Constructing Hamiltonian quantum theories from path integrals in a diffeomorphism invariant context

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Abstract

Osterwalder and Schrader introduced a procedure to obtain a (Lorentzian) Hamiltonian quantum theory starting from a measure on the space of (Euclidean) histories of a scalar quantum field. In this paper, we extend that construction to more general theories which do not refer to any background, space-time metric (and in which the space of histories does not admit a natural linear structure). Examples include certain gauge theories, topological field theories and relativistic gravitational theories. The treatment is self-contained in the sense that an a priori knowledge of the Osterwalder-Schrader theorem is not assumed.

I. INTRODUCTION

For scalar field theories in flat space-time, the Osterwalder-Schrader framework provides a valuable link between Euclidean and Minkowskian descriptions of the quantum field. In this paper we will focus only on one aspect of that framework, namely the so-called ‘reconstruction theorem’ [1] which enables one to recover the Hilbert space of quantum states and the Hamiltonian operator, starting from an appropriate measure on the space of Euclidean paths. At least in simple cases, this procedure provides a precise correspondence between the path integral and canonical approaches to quantization. However, since even the basic axioms of the framework are deeply rooted in (Euclidean and) Poincaré invariance, a priori it is not obvious that the construction would go through in diffeomorphism invariant theories, such as general relativity. In particular, the idea of a ‘Wick rotation’, implicit in the original
framework, has no obvious meaning in this context. The purpose of this paper is to show that, in spite of these difficulties, the construction can be generalized to such theories.

While diffeomorphism invariance is our primary concern, we will also address another issue that arises already in Minkowskian field theories. It stems from the fact that the standard Osterwalder-Schrader framework [2] is geared to ‘kinematically linear’ systems —such as interacting scalar field theories— where the space of Euclidean paths has a natural vector space structure. More precisely, paths are assumed to belong to the space of Schwartz distributions and this assumption then permeates the entire framework. Although the assumption seems natural at first, in fact it imposes a rather severe limitation on physical theories that one can consider. In particular, in non-Abelian gauge theories, the space of gauge equivalent connections does not have a natural vector space structure. Therefore, if one wishes to adopt a manifestly gauge invariant approach, the space of histories cannot be taken to be one of the standard spaces of distributions [3,4]. Even if one were to use a gauge fixing procedure, because of Gribov ambiguities, one can not arrive at a genuine vector space if the space-time dimensions greater than two. Our extension of the Osterwalder-Schrader reconstruction theorem will incorporate such ‘kinematically non-linear’ theories.

Let us now return to our primary motivation. As noted above, since the standard formulation of the reconstruction theorem makes a crucial use of a flat, background metric, it excludes diffeomorphism invariant theories. The most notable examples are gravitational theories such as general relativity and topological ones such as Chern-Simons and BF theories, in which there is no background metric at all. To incorporate these cases, one has to generalize the very setting that underlies the Osterwalder-Schrader framework. A natural strategy would be to substitute the Poincaré group used in the original treatment by the diffeomorphism group. However, one immediately encounters some technical subtleties. In certain cases, such as general relativity on spatially compact manifolds, all diffeomorphisms are analogous to gauge transformations. Hence, while they have a non-trivial action on the space of paths, they have to act trivially on the Hilbert space of physical states. In other cases, such as general relativity in the asymptotically flat context, diffeomorphisms which are asymptotically the identity correspond to gauge while those that preserve the asymptotic structure but act non-trivially on it define genuine symmetries. These symmetries should therefore lead to non-trivial Hamiltonians on the Hilbert space of physical states. The desired extension of the Osterwalder-Schrader framework has to cater to these different situations appropriately.

Thus, from a conceptual viewpoint, the extensions contemplated here are very significant. However, it turns out that, once an appropriate setting is introduced, the technical steps are actually rather straightforward. With natural substitutions suggested by this generalized setting, one can essentially follow the same steps as in the original reconstruction [2] with minor technical modifications. In particular, it is possible to cater to the various subtleties mentioned above.

The plan of this paper is as follows. Section II introduces the new setting and section III contains the main result, the generalized reconstruction theorem. Section IV (and the Appendix) discuss examples which illustrate the reconstruction procedure. These examples will in particular suggest a manifestly gauge invariant approach in the non-Abelian context and also bring out different roles that the diffeomorphism group can play and subtleties associated with them. Section V summarizes the main results and briefly discusses their
II. THE GENERAL SETTING

This section is divided into three parts. In the first, we introduce the basic framework, in the second we discuss some subtleties associated with diffeomorphism invariance and in the third we present the the modified axioms.

A. Basic framework

Heuristically, our task is to relate the path integral and canonical approaches for a system which may not have a background metric structure. Let us therefore begin with a differential manifold $M$ of dimension $D+1$ and topology $\mathbb{R} \times \sigma$, where $\sigma$ is a $D$-dimensional smooth manifold of arbitrary but fixed topology. $M$ will serve as the non-dynamical arena for theories of interest. The product topology of $M$ will play an important role in what follows. In particular, it will enable us to generalize the notion of ‘time-translations’ and ‘time-reflections’ which play an important role in the construction. Since our goal is to obtain a Hamiltonian quantum theory, it is not surprising that we have to restrict ourselves to a product topology.

Our generalized Osterwalder-Schrader axioms will require the use of several structures associated with $M$. The fact that $M$ is diffeomorphic to $\mathbb{R} \times \sigma$ in particular means that it can be foliated by leaves diffeomorphic to $\sigma$. To be precise, consider the set $\text{Emb}(\sigma, M)$ of all embeddings of $\sigma$ into $M$. A foliation $E = \{E_t\}_{t \in \mathbb{R}}$ is a one-parameter family of elements of $\text{Emb}(\sigma, M)$, $E_t \in \text{Emb}(\sigma, M)$, which varies smoothly with $t$ and provides a diffeomorphism between $\mathbb{R} \times \sigma$ and $M$. The set of foliations $\text{Fol}(\sigma, M)$, given by all diffeomorphisms from $\mathbb{R} \times \sigma$ to $M$, will be of special interest to us. Notice that the embedded hyper-surfaces $\Sigma_t = E_t(\sigma)$ have not been required to be ‘space-like, time-like or null.’ Indeed, there is no background metric to give meaning to these labels.

Each foliation $E \in \text{Fol}(\sigma, M)$ enables one to generalize the standard notions of time translation and time reflection. To see this, first note that since $E$ is of the form $E : \mathbb{R} \times \sigma \to M; (t, x) \mapsto X = E_t(x)$, the inverse map $E^{-1}$ defines functions $t_E(X)$ and $x_E(X)$ from $M$ to $\mathbb{R}$ and $\sigma$ respectively. The time translation $\varphi^\Delta$, with $\Delta \in \mathbb{R}$, is the diffeomorphism on $M$, $(t_E(X), x_E(X)) \mapsto (t_E(X) + \Delta, x_E(X))$, which is simply a shift of the time coordinate $t_E$ by $t$, holding $x_E(X)$ fixed. Similarly, the time reflection $\theta_E$ is the diffeomorphism of $M$ defined by $(t_E(X), x_E(X)) \mapsto (-t_E(X), x_E(X))$. We also consider the positive and negative half spaces $S^E_\pm$, defined by $X \in S^E_\pm$ if and only if $\pm t_E(X) \geq 0$. Although these notions are tied to a specific foliation, our final constructions and results will not refer to a preferred foliation.

We now turn to the structures associated with the particular quantum field theory under consideration. Let us assume that our theory is associated with a classical Lagrangian density which depends on a collection of basic (bosonic) fields $\phi$ on $M$ and their various partial derivatives. We will not explicitly display discrete indices such as tensorial or representation space indices, so that the symbol $\phi$ may include, in addition to scalar fields, higher spin fields which may possibly take values in a representation of the Lie-algebra of a structure group.
The fields $\phi$ belong to a space $\mathcal{C}$ of \textit{classical histories} which is typically a space of smooth (possibly Lie algebra-valued) tensor fields equipped with an appropriate Sobolev norm. In the case of a gauge theory, there will be an appropriate gauge bundle over $M$. We assume that the action of $\text{Diff}(M)$, the diffeomorphism group of $M$, has a lift to this bundle, from which an action on $\mathcal{C}$ follows naturally. For notational simplicity, we will denote this action of $\text{Diff}(M)$ on $\mathcal{C}$ simply by $\phi \mapsto \varphi \phi$ for any $\varphi \in \text{Diff}(M)$.

Of greater interest than $\mathcal{C}$ will be the set $\overline{\mathcal{C}}$ of \textit{quantum histories} which is generally an extension of $\mathcal{C}$. In a kinematically linear field theory, $\mathcal{C}$ is typically the space of Schwartz distributions [2]. The extension from $\mathcal{C}$ to $\overline{\mathcal{C}}$ is essential because, while $\mathcal{C}$ is densely embedded in $\overline{\mathcal{C}}$ in the natural topology, in physically interesting cases, $\mathcal{C}$ is generally of measure zero, whence the genuinely distributional paths in $\overline{\mathcal{C}}$ are crucial to path integrals. In the more general case now under consideration, we leave the details of the extension unspecified, as they depend on the particulars of the theory being considered, and refer to elements of $\overline{\mathcal{C}}$ simply as \textit{generalized fields}. For example, in a gauge theory these might include generalized connections discussed briefly in Section IV (and in detail in [3]). For notational simplicity, the symbol $\phi$ will be used to denote generalized fields (elements of $\overline{\mathcal{C}}$) as well as smooth fields in $\mathcal{C}$; the context will remove the ambiguity.

Consider then a suitable collection of subsets of $\overline{\mathcal{C}}$ and denote by $\mathcal{B}$ the $\sigma$-algebra it generates. This equips $\overline{\mathcal{C}}$ with the structure of a measurable space. Let us further consider the set $\mathcal{F}(\overline{\mathcal{C}})$ of measurable functions on this space (that is, functions for which the pre-image of any Lebesgue measurable set is a measurable set).

With this background material at hand, we can now introduce a key technical notion, that of a \textit{label set} $\mathcal{L}$, which in turn will enable us to define the basic random variables and stochastic process. $\mathcal{L}$ is to be regarded as being ‘dual’ to the space $\overline{\mathcal{C}}$ of generalized fields. That is, it must be chosen to match the structure of $\mathcal{C}$ so that there is a well-defined ‘pairing’ $P$:

$$P : \mathcal{L} \rightarrow \mathcal{F}(\overline{\mathcal{C}}); \ f \mapsto P_f . \quad (2.1)$$

For example, in kinematically linear field theories on $\mathbb{R}^{D+1}$, typically $\overline{\mathcal{C}}$ is taken to be the space of Schwartz distributions with appropriate tensor and internal indices. Then, $\mathcal{L}$ consists of smooth, rapidly decreasing (test) functions $f$ on $\mathbb{R}^{D+1}$, with the pairing $P$ defined by $P_f(\phi) = \exp(i\phi(f))$, where $\phi(f) := \int d^{D+1}X \, \phi(X)f(X)$. In $SU(2)$ gauge theories, a natural candidate for $\mathcal{L}$ is the space of loops on $M$ and the pairing is then defined by $P_f(\phi) = \text{Tr} \, h_f(\phi)$, the trace of the holonomy $h_f(\phi)$ of the generalized connection $\phi$ around the loop $f$ in a suitable representation of the structure group [4, 3]. In general, we will assume that each $f \in \mathcal{L}$ is ‘associated with’ a set $\text{supp}(f) \subset M$, which we call the support of $f$. The pairing defines a \textit{stochastic process} $f \rightarrow P_f(\phi)$, and we refer to $P_f$ as a \textit{random variable}.

In the general framework, we will not be concerned with the details of the pairing $P$, but merely ask that it satisfies the following three properties:

\begin{itemize}
  \item[(A1)] The pairing is diffeomorphism covariant, in the sense that there exists a left action of $\text{Diff}(M)$ on $\mathcal{L}$, which we denote $f \mapsto (\varphi^{-1})f$, such that $P_f(\varphi \phi) = P_{(\varphi^{-1})f}(\phi)$ for any $\phi \in \mathcal{C}$. Furthermore, we require that $\text{supp}((\varphi^{-1})f) = \varphi(\text{supp}(f))$.
\end{itemize}
We also introduce a left action of $\varphi$ on random variables: $\varphi(P_f) = P_{\varphi^{-1}f}$. Note that in the familiar case of scalar field theories where the label set is taken to be the set of Schwarz space functions ($f \in \mathcal{S}$), the action of $\varphi$ on $\mathcal{L}$ is $(\varphi^{-1})f = f \circ \varphi^{-1}$.

Let us denote by $\mathcal{A}$ the set of finite linear combinations ($N < \infty$) of random variables $P_f$:

$$\psi(\phi) := \sum_{I=1}^{N} z_I P_{f_I}(\phi) \quad (2.2)$$

with $z_I \in \mathbb{C}$ and $f_I \in \mathcal{L}$. The second assumption about the pairing $P$ is:

$(A2)$ The vector space $\mathcal{A}$ is in fact a $\star$-algebra with unit, whose $\star$ operation is complex conjugation of functions on $\mathcal{C}$. The algebraic operations of $\mathcal{A}$ must commute with the action of diffeomorphisms in the sense that:

$$\text{For } a,b \in \mathcal{A}, \quad \varphi(ab) = [\varphi(a)][\varphi(b)], \quad \text{and} \quad [\varphi(a^*)] = [\varphi(a)]^*. \quad (2.3)$$

The first part of this property will allow us to calculate scalar products between elements of $\mathcal{A}$ with respect to suitable measures on $\mathcal{C}$ purely in terms of expectation values of the random variables $P_f$. Note that, in the kinematically linear theories as well as the gauge theories referred to above, this assumption is automatically satisfied.

Next, let us consider a $\sigma$-additive probability measure $\mu$ on the measurable space $(\mathcal{C}, \mathcal{B})$, thus equipping it with the structure of a measure space $(\mathcal{C}, \mathcal{B}, \mu)$. This structure naturally gives rise to the so called ‘history Hilbert space’

$$\mathcal{H}_{D+1} := L_2(\mathcal{C}, d\mu) \quad (2.4)$$

of square integrable functions. We denote the inner product between $\psi, \psi' \in \mathcal{A}$ by

$$\langle \psi, \psi' \rangle := \int_{\mathcal{C}} d\mu \, \bar{\psi}(\phi) \psi'(\phi). \quad (2.5)$$

Our third requirement on $P$ is that:

$(A3)$ The space $\mathcal{A}$ is dense in $\mathcal{H}_{D+1}$ for some measure $\mu$ on $\mathcal{C}$.

As mentioned above, we are primarily interested in diffeomorphism invariant theories. The pairing allows us to define a representation $\hat{U}(\varphi)$ of $\text{Diff}(M)$ on the dense subspace $\mathcal{A}$ of $\mathcal{H}_{D+1}$:

$$[\hat{U}(\varphi) P_f](\phi) := P_{(\varphi^{-1})f}(\phi) = (\varphi P_f)(\phi). \quad (2.6)$$

At this point, $\hat{U}(\varphi)$ is a densely defined operator on $\mathcal{H}_{D+1}$. When the measure $d\mu$ is invariant under diffeomorphisms, we will see that this operator is in fact unitary and extends to all of $\mathcal{H}_{D+1}$. 

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Finally, in the formulation of the key, ‘reflection positivity’ axiom and in the proof of the reconstruction theorem, we will need certain subsets $\mathcal{A}_E^\pm$ of $\mathcal{A}$. These are defined by restricting the supports of the $f_I$ in (2.2) to be contained in half spaces $S_E^\pm$ on which $\pm t_E \geq 0$.

**B. Subtleties**

We are nearly ready to formulate our extension of the Osterwalder-Schrader axioms. However, our emphasis on diffeomorphism invariant systems will cause a certain change of perspective from the familiar case, e.g. of a kinematically linear field theory in flat space-time. In these simpler theories, the Hamiltonian is an object of primary concern, and its construction is central to the Osterwalder-Schrader reconstruction theorem. Now, we no longer have a background metric and therefore no a priori notion of time translations which Hamiltonians normally generate. An obvious strategy is to treat all diffeomorphisms as symmetries and seek the corresponding Hamiltonians on the Hilbert space of physical states. However, this turns out not to be the correct procedure because of two subtleties.

First, in many diffeomorphism invariant systems, the structure of the classical theory tells us that all diffeomorphisms should be regarded as gauge transformations. This can follow from one of the following three related considerations: i) The initial value formulation could show that the initial data can determine a classical solution only up to diffeomorphisms; or, ii) For fixed boundary values of fields, the variational principle may provide infinitely many solutions, all related to one another by diffeomorphisms which are identity on the boundary; or, iii) In the Hamiltonian formulation, there may be first class constraints whose Hamiltonian flows correspond to the induced action of $\text{Diff}(M)$ on the phase space. (Typically, one of these implies the other two.) An example where this is the case is general relativity on a spatially compact manifold. In these cases, one expects quantum states to be diffeomorphism invariant, i.e., the corresponding Hamiltonian operators to vanish identically on the physical Hilbert space. In these theories, then, the reconstruction problem should reduce only to the construction of the Hilbert space of physical states starting from a suitable measure on $\mathcal{C}$.

The second subtlety is that many interesting theories use a background structure. They are therefore not invariant under the full diffeomorphism group but only under the subgroup which preserves the background structure. An interesting example is provided by the Yang-Mills theory in two dimensional space-times which requires an area form, but not a full metric, for its formulation. The theory is therefore invariant under the group of all area preserving diffeomorphisms, a group which is significantly larger than, say, the Poincaré group but smaller than the group of all diffeomorphisms. (Since the area form is a symplectic form in two dimensions, the group of area preserving diffeomorphisms coincides with the group of symplectomorphisms.) A more common situation is illustrated by general relativity in any space-time dimension, subject to asymptotically flat (or anti-de Sitter) boundary conditions. Here, the background structure consists of a flat (or anti-de Sitter) geometry at infinity. One must therefore restrict oneself to those diffeomorphisms which preserve the specified asymptotic structure. In the general case, we will denote the background structure
by $s$ and the sub-group of $\text{Diff}(M)$ preserving this structure by $\text{Diff}(M, s)$.

In the presence of a background structure $s$, our foliations will also be restricted to be compatible with $s$ in the sense that the associated generalized time translations $\varphi^t_E$ and time reflections $\theta_E$ constructed above preserve $s$. Now, given any two foliations $E, \tilde{E}$, there is a unique diffeomorphism $\varphi_{E\tilde{E}}$ on $M$ which maps $E$ to $\tilde{E}$. Note however that even if $E$ and $\tilde{E}$ are compatible with $s$, $\varphi_{E\tilde{E}}$ need not preserve $s$. This leads us to the following important definitions:

**Definition 1**

(a) Two foliations $E$ and $\tilde{E}$ are strongly equivalent if $\tilde{E} = \varphi_{E\tilde{E}} \circ E$ for some $\varphi_{E\tilde{E}} \in \text{Diff}(M, s)$.

(b) Two foliations $E, \tilde{E}$ will be said to be weakly equivalent if there exists foliations $E', \tilde{E}'$ which are strongly equivalent to $E, \tilde{E}$ respectively such that the time-reflection maps of $E'$ and $\tilde{E}'$ coincide, i.e. $\theta_{E'} = \theta_{\tilde{E}'}$.

Note that strong equivalence trivially implies weak equivalence but the converse is not true. A simple example which illustrates the difference between these two notions of equivalence is provided by setting $M = \mathbb{R}^4$ and choosing the background structure $s$ to be a Minkowskian metric $\eta$. Define $E, \tilde{E}$ as follows: $E : \mathbb{R} \times \mathbb{R}^3 \to M; (t, x) \mapsto E_t(x) = (t, x)$ and $\tilde{E} : \mathbb{R} \times \mathbb{R}^3 \to M; (t, x) \mapsto \tilde{E}_t(x) = (bt, x)$, where $b$ is a positive constant. Both of these foliations are compatible with the background structure. However, since the diffeomorphism $\varphi_{E\tilde{E}}$ is not an isometry of $\eta$, the two foliations are not strongly equivalent. However, they define the same time-reflection map and are therefore weakly equivalent.

Strong equivalence of $E$ and $\tilde{E}$ means that the foliations are in fact related by a symmetry $\varphi_{E\tilde{E}}$ of the theory and we will see that this symmetry defines a unitary mapping of the physical Hilbert space associated with $E$ to that associated with $\tilde{E}$ which takes the Hamiltonian generator of $\varphi^t_E$ to that of $\varphi^t_{\tilde{E}}$. In the case of weak equivalence, the foliations are not related by a symmetry and we should expect no correspondence between the Hamiltonians. The point of this definition, however, is that the construction of the physical Hilbert space itself will depend only on the time inversion map $\theta_E$ induced by the foliation $E$. Thus, when $E$ and $\tilde{E}$ are weakly equivalent, we will still be able to show that the physical Hilbert spaces are naturally unitarily equivalent, though this equivalence will not of course map the generator of $\varphi^t_E$ to that of $\varphi^t_{\tilde{E}}$.

Finally, it is typical in such theories that certain diffeomorphisms play the role of genuine symmetries while others play the role of gauge. Generally, there is a normal subgroup $\text{Diff}_G(M, s)$ of $\text{Diff}(M, s)$ which acts as gauge while the quotient, $\text{Diff}(M, s)/\text{Diff}_G(M, s)$, acts as a symmetry group. (In asymptotically flat general relativity, for example, $\text{Diff}_G(M, s)$ consists of asymptotically trivial diffeomorphisms and the quotient is isomorphic to the Poincaré group.) In these contexts, we have a mixed situation: $\text{Diff}_G(M, s)$ should have a trivial action on the physical Hilbert space, while the action of a symmetry diffeomorphism should be generated by a genuine Hamiltonian as in the original reconstruction theorem.

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1 If there is no background structure, as for example in general relativity or topological field theories on a spatially compact manifold, by $\text{Diff}(M, s)$ we will mean simply $\text{Diff}(M)$. 
C. Generalized Osterwalder-Schrader Axioms

With these subtleties in mind, we can now state our generalization of the Osterwalder-Schrader axioms. Our numbering of the axioms below is chosen to match that of [2]. For flexibility, we wish to allow the possibility that a quantum theory may satisfy only a subset of the following axioms. We will be careful in what follows to explicitly state which axioms are required in order that the various conclusions hold.

As in the original construction, the key mathematical object will be a measure \( \mu \) on the space of quantum histories. In Minkowskian field theories, \( \mu \) can be thought of as a rigorous version of the heuristic measure \( \exp(-S(\phi))D\phi \) constructed from the Euclidean action. In the standard Osterwalder-Schrader framework, there are two axioms which are central to the construction of the Hilbert space of states and both are restrictions on the measure \( \mu \). The first asks that \( \mu \) be Euclidean invariant and the second asks that it satisfy a technical condition called ‘reflection positivity’ formulated in terms of the ‘time-reflection’ operator \( \theta \) in the Euclidean space. Given a measure \( \mu \) with these properties, one can quotient the space \( L^2(\mathcal{C},d\mu) \) of square-integrable functions on quantum histories by a certain sub-space, defined by \( \theta \), to obtain the Hilbert space of quantum states. Heuristically, the restrictions of quantum histories to the \( D \)-dimensional, \( t = 0 \) slice in the Euclidean space define the ‘quantum configuration space’ \( \mathcal{C}_{t=0} \) and the quotient enables one to pass from \( L^2(\mathcal{C},d\mu) \) to the space of square-integrable functions on \( \mathcal{C}_{t=0} \). The remaining axioms ensure the existence of a Hamiltonian operator and existence and uniqueness of the vacuum state.

In the present context, the time reflection operator \( \theta \) is replaced by its generalization \( \theta_E \) associated with a foliation and the Poincaré group is replaced by \( \text{Diff}(M,s) \). Thus, given any foliation \( E \), one can essentially repeat the original construction to obtain a Hilbert space \( \mathcal{H}^E_{D} \) of physical quantum states. The diffeomorphism invariance of \( \mu \) then provides unitary maps relating physical Hilbert spaces constructed from equivalent foliations.

**Definition 2** A quantum theory of fields \( \phi \in \mathcal{C} \) on a space-time \( M \) diffeomorphic to \( \mathbb{R} \times \sigma \) is defined by a probability measure \( \mu \) on \( \mathcal{C} \) and a pairing \( P \) satisfying (A1), (A2) and (A3) above. The generating functional \( \chi \) defined by

\[
\chi(f) := \langle P_f \rangle := \int_{\mathcal{C}} d\mu(\phi) P_f(\phi).
\]

(2.7)

should satisfy at least the first two of the following axioms:

(II) **DIFFEOMORPHISM INVARIANCE**

The measure is diffeomorphism invariant\(^2\) in the sense that, for any \( \varphi \in \text{Diff}(M,s) \), \( \chi \) satisfies:

\[
\chi(f) = \chi(\varphi^{-1}f).
\]

(2.8)

\(^2\) In the case when there are symmetries in the quantum field theory other than diffeomorphism invariance, it would be natural to ask that the measure be invariant under these symmetries as well.
(III) REFLECTION POSITIVITY
Consider the sesquilinear form on $A^+_E$ defined for any $E \in \text{Fol}(\sigma, (M, s))$ by
\[ \langle \psi, \psi' \rangle_E := \langle \hat{U}(\theta_E)\psi, \psi' \rangle. \quad (2.9) \]
We require $(\psi, \psi)_E \geq 0$ for any $\psi \in A^+_E$.

(GI) GAUGE INVARIANCE
For all $\varphi \in \text{Diff}_G(M, s)$ and all $\psi \in A^+_E$ we require
\[ \|\left(\hat{U}(\varphi) - 1\right)\psi\|_E = 0, \quad (2.10) \]
where $\| \|_E$ denotes the norm associated with the inner product introduced in axiom III.

(I) CONTINUITY
For any $E \in \text{Fol}(\sigma, (M, s))$ for which the one-parameter group of diffeomorphisms $\varphi^t_E$ does not belong to $\text{Diff}_G(M, s)$, it acts strongly continuously by operators $\hat{U}(\varphi^t_E)$ on $\mathcal{H}_{D+1}$.

(IV) CLUSTERING
For any $E \in \text{Fol}(\sigma, (M, s))$ for which $\varphi^t_E$ does not belong to $\text{Diff}_G(M, s)$, we have that
\[ \lim_{t \to \infty} \langle \psi, \hat{U}(\varphi^t_E)\psi' \rangle = \langle \psi, 1 \rangle \langle 1, \psi' \rangle \quad (2.11) \]
for any two $\psi, \psi' \in \mathcal{A}$.

Note that if axiom (III), (GI), (I) or (IV) holds for some foliation $E \in \text{Fol}(\sigma, (M, s))$ then, because of the diffeomorphism invariance (II) of the measure, the axiom also holds for any $\tilde{E}$ which is strongly equivalent to $E$. Furthermore, if axiom (III) or (GI) holds for some foliation $E \in \text{Fol}(\sigma, (M, s))$ then it in fact holds for any $\tilde{E}$ which is weakly equivalent to $E$.

We will conclude this sub-section by comparing these axioms with the standard ones of Osterwalder-Schrader [2]. In axiom (II), we have replaced the Euclidean group in the standard formulation by $\text{Diff}(M, s)$, and in (I), the time translation group by $\varphi^t_E$. Axiom (I) is usually referred to as the ‘regularity’ axiom and typically phrased as a more technical condition specific to scalar field theories on a flat background [2]. However, as its essential role in that case is to ensure strong continuity of time translations, and as this condition is straightforward to state in the diffeomorphism invariant context, we have chosen to promote this condition itself to the axiom. Finally, note that we have discarded the zeroth ‘analyticity axiom’ of [2] which, roughly speaking, requires the generating functional $\chi$ to be analytic in $f \in \mathcal{L}$. However, it is not clear that a label space $\mathcal{L}$ appropriate to a diffeomorphism invariant or kinematically non-linear field theory should carry an analytic structure. In the standard formulation this axiom allows one to define Schwinger n-point functions in terms of $\chi(f)$. Fortunately, Schwinger functions are not essential to our limited goal of defining the Hilbert space theory.

\[ ^3 \text{Again, if there exist gauge symmetries in addition to gauge diffeomorphisms, we ask that these be represented trivially on the physical Hilbert space as well.} \]
III. RECOVERY OF THE HILBERT SPACE THEORY

The central subject of this section is the following straightforward extension of the classical Osterwalder-Schrader reconstruction theorem [2].

Theorem 1  

i) For each \( E \in \text{Fol}(\sigma, (M, s)) \), axioms (II) and (III) imply the existence of a Hilbert space \( \mathcal{H}_D^E \) of physical states. There is a natural class of unitary equivalences between \( \mathcal{H}_D^E \) and \( \mathcal{H}_D^{\tilde{E}} \) for all \( E \) and \( \tilde{E} \) in the (weak) equivalence class of \( E \).

ii) Axioms (I), (II) and (III) imply the existence of self-adjoint operators \( \hat{H}^E \) on \( \mathcal{H}_D^E \) which generate time translations and which have \( [1]_E \) as a vacuum state. If the foliations \( E \) and \( \tilde{E} \) are strongly equivalent, then the operators \( \hat{H}^E \) and \( \hat{H}^{\tilde{E}} \) are mapped to each other by a unitary equivalence of the Hilbert spaces \( \mathcal{H}_D^E \) and \( \mathcal{H}_D^{\tilde{E}} \).

iii) Axioms (I), (II), (III) and (IV) imply that the vacuum \( [1]_E \) is unique in each \( \mathcal{H}_D^E \). These states are mapped to each other by the unitary equivalence of ii) above.

We break the proof of this theorem into several lemmas. As noted in the Introduction, the essence of the proof is the same as that in the original Osterwalder-Schrader reconstruction but we present it here for completeness. In the following, \( E \) is an arbitrary but fixed foliation compatible with the background structure (if any).

Lemma 1  

By axiom (II), the family of operators, densely defined on \( \mathcal{H}_{D+1} \) for any \( \varphi \in \text{Diff}(M, s) \) by

\[
\hat{U}(\varphi)P_f := P_{(\varphi^{-1})f}
\]

(3.1)

can be extended to a unitary representation of \( \text{Diff}(M, s) \).

Proof of lemma [4]:

By (A3), \( \mathcal{A} \) is in fact dense in \( \mathcal{H}_{D+1} \). Since \( \hat{U}(\varphi) \) has the inverse \( \hat{U}(\varphi^{-1}) \) on \( \mathcal{A} \), it will be sufficient to show that \( \hat{U}(\varphi) \) is norm-preserving on \( \mathcal{A} \) for any \( \varphi \in \text{Diff}(M, s) \) and then to use continuity to uniquely extend it to \( \mathcal{H}_{D+1} \). Recalling condition (A2), for states \( a, b \in \mathcal{A} \), we have

\[
\langle \hat{U}(\varphi)a, \hat{U}(\varphi)b \rangle = \int_{C} d\mu \varphi(a^*b).
\]

(3.2)

But, since the measure is diffeomorphism invariant, this is just the expectation value of \( a^*b \), which is the inner product \( \langle a, b \rangle \).

\[\square\]

Lemma 2  

By axioms (II), (III) the sesquilinear form \( (2.3) \) defines a non-negative hermitian form on \( \mathcal{A}^+ \).

\[\text{That is, a state annihilated by } [\hat{H}^E]. \text{ The brackets on } [\hat{H}^E] \text{ denote an operator on } \mathcal{H}_D^E \text{ as opposed to } L^2(C, d\mu) \text{ and } [1]_E \in \mathcal{H}_D^E \text{ is the equivalence class of elements of } \mathcal{H}_{D+1}^E \text{ to which the unit function belongs. See the observation below Eq. (3.5).}\]

\[10\]
Proof of lemma 2:
The hermiticity follows easily from the fact that $\theta_E \circ \theta_E = \text{id}_M$ (so that $[\hat{U}(\theta_E)]^\dagger = \hat{U}(\theta_E)$), unitarity of the $\hat{U}(\varphi)$ as established in lemma 1, and the hermiticity of $\langle \ldots \rangle$. We have
\[
\begin{align*}
\langle \psi, \psi' \rangle_E &= \langle \hat{U}(\theta_E)\psi, \psi' \rangle \\
&= \langle \hat{U}(\theta_E)\psi', \psi \rangle = \langle \psi', \psi \rangle_E
\end{align*}
\] (3.3)

Non-negativity is the content of axiom (III).
\[\Box\]

**Lemma 3** The null space $N_E := \{ \psi \in A_E^+; (\psi, \psi)_E = 0 \}$ is in fact a linear subspace of $A_E^+$ owing to axioms (II), (III). As a result, the form $(\ldots)_E$ is well-defined and positive definite on
\[\mathcal{H}_D^E := \overline{A_E^+/N_E}, \] (3.4)
where the over-line denotes completion with respect to $(\ldots)_E$.

Proof of lemma 3:
This is a consequence of the Schwarz inequality for positive semi-definite, hermitian, sesquilinear forms.
\[\Box\]

**Lemma 4** The map $[\ldots]_E : A_E^+ \to A_E^+/N_E$ is a contraction, that is, $||[\psi]_E||_{\mathcal{H}_D^E} \leq ||\psi||_{\mathcal{H}_{D+1}}$ owing to axioms (II), (III).

Proof of lemma 4:
By the Schwarz-inequality for $\langle \ldots \rangle$ and the unitarity of $\hat{U}(\theta_E)$, we have
\[
\langle [\psi]_E, [\psi]_E \rangle_{\mathcal{H}_D^E}^2 = \langle \hat{U}(\theta_E)\psi, \psi \rangle^2 \\
\leq \langle \hat{U}(\theta_E)\psi \rangle_{\mathcal{H}_{D+1}} \langle \psi \rangle_{\mathcal{H}_{D+1}} = ||\psi||_{\mathcal{H}_{D+1}}^2.
\] (3.5)
\[\Box\]

We will also need the following observation: consider an operator $\hat{A}$ on $\mathcal{H}_{D+1}$ with dense domain $\mathcal{D}(\hat{A})$ satisfying the following properties:

Pi) $\mathcal{D}_E(\hat{A}) := (\mathcal{D}(\hat{A}) \cap A_E^+)/N_E$ is dense in $\mathcal{H}_D^E$.

Pii) $\hat{A}$ maps $\mathcal{D}(\hat{A}) \cap A_E^+$ into $A_E^+$ and

Piii) $\hat{A}$ maps $\mathcal{D}(\hat{A}) \cap N_E$ into $N_E$.

Then the operator $[\hat{A}] : \mathcal{D}_E(\hat{A}) \to \mathcal{H}_D^E$ defined by $[\hat{A}][\psi] := [\hat{A}\psi]$ is well-defined since $[A\psi] = [0]$ for any $\psi \in N_E$.

We have now prepared all the tools necessary to complete the proof of the theorem. Notice that lemmas 1 through 4 have so far used only the axioms (II) and (III).

Proof of theorem 1:
i) For every \( E \in \text{Fol}(\sigma, (M, s)) \) we have already constructed \( \mathcal{H}^E_D \). Now, let us suppose two foliations \( E \) and \( \bar{E} \) are weakly equivalent. Then, there exist \( \varphi_{EE'}, \varphi_{ar{E}E'} \in \text{Diff}(M, s) \) such that \( E' = \varphi_{EE'} \circ E \) and \( \bar{E}' = \varphi_{ar{E}E'} \circ \bar{E} \) define the same time-reflection map. Therefore, \( \mathcal{H}^E_D = \mathcal{H}^{ar{E}}_D \).

On the other hand, because of the diffeomorphism \( \varphi_{EE'} \), any vector \( \psi' \in A^+_E \) is of the form \( \hat{U}(\varphi_{EE'})\psi \) for some vector \( \psi \in A^+_E \). Since \( \hat{U}(\varphi_{EE'}) \) maps \( N_E \) to \( N_{E'} \), \( \hat{U}(\varphi_{EE'}) \) respects the quotient construction and we can define a norm-preserving operator \( [\hat{U}_{EE'}] : \mathcal{H}^E_D \to \mathcal{H}^{E'}_D \) by

\[
[\hat{U}_{EE'}][\psi]_E := [\hat{U}(\varphi_{EE'})\psi]_{E'}.
\] (3.6)

Thus, the Hilbert spaces \( \mathcal{H}^E_D, \mathcal{H}^{E'}_D \) are unitarily equivalent in a natural way. Similarly, there exists a unitary map \( [\hat{U}_{EE'}] : \mathcal{H}^{E'}_D \to \mathcal{H}^E_D \). Hence,

\[
[\hat{U}_{EE'}] := [\hat{U}_{EE'}]^{-1} [\hat{U}_{EE'}]
\] (3.7)

is a natural isomorphism between \( \mathcal{H}^E_D \) and \( \mathcal{H}^{E'}_D \).

ii) We will first show the following:

**Lemma 5** For any \( E \in \text{Fol}(\sigma, (M, s)) \) and any \( t \geq 0 \) the operator \( \hat{U}(\varphi^t_E) \) satisfies the properties Pi), Pii) and Piii) above and gives rise to a self-adjoint, one-parameter contraction semi-group \( \hat{C}^t_E \) on \( \mathcal{H}^E_D \). If \( \varphi^t_E \not\in \text{Diff}_0(M, s) \) then the contraction semi-group is also strongly continuous and its generator \( [\hat{H}^E] \) is a self-adjoint, positive semi-definite operator on \( \mathcal{H}^E_D \) with \([1]_E\) a vacuum state for \([\hat{H}^E] \).

Proof of lemma 5:

First we show that \( U(\varphi^t_E) \) has the required properties Pi), Pii), Piii). Consider \( f \in \mathcal{L} \) with \( \text{supp}(f) \subset S^E_+ \). Then \( \text{supp}(\varphi^t_E(f)) = \varphi^t_E(\text{supp}(f)) \subset \varphi^t_E(S^E_+) = S^E_+ \) since for positive \( t \)

we have a time translation into \( S^E_+ \) (‘the future of \( E_0(\sigma) \)’). Thus, \( U(\varphi^t_E) \) maps \( A^+_E \) into itself. From the Schwarz inequality we infer that it also maps \( N_E \) into itself and, finally, since \( A^+_E \subset A = D(U(\varphi_E)) \) we have \( D_E(U(\varphi^t_E)) = A^+_E/N_E \) which is dense in \( \mathcal{H}^E_D \). Thus, the operator

\[
[\hat{C}^t_E][\psi]_E := [\hat{U}(\varphi^t_E)\psi]_E
\] (3.8)

is well defined on a dense domain \( D_E([\hat{C}^t_E]) := A^+_E/N_E \) of \( \mathcal{H}^E_D \) independent of \( t \).

That it defines a semi-group follows from the definition (3.8), the fact that the \( \varphi^t_E \) form a group under composition and the fact that \( \hat{U} \) defines a unitary representation of \( \text{Diff}(M, s) \) on \( \mathcal{H}^{D+1}_D \). We have

\[
[\hat{C}^t_E][\hat{C}^s_E][\psi]_E = [\hat{U}(\varphi^t_E)\hat{U}(\varphi^s_E)\psi]_E = [\hat{U}(\varphi^t_E \circ \varphi^s_E)\psi]_E = [\hat{U}(\varphi^{t+s}_E)\psi]_E = [\hat{C}^{t+s}_E][\psi]_E. \] (3.9)

Next we show that the \( [\hat{C}^t_E] \) are Hermitian on \( \mathcal{H}^E_D \). For any \( \psi, \psi' \in D_E([\hat{C}^t_E]) \) we have

\[
([\hat{C}^t_E][\psi]_E, [\psi']_E)_E = ([\hat{U}(\varphi^t_E)\psi]_E, [\psi']_E)_E = ([\hat{U}(\theta_E)\hat{U}(\varphi^t_E)\psi]_E, [\psi']_E)_E = ([\hat{U}(\varphi^t_E)\psi, \psi']_E)_E = ([\hat{U}(\varphi^t_E)\psi]_E, [\psi']_E)_E = ([\psi]_E, [\hat{C}^t_E][\psi']_E)_E \] (3.10)

where we have used \( \theta_E \circ \varphi^t_E = \varphi^t_E \circ \theta_E \) and the unitarity of the representation of the diffeomorphism group on \( \mathcal{H}^{D+1}_D \). In particular, we see that \( D_E([\hat{C}^t_E]) \) is contained in \( D_E([\hat{C}^t_E]^\dagger] \).
The hermiticity of \( \hat{n} \) we have existence of a generator for \( \hat{H} \) of one-parameter self-adjoint contraction semi-group on \( \mathcal{H} \) theorem [5] we infer that \( \hat{a} \) is a positive semi-definite operator. Iterating \( n \)-times we arrive at

\[
0 \leq ||[\hat{C}^t_E][\psi]_E||_{\mathcal{H}^E_D} = (||\psi||_E, [\hat{C}^{2t}_E][\psi]_E)_{\mathcal{H}^E_D}^{1/2}
\]

\[
\leq ||[\psi]_E||_{\mathcal{H}^E_D}^{1/2} ||[\hat{C}^{2t}_E][\psi]_E||_{\mathcal{H}^E_D}^{1/2}
\]

(3.11)

from the Schwarz inequality. We see, in particular from the first line of (3.11) that \( \hat{C}^t_E \) is a positive semi-definite operator. Iterating \( n \)-times we find the limit of (3.15) vanishes as \( 2 = (\hat{C}^{2n}_E[\psi]_E, [\hat{C}^{2n}_E][\psi]_E)_{\mathcal{H}^E_D}^{1/2} \cdot \)

(3.12)

Now, using lemma [4] and again the unitarity of the representation of the diffeomorphism group on \( \mathcal{H}_{D+1} \) we have \( ||[\hat{C}^{2n}_E][\psi]_E||_{\mathcal{H}^E_D} \leq ||\psi||_{\mathcal{H}_D} \) and finally since \( \sum_{k=1}^{n} (1/2)^k = 1 - (1/2)^n \), we find

\[
||[\hat{C}^t_E][\psi]_E||_{\mathcal{H}^E_D} \leq ||[\psi]_E||_{\mathcal{H}_D}^{1-(1/2)^n} ||\psi||_{\mathcal{H}_{D+1}}^{(1/2)^n} .
\]

(3.13)

Taking the limit \( n \to \infty \) of (3.13) we find the desired result

\[
||[\hat{C}^t_E][\psi]_E||_{\mathcal{H}^E_D} \leq ||[\psi]_E||_{\mathcal{H}^E_D} .
\]

(3.14)

The hermiticity of \( \hat{C}^t_E \) together with its boundedness implies that it can be extended to all of \( \mathcal{H}_D^E \) as a self-adjoint, positive semi-definite operator.

So far we have made no use of axiom (I). However, it is this axiom that guarantees the existence of a generator for \( [\hat{C}^t_E] \). Note that, for any \( \psi \in \mathcal{H}_D^E \), using the hermiticity of \( [\hat{C}^t_E] \), we have

\[
0 \leq ||[\hat{C}^t_E][\psi]_E - [\psi]_E||_{\mathcal{H}^E_D}^2
\]

\[
= (||\psi||_E, [\hat{C}^{2t}_E][\psi]_E)_{\mathcal{H}^E_D} + (||\psi||_E, [\psi]_E)_{\mathcal{H}^E_D} - 2([\psi]_E, [\hat{C}^t_E][\psi]_E)_{\mathcal{H}^E_D}
\]

\[
= \langle \hat{U}(\theta_E)^t \psi, (\hat{U}(\varphi^{2t}_E)^2 + 1 - 2\hat{U}(\varphi^t_E)) \psi \rangle
\]

\[
\leq \langle \hat{U}(\theta_E)^t \psi, (\hat{U}(\varphi^{2t}_E)^2 - 1) \psi \rangle + 2\langle \hat{U}(\theta_E)^t \psi, (\hat{U}(\varphi^t_E) - 1) \psi \rangle
\]

\[
\leq ||\psi||_{\mathcal{H}_{D+1}} ||(\hat{U}(\varphi^{2t}_E)^2 - 1) \psi||_{\mathcal{H}_{D+1}} + 2||\hat{U}(\varphi^t_E) - 1) \psi||_{\mathcal{H}_{D+1}} .
\]

(3.15)

Using axiom (I), strong continuity of the one parameter group of unitary operators \( \hat{U}(\varphi^t_E) \), we find that the limit of (3.15) vanishes as \( t \to 0 \), establishing strong continuity of the one-parameter self-adjoint contraction semi-group on \( \mathcal{H}_D^E \). Therefore, using the Hille-Yosida theorem [5] we infer that \( [\hat{C}^t_E] = \exp(-t[\hat{H}^E]) \) where the generator \( [\hat{H}^E] \) is a positive semi-definite operator on \( \mathcal{H}_D^E \). It must annihilate the state \( [1]_E \) as \( [\hat{C}^t_E][1]_E = [\hat{U}(\varphi^t_E)]_E = [1]_E \) for any \( t \geq 0 \).

\( \square \)

Clearly, if foliations \( E, \tilde{E} \in \text{Fol}(\sigma, (M, s)) \) are strongly equivalent, their generators are related by
\[
[\hat{H}^E] = [\hat{U}(\varphi_{EE'})][\hat{H}^E][\hat{U}(\varphi_{EE'}^{-1})].
\] (3.16)

iii) So far, axiom (IV) has not been invoked. Axiom (IV) tells us that the limit
\[
\lim_{t \to \infty} \hat{U}(\varphi_t^E)
\]
becomes the projector \[|1\rangle\langle 1|\] in the weak operator topology. Suppose that there exists a state \(\Omega_E\) which is orthogonal to \[|1\rangle\] and satisfies \(\hat{U}(\varphi_t^E)\Omega_E = \Omega_E\) for any \(t \geq 0, E \in \text{Fol}(\sigma, (M, s))\). We then have
\[
||\Omega_E||_{\mathcal{H}_{D+1}}^2 = \lim_{t \to \infty} \langle \Omega_E, \hat{U}(\varphi_t^E)\Omega_E \rangle = ||\langle 1, \Omega_E \rangle||^2 = 0.
\] (3.17)

This demonstrates the uniqueness of the vacuum and concludes the proof of the theorem.

\(\square\)

We will conclude this section with a few remarks.

a) The uniqueness result in part ii) of the theorem can be slightly extended. Let \(E\) and \(\tilde{E}\) be weakly equivalent (rather than strongly, as required in part ii)). Then, there exist \(\varphi_{EE'}, \varphi_{E\tilde{E}}' \in \text{Diff}(M, s)\) such that \(\theta_{EE'} = \theta_{E\tilde{E}}\). Although, in general the diffeomorphism \(\varphi_{EE'} \circ \varphi_{E\tilde{E}}^{-1}\) will not map the foliation \(E\) to \(\tilde{E}\), it may map the time translation \(\varphi_t^E\) to the time translation \(\varphi_t^{\tilde{E}}\). In this case, the unitary map \(U_{E, \tilde{E}}\) of part i) of the Theorem will map \([\hat{H}^E]\) to \([\hat{H}^{\tilde{E}}]\) as in (3.16).

b) At first it may seem surprising that the uniqueness result for the Hamiltonian is not as strong as in the standard Osterwalder-Schrader construction. However, this is to be expected on general grounds. In the standard construction, the notion of time translation is rigid. In the present context, there is much more freedom. As shown by the example at the end of section II B, from the viewpoint of the general framework, already in Minkowski space we are led to allow both \(\partial/\partial t\) and \(b\partial/\partial t\) as time evolution vector fields for any positive constant \(b\). Clearly, the Hamiltonian operators must also differ by a multiplicative constant in this case. More generally, agreement between Hamiltonians can be expected only if the two generate the same (or equivalent) ‘translations’ in space-time. This is precisely what our uniqueness result guarantees.

c) If two foliations \(E, \tilde{E}\) are not even weakly equivalent, there may be no natural isomorphism between the physical Hilbert spaces \(\mathcal{H}_D^E\) and \(\mathcal{H}_D^{\tilde{E}}\). In the specific examples we will consider in the next section, all foliations will in fact be weakly equivalent. However, due to the so-called ‘super-translation freedom’ \(\text{[6]}\), the situation may well be different in asymptotically flat general relativity in four space-time dimensions. It would be interesting to find explicit examples in which this inequivalence occurs and to understand its physical significance.

d) Note that, as in the original Osterwalder-Schrader reconstruction theorem, the Hilbert space theory obtained here is not as complete as one would ideally like it be. In particular, no prescription has been given to construct quantum operators corresponding to the classical (Dirac) observables.
IV. EXAMPLES

In this section we discuss three examples of measures on spaces of quantum histories. The first example is natural from a mathematical viewpoint but does not obviously come from the path integral formulation of a theory of direct physical interest. (However, we show in the Appendix that, if \( D = 1 \), this measure is naturally associated with a universality class of generally covariant quantum gauge field theories. See [19] for further details.) This measure satisfies some of our axioms. The other two measures satisfy all of our axioms and come from the following systems: Yang-Mills theory in two space-time dimensions and general relativity in three space-time dimensions (or, B-F theory in any space-time dimension). In all three cases, the space \( \mathcal{C} \) of quantum histories is kinematically non-linear and there is no background metric. These examples serve to bring out different aspects of the generalization of the reconstruction theorem.

A. The Uniform Measure for Gauge Theories

The space of ‘generalized connections’ admits a natural diffeomorphism invariant measure in any space-time dimension, which we will refer to as the uniform measure and denote by \( \mu_0 \). It plays a crucial role in the kinematical part of a non-perturbative approach to the quantization of diffeomorphism invariant theories of connections.\(^5\) It has also led to a rich quantum theory of geometry [9].

As remarked above, if \( D \neq 1 \), it is unlikely that \( \mu_0 \) would arise as the measure on the space of quantum histories in a theory of direct physical interest. Nonetheless, we discuss it here as a simple example of a measure which satisfies axioms II,III and IV above and to illustrate the construction of the Hilbert space \( \mathcal{H}_D^C \). (For a more complete discussion of this measure, see [10, 8].)

Suppose we are interested in a theory of connections based on a compact structure group \( K \) on a \( D+1 \) dimensional space-time manifold. For simplicity, in this brief account we will set \( K = SU(2) \), assume that the principal \( K \)-bundle over \( M \) is trivial, and work with a fixed trivialization. There is no background structure and so \( \text{Diff}(M,s) \) is just \( \text{Diff}(M) \) and any two foliations are strongly equivalent. The gauge subgroup \( \text{Diff}_G(M,s) \) of \( \text{Diff}(M,s) \) depends on the specific theory under consideration.

The space \( \mathcal{C} \) of quantum histories is now the moduli space of ‘generalized connections’ defined as follows [7]. Let \( \mathcal{A}_W \) denote the \( C^* \) algebra generated by Wilson loop functions (i.e. traces of holonomies of smooth connections around closed loops in \( M \)). \( \mathcal{C} \) is the Gel’fand spectrum of \( \mathcal{A}_W \). Therefore, it is naturally endowed with the structure of a compact, Hausdorff space and one can show that the moduli space of smooth connections \( \mathcal{C} \) is naturally and densely embedded in \( \mathcal{C} \).

The label space \( \mathcal{L} \) consists of triples \( f = (\gamma, \vec{j}, \vec{I}) \) where \( \gamma \) is a graph in \( M \), \( \vec{j} \) a labeling of its edges with non-trivial irreducible equivalence classes of representations of \( K \), and \( \vec{I} \) a

\(^5\)See [10] for a summary and a discussion of the mathematical details. This approach was motivated in large measure by ideas introduced in [11] and [12].
labeling of its vertices with intertwiners. The stochastic process is defined by \( f \mapsto P_f(\phi) := T_{\gamma,\vec{j},\vec{I}}(\phi) \) where the latter is a spin-network function on \( C \). (Roughly, each \( \phi \in C \) assigns to every edge of \( \gamma \) a group element, the representations \( \vec{j} \) convert these elements into matrices and the function \( T_{\gamma,\vec{j},\vec{I}} \) arises from contractions of indices of these matrices and intertwiners \( \vec{I} \). For details, see \([13,14]\).) Thanks to this judicious choice of \( L \), the measure \( \mu_0 \) can be defined quite simply:

\[
< P_f > = 0 \text{ for all } f \text{ except } f_0 = (\emptyset, \vec{0}, \vec{0}), \quad \text{and} \quad < P_{f_0} > = 1. \quad \text{(4.1)}
\]

It is easy to see that this measure satisfies axioms (II) and (III) : The only property that one needs to use is that spin-network functions form an orthogonal basis for \( \mathcal{H}_{D+1} \). Given a foliation \( E \), the equivalence classes under the quotient by the null vectors are in one to one correspondence with finite linear combinations of spin-network states whose graph lies entirely in the surface \( E_0(\sigma) \). This then defines the Hilbert space \( \mathcal{H}_E \) which is easily seen to be isomorphic to the Hilbert space defined by the quantum configuration space \([7]\) over \( \sigma \) and the corresponding uniform measure \( d\mu_0,\sigma \).

While the uniform measure is associated with certain mathematical models introduced by Husain and Kuchař \([15]\), it does not capture the dynamics of a physical system. Therefore, we have some freedom in the choice of \( \text{Diff}_G(M,s) \). However, if we want to satisfy both the remaining axioms, (GI) and (I), no choice is entirely satisfactory. For example, every diffeomorphism in \( \text{Diff}(M,s) \) has a non-trivial (unitary) action on \( \mathcal{H}_E^F \) so that axiom (GI) is satisfied only if we take the group of gauge diffeomorphisms to be trivial. For this choice of \( \text{Diff}_G(M,s) \), and thus for any other, the measure \( d\mu_0 \) also satisfies axiom (IV) \([16]\). However, while \( \text{Diff}(M,s) \) has an unitary action on \( \mathcal{H}_{D+1} \), there is no one-parameter group of diffeomorphisms that acts strongly continuously on the Hilbert space \( L^2(d\mu_0) \) \([10]\). Thus, axiom (I) is satisfied only for the complementary trivial choice \( \text{Diff}_G(M,s) = \text{Diff}(M,s) \). Nonetheless, it is true that \( P_{f_0} = 1 \) is the only state invariant under all time translations.

**B. Two-dimensional Yang-Mills Theory**

As mentioned in Section \([14,13]\) in two space-time dimensions, Yang-Mills action requires only an area 2-form rather than a full space-time metric. Therefore, the theory is invariant under all area preserving diffeomorphisms and thus provides an interesting example for our general framework.

Since connected, one-dimensional manifolds without boundary are diffeomorphic either to the circle \( S^1 \) or to \( \mathbb{R} \), let us consider Yang-Mills theory with structure group \( K = SU(N) \) on space-time manifolds \( M = \mathbb{R} \times \sigma \) where \( \sigma = S^1 \) or \( \sigma = \mathbb{R} \). The background structure \( s \) is an area two-form \( \omega \) on \( M \) and the action reads

\[
S(A) = -\frac{1}{g^2} \int_M \text{Tr}(F \wedge *F) = -\frac{1}{g^2} \int_M \frac{dx^0dx^1}{\omega_{01}} tr(F_{01}^2), \quad \text{(4.2)}
\]

where \( F \) denotes the curvature two-form of the connection \( A \) and \((x^0,x^1) = (t,x)\) are the standard coordinates on \( \mathbb{R} \times \sigma \). In this case, the group \( \text{Diff}(M,s) \) is the group \( \text{Diff}(M,\omega) \) of area preserving diffeomorphisms. The classical Hamiltonian formulation shows that the
gauge transformations of the theory correspond only to local \( SU(N) \)-rotations. Thus, \( \text{Diff}_G(M, \omega) \) contains only the identity diffeomorphism. Finally, the example given in section IIB can be trivially adapted to the case under consideration (simply by replacing \( \mathbb{R}^3 \) by \( S^1 \) and \( \eta \) by \( \omega \)) to show that there exist compatible foliations which fail to be strongly equivalent. However, it is not difficult to show that all compatible foliations are in fact weakly equivalent. We will now construct a quantum field theory for this system, satisfying all our axioms.

For the standard area form, the reconstruction of the Hamiltonian formalism from the Euclidean measure was obtained in [3]. The particular Euclidean measure utilized was the limit as the lattice spacing \( a \) goes to zero of the Wilson lattice action for Yang-Mills theory. Let us recall some of the results adapted to the case of a general area form. As label space \( \mathcal{L} \) we use \( N \)-1-tuples of loops in \( M \), \( f = (\alpha_1, \ldots, \alpha_{N-1}) \). Let \( \mathcal{C} \) be the moduli space of generalized connections as in the previous example and the random process be given by

\[
\mathcal{L} \rightarrow \mathcal{F}(\mathcal{C})
\]

\[
f = (\alpha_1, \ldots, \alpha_{N-1}) \mapsto P_{(\alpha_1, \ldots, \alpha_{N-1})}(\phi) := T_{\alpha_1}(\phi) \cdots T_{\alpha_{N-1}}(\phi).
\]

Here \( T_\alpha \) denotes the Wilson function,

\[
T_\alpha(\phi) = \frac{1}{N} \text{Tr}(h_\alpha(\phi)),
\]

where \( h_\alpha \) is the holonomy corresponding to the loop \( \alpha \) and the (generalized) connection \( \phi \).

To begin with, let us consider any compatible foliation \( E \). In order to adapt the calculations of [3] we consider a ultraviolet regulator \( a \) by taking a (possibly) curved lattice in \( M \) made of plaquets diffeomorphic (as manifolds with boundary) to rectangles, with area \( a^2 \) and such that the time-zero slice \( \gamma_E = E_0(\sigma) \) is a union of edges of plaquets. Notice that if \( \sigma = S^1 \), \( \gamma_E \) is a (homotopically non-trivial) loop in \( M \).

It is easy to verify that the calculations and results of [3] remain essentially the same and that we can take the ultraviolet limit \( a \rightarrow 0 \) in the expression for the generating functional \( \chi(\alpha_1, \ldots, \alpha_{N-1}) = \langle T_{\alpha_1}(\phi) \cdots T_{\alpha_{N-1}}(\phi) \rangle \). Axioms II and III hold and so we can construct the physical Hilbert spaces. Irrespective of the choice of the compatible foliation \( E \), the physical Hilbert space is one-dimensional if \( \sigma = \mathbb{R} \) and is \( L^2(\text{SU}(N)/\text{Ad}_{\text{SU}(N)}), d\tilde{\mu}_H) \) if \( \sigma = S^1 \), where \( \tilde{\mu}_H \) is the measure induced on \( \text{SU}(N)/\text{Ad}_{\text{SU}(N)} \) by the Haar measure on \( \text{SU}(N) \).

Let us concentrate on the more interesting case of \( \sigma = S^1 \). From [3] we obtain that the time evolution operator \( \hat{C}^t_E \) is given by

\[
[\hat{C}^t_E] = e^{\frac{1}{2}g^2 \text{Area}(E,t)\Delta}, \tag{4.3}
\]

where \( \Delta \) denotes the invariant Laplacian on \( \text{SU}(N) \) (functions on \( \text{SU}(N)/\text{Ad}_{\text{SU}(N)} \) can be thought as \( \text{Ad}_{\text{SU}(N)} \)-invariant functions on \( \text{SU}(N) \)) and \( \text{Area}(E,t) \) denotes the area inclosed between the loops \( \gamma_E = E_0(S^1) \) and \( E_t(S^1) \). Thus:

\[
\text{Area}(E,t) = \int_0^t dt' \int_0^1 dx (E^* \omega)_{01}(x,t'). \tag{4.4}
\]
Since \( \varphi^u = E \circ \varphi^u \circ E^{-1} \in \text{Diff}(M, \omega) \), \( \forall u \in \mathbb{R} \) (where \( \varphi^u \) denotes the standard time translation on \( \mathbb{R} \times S^1 \)) or, equivalently, \( \varphi^u \in \text{Diff}(\mathbb{R} \times S^1, E^* \omega) \), \( \forall u \in \mathbb{R} \), the component \( (E^* \omega)_{01} \) does not depend on \( t \). Therefore the area in (4.4) is linear in \( t \). Now, we showed in Lemma 5 that \( [\hat{C}^E_t] = \exp(-t[\hat{H}^E]) \). Hence, the Hamiltonian can now be read-off as:

\[
[\hat{H}^E] = -\frac{1}{2}g^2 L_E \Delta ,
\]

(4.5)

where

\[
L_E = \int_0^1 dx (E^* \omega)_{01}(x) .
\]

(4.6)

Notice that if \( \tilde{E} \) is strongly equivalent to \( E \), \( L_{\tilde{E}} = L_E \) and the two Hamiltonians agree as expected from Theorem 1.

What would happen if we use a foliation \( \tilde{E} \) which is not strongly equivalent to \( E \)? Then, the value of \( L_E \) (and therefore also the Hamiltonian \( [\hat{H}^E] \)) will in general change. For example, if we choose, as in section IIB, \( \tilde{E} : \mathbb{R} \times S^1 \to M, (t, x) \mapsto \tilde{E}(x) = (bt, x) \) with \( b > 0 \), then \( L_{\tilde{E}} = bL_E \) and therefore \( [\hat{H}^E] = b[\hat{H}^E] \). This is, however, exactly what one would expect since the vector field generating the time translation on \( M \) defined by \( E \) is \( b \) times that defined by \( E \). This is a concrete illustration of remark b) at the end of sec III.

Finally, in this model, the axiom (GI) holds trivially since \( \text{Diff}_G(M, \omega) \) contains only the identity diffeomorphism. Furthermore, all physical states are manifestly \( SU(N) \)-gauge invariant. The existence of a Hamiltonian operator \( [\hat{H}^E] \) implies that axiom (I) holds. Axiom (IV) also holds and the vacuum state is unique.

**C. 2+1 gravity and BF-Theories**

Fix a 3-manifold \( M \) with topology \( \mathbb{R} \times \sigma \), where \( \sigma \) is a compact 2-manifold. In the first order form, the basic fields for general relativity can be taken to be a connection \( A \) and a Lie-algebra-valued 1-form \( e \). The action is given by

\[
S(A, e) = \int_M \text{Tr} \ e \wedge F
\]

(4.7)

where the trace is taken in the fundamental representation. If the structure group \( K \) is \( SO(3) \), we obtain general relativity with signature +,+,+ while if the structure group is \( SO(2,1) \), we obtain general relativity with signature -,+,+. The field \( e \) can be thought of as a triad, and when \( e \) satisfies the equation of motion, the field \( A \) is the spin-connection compatible with the triad. The equations of motion on \( A \) say that \( F \) vanishes. In this case, there is no background structure \( s \), \( \text{Diff}_G(M) \) is the connected component of the identity of \( \text{Diff}(M) \), and all foliations are strongly equivalent.

The heuristic measure on the space of paths \( (A, e) \) is given by \( \mathcal{D}A \mathcal{D}e \exp i S(A, e) \) and if we integrate out the \( e \) fields we obtain the measure \( \delta(F) \mathcal{D}A \) on the space of connections. This suggests that the rigorous measure should be concentrated on flat connections. It turns
out that the moduli space of flat connections is a finite dimensional symplectic manifold\(^6\) and therefore has a natural Liouville measure.

With this intuitive picture in mind, we will now construct a quantum field theory for this system, satisfying all our axioms. Choose for \(\mathcal{C}\) the moduli space of smooth connections (or a suitable completion thereof. For example, in the \(SO(3)\) theory one can use the completion used in example 1.) For \(\mathcal{L}\) we use the space of closed loops \(f\) on \(M\). The stochastic process is defined by \(f \mapsto P_f(\phi) = \text{Tr} h_f(\phi)\) where \(h_f(\phi)\) is the holonomy of \(\phi \in \mathcal{C}\) around the closed loop \(f\) in \(M\) and trace is taken in the fundamental representation. The measure is defined by

\[
\langle \chi_f \rangle = \langle P_f(\phi) \rangle = \int_{\mathcal{M}_o} d\mu_L P_f(\phi),
\]

where \(\mathcal{M}_o\) is the moduli space of flat connections and \(\mu_L\) is the Liouville measure thereon. Note incidentally that, in the resulting history Hilbert space \(\mathcal{H}_{D+1}\), \(P_f\) and \(P_f'\) define the same element if \(f\) and \(f'\) are homotopic to each other. Hence \(\text{Diff}_G(M)\) is represented by the identity operator on \(\mathcal{H}_{D+1}\).

It is straightforward to check that the axioms (II), (III), (GI), (I) and (IV) are all satisfied. (In fact (I) and (IV) hold trivially because \(\text{Diff}_G(M,s)\) is so large.) The Hilbert space \(\mathcal{H}_D^k\) is isomorphic to \(L^2(\mathcal{M}_o, d\mu_L)\). The Hamiltonian theory can be constructed independently through canonical quantization \cite{18} and yields precisely the same Hilbert space of physical states. Note that the correct correspondence between the path integral and canonical quantization holds for both signatures, \(-,+,+\) and \(+,+,+\). However, one has to use measures whose heuristic analogs involve \(\exp iS\) in \textit{both} cases, so that the signature of the associated metric is not fundamental to determining the heuristic form of the measure. In particular, the Wick rotation has no obvious role in the diffeomorphism invariant context. (For further discussion, see \cite{21}.)

This viewpoint is also supported by the fact that 2+1 dimensional general relativity is a special case of B-F theories which can be defined in any dimension and in which there is no natural metric at all; the presence of a metric can thus be regarded as an ‘accident’ of 2+1 dimensions. In these theories, the basic fields are a connection \(A\) and a D-1 form \(B\) with values in the dual of the Lie algebra. The action has the same form as (4.7) with \(e\) replaced by \(B\). One can repeat essentially the same construction for all of these theories.

\textbf{V. DISCUSSION}

In this paper, we introduced an extension of the Osterwalder-Schrader framework to diffeomorphism invariant theories. The key idea was to generalize the standard setting by dropping all references to the space-time metric. We considered \(D + 1\) dimensional

\(^6\)There are certain technical subtleties in the \(SO(2,1)\) case \cite{17}. In what follows we will assume that a Hausdorff manifold has been obtained by deleting suitable points. The resulting moduli space has disconnected components. By moduli space we will refer either to the ‘time-like’ or ‘space-like’ components.
space-times $M$ with topology $\mathbb{R} \times \sigma$, where $\sigma$ is allowed to be an arbitrary, $D$-dimensional manifold. Heuristically, $\mathbb{R}$ serves as a generalized ‘time direction’. More precisely, using foliations $E$ of $M$, with leaves transverse to the $\mathbb{R}$-direction, we were able to extend the standard notions of time translation and time reflection without any mention of a space-time metric. This in turn enabled us to generalize the Osterwalder-Schrader axioms and construct a Hamiltonian quantum theory starting from a path integral. While $M$ is required to have a product topology, given our goal of constructing a Hamiltonian framework, this restriction is unavoidable.

As in the original Osterwalder-Schrader framework, the key mathematical object in the path integral formulation is the measure $\mu$ on the space $\mathcal{C}$ of quantum histories and the axioms are restrictions on permissible $\mu$. In the construction of a bridge from the path integral to the Hilbert space theory, two of these axioms play a central role: reflection positivity (axiom III, unchanged from the original Osterwalder-Schrader treatment) and diffeomorphism invariance (axiom II, which replaces the Euclidean invariance of the standard treatment). Given a foliation $E$ of $M$ and a measure $\mu$ satisfying reflection positivity, one can construct the Hilbert space $H_E$ of quantum states. The diffeomorphism invariance of $\mu$ then ensures that the Hilbert space is essentially insensitive to the choice of the foliation $E$. The remaining axioms ensure the existence of the Hamiltonian operators generating (generalized) time-translations which are true (i.e. non-gauge) symmetries of the theory and the existence and uniqueness of a vacuum state.

Perhaps the most striking feature of the present framework is its generality. We did not have to restrict ourselves to specific space-time manifolds and the Lagrangian — indeed even the matter content — of the theory was left arbitrary. In particular, our generalized setting allows theories of interacting gauge and tensor fields with arbitrary index structure, general relativity, higher derivative gravity theories, etc. However, this generality comes at a price. As with the original Osterwalder-Schrader reconstruction theorem, the results of this paper only tell us how to obtain a Hilbert space theory from a given measure satisfying certain axioms. It does not tell us how to construct this measure from a given classical theory. For familiar field theories (without diffeomorphism invariance), $\exp -S_E$, with $S_E$, the Euclidean action, generally provides a heuristic guide in the construction of this measure. We saw in Sections IV B and IV C that, in the absence of a space-time metric, the distinction between the usual Euclidean and Lorentzian prescriptions become blurred. In some cases, the heuristic guide is again provided by $\exp -S$ while in other cases it is provided by $\exp iS$. Thus, the construction of the measure now acquires a new subtle dimension. Furthermore, because our setting is much more general than the original one, even the ‘kinematical structure’ — the spaces $\mathcal{C}$ of quantum histories, the label set $\mathcal{L}$ and pairings $P$ of Section II — can vary from one theory to another and have to be constructed case by case. However, for diffeomorphism theories of connections, including general relativity in three space-time dimensions, we were able to provide natural candidates for these structures and find the appropriate measures. In [21], we will extend these considerations to more general contexts, albeit at a more heuristic level. In particular, starting from the classical Hamiltonian framework, we will discuss how one can construct heuristic measures. We will find some subtle but important differences from the familiar cases.
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APPENDIX A: A UNIVERSALITY CLASS OF GENERALLY COVARIANT QUANTUM GAUGE FIELD THEORIES IN TWO SPACETIME DIMENSIONS

The purpose of this appendix is to show that, if \( D = 2 \), the uniform measure \( \mu_0 \) on the space \( C \) of generalized connections considered in section IV A naturally arises in the path integral formulation of a class of diffeomorphism invariant gauge theories. For generalizations and further discussion, see [19].

Let \( M \) be a two-dimensional manifold with topology \( \mathbb{R}^2 \) or \( S^1 \times \mathbb{R} \), and let \( G \) be a compact, connected, semi-simple gauge group. We will denote by \( A \) the pull-back by local sections of a connection on a principal \( G \)-bundle over \( M \) and by \( F \) its curvature. We will use \( a, b, c,.. = 1, 2 \) as the tensorial indices and \( i, j, k,.. = 1,..,\dim(G) \) as the Lie algebra indices. Choose a basis \( \tau_i \) of the Lie algebra \( \text{Lie}(G) \) of \( G \) normalized such that \( \text{tr}(\tau_i \tau_j) = -N \delta_{ij} \) and structure constants are defined by \( [\tau_i,\tau_j] = 2f_{ij}^k \tau_k \). Finally, let \( \epsilon^{ab} \) be the metric independent totally skew tensor density of weight one and use it to represent the curvature as a Lie-algebra valued scalar density \( F^i := \frac{1}{2} \epsilon^{ab} F_{ab}^i \).

Let us consider the following action
\[
S = \int_M d^2x \left[ F^i F^i \right]^{\frac{1}{2}} \equiv \int_M d^2x \left[ P_2(F) \right]^{\frac{1}{2}} \tag{A1}
\]
where \( F^i F^i = k_{ij} F^i F^j \) is the norm of \( F^i \) with respect to the Cartan-Killing metric. Since the quantity under the square-root is a scalar density of weight two, the action (A1) is diffeomorphism invariant. This is perhaps the simplest of such actions for \( G \)-connections. Other, more general diffeomorphism invariant actions can be constructed by taking n-th roots of suitable n-nomials in \( F^i \) and their covariant derivatives.

Assuming appropriate boundary conditions, the equations of motion that follow from (A1) are
\[
D_a \frac{F^i}{\sqrt{F^j F_j}} = 0 \tag{A2}
\]
These equations are consistent since the integrability condition \( \epsilon^{ab} D_a D_b (F^i / \sqrt{F^j F_j}) = f^{i}_{jk} F^j F^k / \sqrt{F^l F_l} = 0 \) is identically satisfied. The general solution is \( F^i = s b^i \) where \( s \) is an arbitrary, nowhere negative, scalar density of weight one and \( b^i \) a covariantly constant vector of unit norm. In the special case \( G = U(1) \), we have: \( b^i(x) = b^i(x) = \pm 1 \) must be constant, say \( b^1(x) = 1 \) and thus \( F(x) = e^{ab} (\partial_a A_b)(x) \geq 0 \) is the general solution. The case \( G = U(1) \) also gives a simple class of solutions for general \( G \): Suppose we choose
the connection to be of the form $A^i_a = a_i t^i$ where $t^i$ is a constant unit norm vector. Then $F^i = F t^i$ where $F = \epsilon_{ab} \partial_a a_b$ and $D_a F^i / \sqrt{\epsilon_{jF}} = t^i \partial_a F / |F|$ and this reduces the problem to $G = U(1)$. Thus, the space of solutions to the field equations is infinite dimensional.

We now wish to derive a path integral for this theory, i.e., construct a continuum measure along the constructive approach of [3], using the action (A1) as the classical input. Let us focus on the non-trivial case when $M_T$ is topologically $\mathbb{R} \times S^1$. Let us introduce a system of coordinates $(x, t)$ where $x \in [-a, a]$ denotes the compact direction and $a$ is an arbitrary parameter of dimension of length. Next we introduce a foliation of the cylinder by circles of constant ‘time’ $t$ coordinate. We introduce a dimensionless UV cut-off $\epsilon$ and an IR cut-off $T$ of dimension of length. Consider $1 + 2N, N = T/(\epsilon a)$ circles at values of $t$ given by $t/a = l\epsilon, l = -N, -N + 1, \ldots, N$ and similarly $2N' + 1, N' = 1/\epsilon$ coordinate lines at constant values of $x$ given by $x/a = k\epsilon, k = -N', -N' + 1, \ldots, N'$ with $x/a = \pm 1$ identified. Thus we have covered a portion $M_T$ of the cylinder $M_T$, corresponding to $(x,t) \in [-a,a] \times [-T,T]$ by a lattice $\gamma_{e,T}$ of cubic topology and can consider the usual plaquette loops $\square$ of that lattice based at the point $p$ with coordinates $x = t = 0$, say. We can label plaquettes by two integers $(k,l)$ in the obvious way and have

$$\square_{k,l} = \rho_{k,l} \circ e_{k,l} \circ f_{k+1,l} \circ e_{k+1,l}^{-1} \circ f_{k,l}^{-1} \circ \rho_{k,l}^{-1} \quad (A3)$$

where $e_{k,l}, f_{k,l}$ are respectively edges of the lattice at constant values of $t$ and $x$ respectively and $\rho_{k,l}$ is an arbitrary but fixed lattice path between the base point and the corner of the box corresponding to lowest values of $x,t$ respectively (modulo the identification of $x/a = \pm 1$).

If we now parameterize edges as the image of the interval $[0,a]$ then it is not difficult to see that for a classical connection $A$ the holonomy for a plaquette is to leading order in $\epsilon$ given by

$$h_{\square_{k,l}} = 1 + a^2 \epsilon^2 \epsilon_{ab} \epsilon^{a_b}(0) f_{k,l}^b(0) F^j(x = k\epsilon a, t = l\epsilon a) \tau_j/2 \quad (A4)$$

Then,

$$S_{\epsilon,T} := \sum_{k,l} \left[ P_2 \left( -\frac{2}{N} \text{tr}(\tau_i(h_{\square_{k,l}} - 1))) \right) \right]^{\frac{1}{2}} = \sum_{k,l} s(h_{\square_{k,l}}) \quad (A5)$$

can be taken to be a Wilson-like action that approximates (A1) in the sense that

$$\lim_{\epsilon \to 0} S_{T,\epsilon} = \int_{M_T} d^2 x \left[ P_2(F) \right]^{\frac{1}{2}} \quad (A6)$$

The elementary but important observation is that function $s(h)$ defined in (A3) is the same for all plaquette loops.

Let us choose as our random variables the ‘loop network functions’ $T_{\tilde{\alpha},\tilde{\pi},c}$ [20]. These are similar to the $T_{\gamma_{e,T}}$ considered in section [VA], except that: i) now the graphs $\alpha$ are replaced by a finite collection, say $L$, of mutually non-overlapping loops $\tilde{\alpha}$ based at $p$ (possibly including the homotopically non-trivial one that wraps once around the cylinder at $t = 0$); ii) $\tilde{\pi}$ now denotes a collection of $L$ equivalence classes of non-trivial irreducible representations of $G$ subject to the constraint that if loops $\alpha_1, \ldots, \alpha_r, r \leq L, I_1 < \ldots < I_r$ share a segment, then the tensor product $\pi_{I_1} \otimes \ldots \otimes \pi_{I_r}$ does not contain a trivial representation; and, iii) $c$
is an intertwiner between the trivial representation and $\pi_1 \otimes \ldots \otimes \pi_L$. The function $T_{\vec{\alpha},\vec{\pi},c}(A)$ depends on $A$ through the holonomies $h_{\alpha_I}(A)$, $I = 1, \ldots, L$ only. Given any measure on the moduli space $\mathcal{C}$ of generalized connections, one can compute the the expectation values of these loop-network functions. These provide us with the characteristic function of the underlying measure [8, 3].

Using this machinery, we can write down the regularized measure on $\mathcal{C}$ by specifying its characteristic function:

$$\langle T_{\vec{\alpha},\vec{\pi},c} \rangle_{T,\epsilon} := \frac{1}{Z_{\epsilon,T}} \prod_{k,l} \left( \int_G d\mu_H(h_{e_{k,l}}) \right) \left( \int_G d\mu_H(h_{e_{k,l}}) \right) e^{-S_{T,\epsilon} T_{\vec{\alpha},\vec{\pi},c}},$$

(A7)

where,

$$Z_{\epsilon,T} := \prod_{k,l} \left( \int_G d\mu_H(h_{e_{k,l}}) \right) \left( \int_G d\mu_H(h_{e_{k,l}}) \right) e^{-S_{\epsilon,T}}.$$

(A8)

Here, of course, all the loops in question live on our lattice.

In order to explicitly compute the expectation value (A7) in the limit $\epsilon \to 0$ and $T \to \infty$ we can essentially follow [3]. This is due to a peculiarity of $D = 1$ and the planar or cylindrical topology of $M$, namely, that the plaquette loops are holonomically independent. Let then, at fixed $\epsilon, T$, $|\alpha|$ be the number of plaquettes contained in the surface bounded by $\alpha$. Then, repeating literally all the calculations performed in [3] we find

$$\langle T_{\vec{\alpha},\vec{\pi},c} \rangle_{T,\epsilon} = T_{\vec{\alpha},\vec{\pi},c}(A = 0) \prod_{I=1}^{L} \left( \frac{J_{\pi_I}}{J_{\pi_0}} \right)^{|\alpha_I|}$$

$$J_{\pi} = \frac{1}{\dim(\pi)} \int_G d\mu_H(h) \chi_\pi(h) e^{-s(h)}$$

(A9)

Here $\chi_\pi$ denotes the character of $\pi$, $\chi_\pi(1) = \dim(\pi)$ and $\pi_0$ denotes the equivalence class of the trivial representation. If one of the loops, say $\alpha_I$, is homotopically non-trivial then $(\frac{J_{\pi_I}}{J_{\pi_0}})^{|\alpha_I|}$ has to be replaced by $\delta_{\pi_I,\pi_0}$.

The UV and IR cut-off can now be trivially removed from (A9): Due to the holonomic independence of the plaquette loops the quantity (A9) is already independent of $T$. Due to the diffeomorphism invariance of the original action, the quantity (A9) depends on $\epsilon$ only through the numbers $|\alpha_I|$ which just counts the number of plaquette loops that one uses in order to approximate the loop $\alpha_I$. In contrast to the situation with two-dimensional Yang-Mills theory [3], the numbers $J_{\pi}$ are already independent of $\epsilon$. The reason for this is, of course, the background independence of the original action (A1). By contrast, as we saw in section [IV B], the Yang-Mills action requires a background area-element. At first the difference seems small. However, it leads one in Yang-Mills theory to the Wilson action $\frac{1}{\epsilon} \sum_{\square} [s(h_{\square})]^2$ —rather than $\sum_{\square} s(h_{\square})$— which makes $J_\pi$ $\epsilon$-dependent in just the right way to produce the area law [3] as $\epsilon \to 0$. In the present case there is no background structure and therefore we cannot have an area law; there is no area 2-form to measure the area with! Instead we have the following. Since the definition of $J_\pi$ implies $|J_\pi| < J_{\pi_0}$ for $\pi \neq \pi_0$ and since in the limit $\epsilon \to 0$ the numbers $|\alpha_I|$ diverge, it follows that $\lim_{\epsilon \to 0} (\frac{J_{\pi_I}}{J_{\pi_0}})^{|\alpha_I|} = \delta_{\pi,\pi_0}$. Consequently, we have:
\[
\lim_{T \to \infty} \lim_{\epsilon \to 0} < T_{\tilde{\alpha}, \tilde{\pi}, c} >_{T, \epsilon} = \begin{cases} 
1 & \text{if } \tilde{\alpha} = \{p\}, \tilde{\pi} = c = \pi_0 \\
0 & \text{otherwise}
\end{cases} \quad (A10)
\]

in other words, we arrive at the characteristic functional of the uniform measure in \( D = 1 \) discussed in section [IV A]. Note that the limit (A10) is completely insensitive to the choice of the regularizing lattice. Hence, the result is independent of the regulator.

Finally, we note that in place of (A1) we could have considered the most general diffeomorphism and gauge invariant action \( \tilde{S}(A) \) which depends on the field strengths \( F^i \) but not on their derivatives. For this, we can begin with globally defined (i.e. gauge invariant) monomials \( F^n := F^{i_1}..F^{i_n}\text{tr}(\tau_{i_1}..\tau_{i_1}) \). (If the rank of \( G \) is \( R \), only the first \( R - 1 \) of these will be independent, others being polynomials in them.) Let \( P_n \) be an arbitrary, gauge invariant, positive semi-definite, homogeneous polynomial of degree \( n \) in the \( F^j \) and set

\[
\tilde{S}(A) = \sum_{n=1}^{\infty} a_n \int_M d^2x \sqrt{P_n[F]} \quad (A11)
\]

where \( a_i \) are non-negative constants all but a finite number of which are zero. This action is diffeomorphism invariant, again because the integrand is a scalar density of weight one. We could have carried out the above procedure for any of these theories. Irrespective of the action in this class, the final characteristic function would have been again (A10). Thus, all these theories lie in the same universality class; their renormalization group flows all reach the same UV fixed point and the final quantum theory is dictated by the uniform measure. This issue, the Hamiltonian formulation, and the canonical quantization of these theories will be discussed in [19].
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