagrangian *foliation* on a symplectic manifold are affine manifolds (cf. [Wei71]); Section 4.1 provides a direct proof of this fact and a first definition of affine manifolds. These are studied in more details in Section 4.2 where several examples and constructions relating to these manifolds are presented. The connection between affine geometry and Lagrangian fibrations runs deeper than the affine structure on the fibres, as the base of a Lagrangian bundle with compact and connected fibres is also an affine manifold which satisfies a special integrality condition. This is proved in Section 4.3, where it is shown that the integral affine geometry of the base of such a bundle is intimately connected to the period net Λ constructed in Section 3.2. Using the results of Sections 4.2 and 4.3, Section 4.4 proves that the Dazord-Delzant homomorphism of equation (35) is determined by the integral affine geometry of the base of a Lagrangian bundle.

Assumption 3. Throughout this section, any Lagrangian fibration $\pi : (M, \omega) \to B$ has connected fibres and is a surjective submersion unless otherwise stated.

4.1. **Affine structure on the fibres of a Lagrangian fibration.** Let $\pi : (M, \omega) \to B$ be a Lagrangian fibration and fix $b_o \in B$. Set $F_{b_0} = \pi^{-1}(b_0)$. For any 1-form α defined locally near b_0 and for each $m \in F$, the vector $X_{\pi^*\alpha}(m) \in \mathrm{T}_m F$; conversely, if $y \in \mathrm{T}_m F$, there exists a 1-form α_Y defined locally near b_0 with $X_{\pi^*\alpha_Y} = Y$ (cf. Proposition 3.4). Define a *connection* ∇ on $\mathrm{T}F$ by setting

$$\nabla_{X_{\pi^*\alpha}} X_{\pi^*\beta} = 0 \tag{36}$$

for all 1-forms α, β defined locally near b_0 and extend to all vector fields on F. The only property that needs checking in order to prove that equation (36) gives a well-defined connection on F is that if f is a smooth function defined locally near b_0, then for all α, β as above

$$\nabla_{X_{\pi^*\alpha}} X_{\pi^*(f\beta)} = 0.$$

This follows from calculations akin to those of Claim 2.4.

Exercise 11. Prove the above statement.

Fixing a choice of 1-forms $\alpha_1, \ldots, \alpha_n$ defined locally near b_0 whose corresponding vector fields

$$X_{\pi^*\alpha_1}, \ldots, X_{\pi^*\alpha_n}$$

give a basis of $\mathrm{T}_m F$ for all $m \in F$ (why do such forms exist?), equation (36) can be seen as setting all Christoffel symbols Γ^k_{ij} of ∇ to be identically zero in the above coordinates (which are, therefore, geodesic coordinates for this connection). This implies that ∇ is torsion-free, since $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, and flat, *i.e.* with zero curvature.

Definition 4.1 ([AM55])**.** A flat, torsion-free affine connection ∇ on $\mathrm{T}F$ is called a *affine* structure on F. The pair (F, ∇) is called an *affine manifold*.

In light of Definition 4.1 and of the above discussion, the following lemma holds.

Lemma 4.1. *Let $\pi : (M, \omega) \to B$ be a Lagrangian fibration. Its fibres admit an affine structure ∇ determined by equation* (36)*.*

Remark 4.2. Note that, for a fixed fibre F, the vector fields $X_{\pi^*\alpha_1}, \ldots, X_{\pi^*\alpha_n}$ form a global frame of $\mathrm{T}F$ by geodesic vector fields.

For an affine manifold (F, ∇) there is a natural notion of completeness which generalises complete Riemannian manifolds.

Definition 4.3. An affine manifold (F, ∇) is *complete* if all the geodesics of ∇ exist for all $t \in \mathbb{R}$.

Recall that the topological and symplectic classification of Lagrangian fibrations studied in Section 3 relies upon an assumption that the fibrations under consideration are *complete* in the sense of Assumption 2. In fact, the following proposition states that this notion of completeness is equivalent to the one given in Definition 4.3.

Proposition 4.2. *A Lagrangian fibration $\pi : (M, \omega) \to B$ is complete if and only if all fibres are complete as affine manifolds with affine structure defined as in equation* (36)*.*

Exercise 12. Prove the above proposition. [Hint: use geodesic coordinates induced by the vector fields $X_{\pi^*\alpha_1}, \ldots, X_{\pi^*\alpha_n}$]

4.2. **Generalities on (integral) affine manifolds.** In light of Definition 4.1, affine manifolds are generalisations of flat Riemannian manifolds. However, it important to notice some crucial differences. Firstly, generally it is not true that, given an affine manifold (F, ∇), its affine structure ∇ is obtained as the Levi-Civita connection of some flat metric. Secondly, a compact affine manifold need *not* be complete. The following exercise illustrates both these observations.

Exercise 13 ([FGH81])**.** Let

$$\mathbb{R}^+ := \{x \in \mathbb{R} : \, x > 0\}.$$

Define a $\mathbb{Z}$-action on $\mathbb{R}^+$ by

$$\begin{aligned} \mathbb{Z} \times \mathbb{R}^+ &\to \mathbb{R}^+ \\ (k, x) &\mapsto 2^k x. \end{aligned}$$

Let d denote the restriction of the standard flat, torsion-free connection on $\mathbb{R}$ to $\mathbb{R}^+$. Prove that

i) the above action preserves d;
ii) the quotient $Q = \mathbb{R}^+/\mathbb{Z}$ is a smooth, compact manifold;
iii) Q inherits an affine structure ∇ from d which is incomplete;
iv) ∇ is not the Levi-Civita connection of any flat Riemannian metric g on Q [Hint: compute the holonomy of ∇].

Notation. Henceforth, $\mathbb{R}^n$ with its standard affine structure d (*i.e.* the Levi-Civita connection of the standard Euclidean metric on $\mathbb{R}^n$) is denoted by $\mathbb{R}^n$ so as to simplify notation (cf. Remark 4.5).

Chronologically, affine manifolds were first defined using affine connections as in Definition 4.1 (cf. [AM55]). More recently, an equivalent definition in terms of local coordinate charts has been preferred (cf. [FGH80, GH84]).

Definition 4.4. Let B be a topological n-dimensional manifold. An *affine structure* on B is a choice of smooth atlas $\mathcal{A} = \{U_i, \chi_i\}$ with coordinate charts $\chi_i : U_i \to \mathbb{R}^n$ whose changes of coordinates $\chi_j \circ \chi_i^{-1}$ are restrictions of elements of

$$\mathrm{Aff}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{R}) \ltimes \mathbb{R}^n,$$

on each connected component of $U_i \cap U_j$. Say the affine structure $\mathcal{A}$ is *integral* if its changes of coordinates are restrictions of elements of

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n,$$

on each connected component. An *(integral) affine manifold* consists of a pair $(B, \mathcal{A})$, where B is a topological manifold and $\mathcal{A}$ an (integral) affine structure.

Exercise 14. Prove the equivalence of Definitions 4.1 and 4.4 (cf. [AM55]).

Remark 4.5.

i) The groups

$$\mathrm{Aff}(\mathbb{R}^n) \qquad \text{and} \qquad \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$$

are called the group of *affine* (resp. *integral affine*) transformations of $\mathbb{R}^n$;
ii) An (integral) affine structure $\mathcal{A}$ on B determines a unique maximal smooth atlas. However, a given topological manifold B which admits an (integral) affine structure may in fact admit several 'inequivalent' such structures (cf. Definition 4.8 for a notion of equivalence and Example 4.10);
iii) Note that some authors define integral affine structures as atlases whose changes of coordinates lie in $\mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{Z}^n$ (cf. [GS06]).

Having defined (integral) affine manifolds, it is natural to consider maps between (integral) affine manifolds which 'preserve' the (integral) affine structures. The next definitions make this notion precise.

Definition 4.6. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* (resp. *integral affine*) if

$$f \in \hom(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^m \qquad (\text{resp. } \hom(\mathbb{Z}^n, \mathbb{Z}^m) \times \mathbb{R}^m).$$

If $n = m$, an (integral) affine map f which admits a smooth inverse f^{-1} is an *(integral) affine diffeomorphism* if both f and f^{-1} are (integral) affine maps.

Remark 4.7. Note that when $n = m$ in Definition 4.6, an (integral) affine map f is required to be an element of the monoid

$$\hom(\mathbb{R}^n, \mathbb{R}^n) \ltimes \mathbb{R}^n \qquad (\text{resp. } \hom(\mathbb{Z}^n, \mathbb{Z}^n) \ltimes \mathbb{R}^n),$$

where the action of $\hom(\mathbb{R}^n, \mathbb{R}^n)$ (resp. $\hom(\mathbb{Z}^n, \mathbb{Z}^n)$) on $\mathbb{R}^n \cong \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$ is the standard one.

Definition 4.8. Let $(B, \mathcal{A})$ and $(B', \mathcal{A}')$ be (integral) affine manifolds of dimension n and m respectively. A smooth map $f : B \to B'$ is said to be *(integral) affine* if it is an (integral) affine map in local (integral) affine coordinates. Denote such a map by $f : (B, \mathcal{A}) \to (B', \mathcal{A}')$. If $n = m$, a diffeomorphism $f : B \to B'$ with inverse f^{-1} is an *(integral) affine diffeomorphism* if both f and f^{-1} are (integral) affine maps. In this case, say that $(B, \mathcal{A})$ and $(B', \mathcal{A}')$ are diffeomorphic as (integral) affine manifolds.

Remark 4.9. While an affine map which admits an inverse is an affine diffeomorphism, this is not true in general for integral affine maps, as the inverse has to be an *integral* affine map.

It is high time for some constructions and examples of (integral) affine manifolds.

Example 4.10.

i) Any open subset $U \subset \mathbb{R}^n$ is an (integral) affine manifold;

ii) More generally, let $(B, \mathcal{A})$ be an (integral) affine manifold and suppose that $f : B' \to B$ is a local diffeomorphism. Then there exists an (integral) affine structure $\mathcal{A}'$ such that $f : (B', \mathcal{A}') \to (B, \mathcal{A})$ is an (integral) affine map. For instance, if $\mathcal{H}$ denotes the upper-half plane with complex coordinate w, then the map

$$(37) \qquad \begin{aligned} \delta : \mathcal{H} &\to \mathbb{R}^2 \\ w &\mapsto \left(\Re(e^{2\pi i w}), \Re\big(e^{2\pi \imath w}(w - \frac{1}{2\pi\imath})\big) \right) \end{aligned}$$

induces an (integral) affine structure on $\mathcal{H}$ (cf. [GS06]);

iii) Let $(B, \mathcal{A})$ be an (integral) affine manifold and let Γ be a discrete group acting on $(B, \mathcal{A})$ by (integral) affine diffeomorphisms and such that B/Γ is a topological manifold. Then B/Γ inherits an (integral) affine structure $\mathcal{A}_\Gamma$ which makes the quotient map

$$q : (B, \mathcal{A}) \to (B/\Gamma, \mathcal{A}_\Gamma)$$

into an (integral) affine map. This generalises the idea of Exercise 13. As an application, note that the standard action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ is by (integral) affine diffeomorphisms. Therefore the quotient (homeomorphic to an n-torus) inherits an (integral) affine structure; denote the resulting (integral) affine manifold by $\mathbb{R}^n/\mathbb{Z}^n$;

iv) Setting $n = 1$ in the construction above, obtain an affine manifold $\mathbb{R}/\mathbb{Z}$ which is complete, as the action of $\mathbb{Z}$ on $\mathbb{R}$ is by isometries on the standard Euclidean metric on $\mathbb{R}$. Therefore, $\mathbb{R}/\mathbb{Z}$ cannot be affinely isomorphic to the example constructed in Exercise 13, thus showing that there may exist several affine structures on a given topological manifold which are not affinely diffeomorphic;

v) As another example of the construction in (iii), consider the (integral) affine manifold $(\mathcal{H}, \tilde{\mathcal{A}})$ defined by equation (37). Consider the action of $\mathbb{Z}$ on $\mathcal{H}$ defined by

$$(38) \qquad \begin{aligned} \mathbb{Z} \times \mathcal{H} &\to \mathcal{H} \\ (k, w) &\mapsto w + k. \end{aligned}$$

It can be checked that the above action is by (integral) affine diffeomorphisms of $(\mathcal{H}, \tilde{\mathcal{A}})$ and that the quotient $\mathcal{H}/\mathbb{Z}$ is a topological manifold. In fact, $\mathcal{H}/\mathbb{Z}$ can be identified with

$$B_0 := \{z \in \mathbb{C} : |z|^2 < 1\} \setminus 0;$$

with this identification, the quotient map is given by

$$q : \mathcal{H} \to B_0 \qquad w \mapsto e^{2\pi \imath w}, \tag{39}$$

(cf. [GS06]). In light of (iii), it follows that B_0 is an (integral) affine manifold with (integral) affine structure $\mathcal{A}$ such that the quotient map q of equation (39) is an (integral) affine map (in fact, a local diffeomorphism);

vi) A theorem proved independently by Benzecri in [Ben60] and Milnor in [Mil58] states that the only closed topological surfaces which admit affine structures are the 2-torus T^2 and the Klein bottle K^2. Affine structures on these surfaces have been classified by Arrowsmith and Furness in [FA75] and [FA76].

Exercise 15. Prove the various claims made in Example 4.10.

The various constructions of Example 4.10 can be used to define a *developing map* associated to a given (integral) affine manifold $(B, \mathcal{A})$. Let $q : \tilde{B} \to B$ be the universal covering; since it is a local diffeomorphism, by (ii) above, there exists an (integral) affine structure $\tilde{\mathcal{A}}$ which makes q into an (integral) affine map. The (integral) affine structure $\tilde{\mathcal{A}}$ also arises from a local diffeomorphism $\delta : \tilde{B} \to \mathbb{R}^n$, which, intuitively, arises from fixing an open set V in $\tilde{B}$ diffeomorphic under q to an (integral) affine coordinate neighbourhood U with coordinate chart $\chi : U \to \mathbb{R}^n$. Define $\delta : V \to \mathbb{R}^n$ to be $\chi \circ q$; this map can be extended uniquely to $\tilde{B}$ by using analyticity of the action of $\mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$) on $\mathbb{R}^n$ (cf. [FGH81, Smi81]). This is entirely akin to analytic continuation of complex maps. In fact, the following theorem, stated below without proof, holds.

Theorem 4.3 ([FGH81])**.** *Let $(B, \mathcal{A})$ be an (integral) affine manifold and let $\tilde{B}$ be its universal cover. There exists a local diffeomorphism $\delta : \tilde{B} \to \mathbb{R}^n$, called* developing *map of $(B, \mathcal{A})$, which induces an (integral) affine structure on $\tilde{B}$ which is (integral) affinely diffeomorphic to $\tilde{\mathcal{A}}$ defined above.*

Remark 4.11.

i) For a given (integral) affine manifold $(B, \mathcal{A})$, there exist several developing maps associated to $(B, \mathcal{A})$. There are two reasons for this. Firstly, if $\delta : \tilde{B} \to \mathbb{R}^n$ is a developing map of $(B, \mathcal{A})$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is an (integral) affine diffeomorphism, then $f \circ \delta$ is also a developing map of $(B, \mathcal{A})$. Secondly, the construction of a developing map δ depends upon a choice of basepoint $b \in B$; if $b' \in B$ is another basepoint with corresponding developing map δ', then there exists an (integral) affine diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\delta' = f \circ \delta$. This proves that any two developing maps $\delta, \delta' : \tilde{B} \to \mathbb{R}^n$ associated to an (integral) affine manifold $(B, \mathcal{A})$ differ by an (integral) transformation of $\mathbb{R}^n$;

ii) The map $\delta : \mathcal{H} \to \mathbb{R}^2$ of equation (37) is a developing map for the (integral) affine manifold $(B_0, \mathcal{A})$ constructed in Example 4.10.(v);

iii) Given an (integral) affine manifold $(B, \mathcal{A})$ and a choice of developing map $\delta : \tilde{B} \to \mathbb{R}^n$ depending on a basepoint $b \in B$, the action of the fundamental group $\pi_1(B; b)$ on $\tilde{B}$ by deck transformations is by (integral) affine diffeomorphisms. This is equivalent to stating that for all $\gamma \in \pi_1(B; b)$ there exists $\mathfrak{a}(\gamma) \in \mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$) which makes the following diagram commutative

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\delta} & \mathbb{R}^n \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\mathfrak{a}(\gamma)} \\ \tilde{B} & \xrightarrow[\delta]{} & \mathbb{R}^n. \end{array} \tag{40}$$

4.2.1. *Holonomy and the radiance obstruction.* The map

$$\mathfrak{a} : \pi_1(B; b) \to \mathrm{Aff}(\mathbb{R}^n)\,(\text{resp. } \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)) \qquad \gamma \mapsto \mathfrak{a}(\gamma) \tag{41}$$

defined by equation (40) is, in fact, a group homomorphism (cf. [AM55]) and it plays an important role in the study of (integral) affine manifolds.

Definition 4.12. The homomorphism $\mathfrak{a} : \pi_1(B;b) \to \mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$) is called the *affine holonomy* associated to $(B, \mathcal{A})$ (with respect to the basepoint $b \in B$).

Note that the groups $\mathrm{Aff}(\mathbb{R}^n)$ and $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ admit surjective homomorphisms onto $\mathrm{GL}(n;\mathbb{R})$ and $\mathrm{GL}(n;\mathbb{Z})$ respectively obtained by taking the linear part of the (integral) affine transformation. Denote this homomorphism by Lin.

Definition 4.13. The composite

$$\mathfrak{l} := \mathrm{Lin} \circ \mathfrak{a} : \pi_1(B;b) \to \mathrm{GL}(n;\mathbb{R})\,(\text{resp. } \mathrm{GL}(n;\mathbb{Z}))$$

is called the *linear holonomy* associated to $(B, \mathcal{A})$ (with respect to the basepoint $b \in B$).

Exercise 16. Compute the linear and affine holonomies of the (integral) affine manifolds (with respect to some basepoint) constructed thus far.

Remark 4.14. The affine (resp. linear) holonomy associated to an (integral) affine manifold $(B, \mathcal{A})$ is precisely the holonomy of the underlying connection ∇ viewed as an affine (resp. linear) connection on $\mathrm{T}B$.

One of the defining characteristics of (integral) affine manifolds is the difference between linear and affine objects. For instance, a natural question to ask is whether the affine holonomy $\mathfrak{a}$ of an (integral) affine manifold $(B, \mathcal{A})$ has a fixed point (like its linear counterpart $\mathfrak{l}$). This is akin to asking whether, given an (integral) affine structure $\mathcal{A}$ on B, there exists another (integral) affine structure $\mathcal{A}'$ whose changes of coordinates are linear and an (integral) affine diffeomorphism $f : (B, \mathcal{A}) \to (B, \mathcal{A}')$. It turns out that there exists a cohomology class which measures the obstruction to being able to give an affirmative answer to the above questions; it is known as the *radiance* obstruction of an (integral) affine manifold $(B, \mathcal{A})$ and plays an important role in the study of (integral) affine geometry. In what follows, the radiance obstruction is presented from several points of view to illustrate the interplay between topology, differential and algebraic geometry that characterises (integral) affine manifolds.

Consider the map

$$\begin{aligned} \mathrm{Trans} : \mathrm{Aff}(\mathbb{R}^n)\,(\text{resp. } \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)) &\to \mathbb{R}^n \\ (A, \mathbf{b}) &\mapsto \mathbf{b}, \end{aligned} \tag{42}$$

where $A \in \mathrm{GL}(n;\mathbb{R})$ (resp. $\mathrm{GL}(n;\mathbb{Z})$) and $\mathbf{b} \in \mathbb{R}^n$.

Exercise 17. For $(A, \mathbf{b}), (A', \mathbf{b}') \in \mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$), show that

$$\mathrm{Trans}((A, \mathbf{b}) \cdot (A', \mathbf{b}')) = \mathrm{Trans}(A, \mathbf{b}) + \mathrm{Lin}(A, \mathbf{b})\mathrm{Trans}(A', \mathbf{b}').$$

The result of Exercise 17 shows that Trans defines a *crossed* homomorphism from $\mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$) into $\mathbb{R}^n$, viewed as an $\mathrm{Aff}(\mathbb{R}^n)-$ (resp. $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)-$) module via the representation Lin. Crossed homomorphisms are 1-cocycles in group cohomology (cf. [Bro94]); therefore equation (42) defines a cohomology class $r_U \in \mathrm{H}^1(\mathrm{Aff}(\mathbb{R}^n); \mathbb{R}^n_{\mathrm{Lin}})$, which, in fact, is non-zero (cf. Exercise 18).

Definition 4.15. r_U is called the *universal* (algebraic) radiance obstruction.

Fix an (integral) affine manifold $(B, \mathcal{A})$, a basepoint $b \in B$ and set $\mathfrak{a}$, $\mathfrak{l}$ to be the associated affine and linear holonomies respectively.

Definition 4.16. The pullback $r_{(B,\mathcal{A})} = \mathfrak{a}^* r_U \in \mathrm{H}^1(\pi_1(B); \mathbb{R}^n_{\mathfrak{l}})$ is called the *radiance obstruction* associated to $(B, \mathcal{A})$.

Remark 4.17. It is important to note that while $r_{(B,\mathcal{A})}$ depends upon a choice of basepoint $b \in B$, its vanishing does not. Moreover, $r_{(B,\mathcal{A})}$ vanishes if and only if $\mathfrak{a}$ is conjugate to a linear representation, *i.e.* there exists $\mathbf{b}_0 \in \mathbb{R}^n$ such that, for all $\gamma \in \pi_1(B;b)$, $\mathfrak{a}(\gamma)(\mathbf{b}_0) = \mathbf{b}_0$.

Exercise 18. Consider the (integral) affine manifold $\mathbb{R}/\mathbb{Z}$ defined in Example 4.10.(iii). Prove that its radiance obstruction $r_{\mathbb{R}/\mathbb{Z}} \neq 0$. Deduce that $r_U \neq 0$.

Thus far the radiance obstruction $r_{(B,\mathcal{A})}$ is, a priori, an invariant of $\pi_1(B)$ and of the affine holonomy $\mathfrak{a}$. In fact, it also defines a topological invariant of B itself (which depends on a choice of basepoint) via the natural isomorphism

$$\mathrm{H}^1(\pi_1(B);\mathbb{R}^n_{\mathfrak{l}}) \cong \mathrm{H}^1(B;\mathbb{R}^n_{\mathfrak{l}}),$$

where the cohomology on the righthand side is taken with respect to the local coefficient system defined by the linear holonomy $\mathfrak{l}$ (cf. [GH84]).

Lemma 4.4 ([GH84])**.** *Let $(B,\mathcal{A})$ be an (integral) affine manifold and let ∇ denote the associated affine flat, torsion-free connection. The radiance obstruction $r_{(B,\mathcal{A})}$ is the obstruction to the existence of a* parallel *section $\sigma : B \to \mathrm{T}B$ for ∇.*

Lemma 4.4 can be seen as a geometric way to define the radiance obstruction. This is because the tangent bundle $\mathrm{T}B \to B$ of an affine manifold $(B,\mathcal{A})$ can be endowed with the structure of a *flat* affine bundle, *i.e.* the structure group can be reduced to $\mathrm{Aff}^{\delta}(\mathbb{R}^n)$, where superscript δ denotes the group endowed with the discrete topology. This can be seen explicitly as follows. Let $\mathcal{A} = \{U_i, \chi_i\}$ denote the given affine structure on B. For each i, define local affine trivialisations

$$\begin{aligned} \psi_i : \mathrm{T}U_i &\to U_i \times \mathbb{R}^n \\ v &\mapsto (\mathrm{pr}(v), (D\phi_i(\mathrm{pr}(v)))(v) + \chi_i(\mathrm{pr}(v))), \end{aligned} \tag{43}$$

where $\mathrm{pr} : \mathrm{T}B \to B$ denotes the footpoint projection.

Exercise 19. If, on a connected component of $U_i \cap U_j$, $\chi_j \circ \chi_i^{-1} = (A_{ji}, \mathbf{b}_{ji}) \in \mathrm{Aff}(\mathbb{R}^n)$, then prove that the transition function $\psi_j \circ \psi_i^{-1}$ is given by $(A_{ji}, \mathbf{b}_{ji})$.

The result of Exercise 19 implies that the local trivialisations of equation (43) endow the tangent bundle to B with the structure of a flat affine bundle, which is henceforth denoted by $\mathrm{T}^{\mathrm{Aff}}B \to B$. Lemma 4.4 simply states that the radiance obstruction $r_{(B,\mathcal{A})}$ can be interpreted as the obstruction to finding a globally defined *flat* section, *i.e.* a section which is constant in the local affine trivialisations given by equation (43) (cf. [GH84]).

There is yet another equivalent description of the radiance obstruction which arises in a more algebro-geometric setting (cf. [GS06]). Fix an affine manifold $(B,\mathcal{A})$ and let $\mathcal{A}ff((B,\mathcal{A});\mathbb{R})$ denote the sheaf of locally defined affine functions $f : U \subset (B,\mathcal{A}) \to \mathbb{R}$. Endow $\mathrm{T}^*B \to B$ with a flat affine structure (cf. equation (43)) and let $\mathcal{F}$ denote the sheaf of flat sections of this flat affine bundle. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}ff((B,\mathcal{A});\mathbb{R}) \xrightarrow{\mathrm{d}} \mathcal{F} \longrightarrow 0, \tag{44}$$

where $\mathbb{R}$ denotes the constant sheaf with values in the real numbers (by abuse of notation) and d denotes the morphism which maps an affine function to its differential. The following lemma, stated below without proof, relates the radiance obstruction of $(B,\mathcal{A})$ with the topology of the above short exact sequence.

Lemma 4.5 ([GS06])**.** *The radiance obstruction $r_{(B,\mathcal{A})}$ is the extension class of the short exact sequence of sheaves of equation* (44)*, i.e. it measures the obstruction to the existence of a global splitting of the above short exact sequence.*

This interpretation of the radiance obstruction is useful to solve the problem of constructing Lagrangian bundles over a given integral affine manifold (cf. Section 4.4).

4.3. **Integral affine geometry of Lagrangian bundles.**

Assumption 4. Henceforth, assume that any Lagrangian fibration $\pi : (M,\omega) \to B$ has compact and connected fibres. This is equivalent to asking that $\pi : (M,\omega) \to B$ is a fibre bundle. Such fibrations are called *Lagrangian bundles*

Remark 4.18. Note that Assumption 4 implies that the fibres of $\pi : (M,\omega) \to B$ are tori (cf. Corollary 3.2).

The aim of this section is to prove that the base space B of a Lagrangian bundle $\pi : (M, \omega) \to B$ inherits an integral affine structure $\mathcal{A}$ defined by the period net $\Lambda \subset (\mathrm{T}^*B, \Omega_{\mathrm{can}})$ associated to $\pi : (M, \omega) \to B$. Conversely, given an integral affine manifold $(B, \mathcal{A})$, it is possible to construct a Lagrangian bundle over B which induces the given integral affine structure $\mathcal{A}$ on B. This bundle is called the *symplectic reference* Lagrangian bundle associated to $(B, \mathcal{A})$; the Liouville-Mineur-Arnol'd theorem can be viewed as stating that any Lagrangian bundle $\pi : (M, \omega) \to B$ is locally fibrewise symplectomorphic to the symplectic reference Lagrangian bundle associated to $(B, \mathcal{A})$, where $\mathcal{A}$ is the integral affine structure on B induced by $\pi : (M, \omega) \to B$.

Lemma 4.6. *Let $\pi : (M, \omega) \to B$ be a Lagrangian bundle as above. Then B admits an integral affine structure $\mathcal{A}$.*

Proof. Let $\Lambda \subset (\mathrm{T}^*B, \Omega_{\mathrm{can}})$ be the period net associated to $\pi : (M, \omega) \to B$. Compactness of the fibres implies that for each $b \in B$ $\Lambda_b \cong \mathbb{Z}^n$, where $n = \dim B$. Moreover, the map $\mathrm{pr} : \Lambda \to B$ is a fibre bundle. In order to see this, note that the proof of Theorem 3.6 implies that there exist local smooth sections $\alpha^1, \ldots, \alpha^n : U \subset B \to \Lambda$ such that for all $b \in U$, $\alpha^1(b), \ldots, \alpha^n(b)$ are a $\mathbb{Z}$-basis of Λ_b. These sections yield a local trivialisation of $\mathrm{pr} : \Lambda \to B$ by setting

$$
\begin{aligned}
U \times \mathbb{Z}^n &\mapsto \Lambda|_U \\
(b, (k_1, \ldots, k_n)) &\mapsto \sum_{l=1}^{n} k_l \alpha^l(b).
\end{aligned}
\tag{45}
$$

Let $\bar{\alpha}^1, \ldots, \bar{\alpha}^n : \bar{U} \to \Lambda$ be a different choice of local smooth frame for $\Lambda|_{\bar{U}}$, where $U \cap \bar{U} \neq \emptyset$ is, without loss of generality, connected. Then there exists a matrix $A \in \mathrm{GL}(n; \mathbb{Z})$ such that for all $b \in U \cap \bar{U}$ and for all l

$$
\bar{\alpha}^l(b) = \sum_{p=1}^{n} A_{lp} \alpha^p(b);
\tag{46}
$$

note that, *a priori*, $A : U \cap \bar{U} \to \mathrm{GL}(n; \mathbb{Z})$ is a smooth function, but since $U \cap \bar{U}$ is connected, then it must be constant.

The construction of equation (45) can be carried out near each point $b \in B$, *i.e.* B can be covered with open sets U_i such that equation (45) yield local trivialisations of $\mathrm{pr} : \Lambda \to B$ over each U_i. Moreover, the transition functions for $\mathrm{pr} : \Lambda \to B$ with respect to this choice of local trivialisations are given by equation (46). Choose each U_i small enough that $\pi_1(U_i)$ is trivial (this can always be achieved by taking a refinement of the original open cover) and, for each i, let $\alpha_i^1, \ldots, \alpha_i^n$ be the local frame of $\Lambda|_{U_i}$ constructed above. Recall that any local section $\alpha : U \to \Lambda$ is a closed 1-form (cf. Corollary 3.9). Since each U_i is simply connected, any closed 1-form is exact, so there exist locally defined functions $a_i^1, \ldots, a_i^n : U_i \to \mathbb{R}$ such that for each l

$$
\mathrm{d}a_i^l = \alpha_i^l.
$$

Note that the fact that $\alpha_i^1(b), \ldots, \alpha_i^n(b)$ are a $\mathbb{Z}$-basis for Λ_b for all $b \in U_i$ implies that the map

$$
\chi_i := (a_i^1, \ldots, a_i^n) : U_i \to \mathbb{R}^n
\tag{47}
$$

is a local diffeomorphism. The maps χ_i of equation (47) are going to give integral affine coordinates on B. For this to be true, the change of coordinates has to be a restriction of an element of $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ on each connected component.

Consider i, j such that $U_i \cap U_j \neq \emptyset$ and, by restricting to a connected component, assume that it is connected. Then equation (46) implies that there exists a matrix $A_{ji} \in \mathrm{GL}(n; \mathbb{Z})$ such that for all $b \in U_i \cap U_j$ and for all l

$$
\mathrm{d}a_j^l = \sum_{p=1}^{n} A_{ji}^{lp} \mathrm{d}a_i^p = \mathrm{d}\left(\sum_{p=1}^{n} A_{ji}^{lp} a_i^p \right).
$$

Therefore, there exist a constant $\mathbf{c}_{ji} \in \mathbb{R}^n$ such that

$$
\mathbf{a}_j = A_{ji} \mathbf{a}_i + \mathbf{c}_{ji},
\tag{48}
$$

where $\mathbf{a}_j = (a_j^1, \ldots, a_j^n)$ and similarly for i. Equation (48) proves that the coordinates $\chi_i : U_i \to \mathbb{R}^n$ define an integral affine structure $\mathcal{A}$ on B as required. □

Conversely, let $(B, \mathcal{A})$ be an integral affine manifold and set $\mathcal{A} = \{U_i, \chi_i\}$, with $\chi_i : U_i \to \mathbb{R}^n$. Recall that there exists a period net of full rank $\Lambda_{\mathbb{R}^n} \subset (\mathrm{T}^*\mathbb{R}^n, \Omega_{\mathrm{can}})$ which is a closed Lagrangian submanifold (cf. Example 2.1.2). The resulting Lagrangian bundle over $\mathbb{R}^n$ is denoted by

$$(\mathrm{T}^*\mathbb{R}^n/\Lambda_{\mathbb{R}^n}, \omega_{0,\mathbb{R}^n}) \to \mathbb{R}^n.$$

For each i, consider the Lagrangian bundle

$$(\chi_i^*\mathrm{T}^*\mathbb{R}^n/\Lambda_{\mathbb{R}^n}, \omega_{0,i}) \to U_i,$$

where $\omega_{0,i}$ denotes the symplectic form obtained by pulling back $\omega_{0,\mathbb{R}^n}$. Set $\Lambda|_{U_i} := \chi_i^*\Lambda_{\mathbb{R}^n}$ and note that

$$\chi_i^*\mathrm{T}^*\mathbb{R}^n/\Lambda_{\mathbb{R}^n} \cong \mathrm{T}^*U_i/\Lambda|_{U_i};$$

the above diffeomorphism identifies $\omega_{0,i}$ with the symplectic form on $\mathrm{T}^*U_i/\Lambda|_{U_i}$ which descends from the canonical symplectic form Ω_{can} on T^*U_i. The fact that $\mathcal{A}$ is an integral affine structure implies that these locally defined Lagrangian bundles patch together to give a smooth, globally defined bundle

$$(\mathrm{T}^*B/\Lambda_{\mathcal{A}}, \omega_0) \to B, \tag{49}$$

where $\Lambda_{\mathcal{A}} \subset (\mathrm{T}^*B, \Omega_{\mathrm{can}})$ is a closed Lagrangian submanifold and ω_0 is the symplectic form induced by Ω_{can}.

Exercise 20.

i) Check that the above construction works;
ii) Check that the integral affine structure induced by the Lagrangian bundle of equation (49) on B is integral affinely diffeomorphic to $\mathcal{A}$.

Definition 4.19. The closed smooth Lagrangian submanifold $\Lambda_{\mathcal{A}}$ constructed above is called the *period net* associated to the integral affine manifold $(B, \mathcal{A})$.

The bundle of equation (49) plays an important role in the classification and construction of Lagrangian bundles.

Definition 4.20. The bundle of equation (49) is called the *symplectic reference Lagrangian* bundle associated to the integral affine manifold $(B, \mathcal{A})$.

Remark 4.21.

i) The symplectic reference Lagrangian bundle associated to (any) $(B, \mathcal{A})$ admits a globally defined Lagrangian section, *e.g.* the zero section;
ii) Henceforth, let $\pi : (M, \omega) \to (B, \mathcal{A})$ denote a Lagrangian bundle which induces the integral affine structure $\mathcal{A}$ on B;
iii) The above results present a different way of understanding integral affine structures on a given manifold B, namely as closed, smooth Lagrangian submanifolds $\Lambda \subset (\mathrm{T}^*B, \Omega_{\mathrm{can}})$ which are the total space of $\mathbb{Z}^n$-bundles over B. This point of view may be generalised to deal with integral affine structures which have *singularities* (work in progress).

4.4. **Construction of Lagrangian bundles.**

5. Singularities of Lagrangian fibrations

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