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Aspects of the connections between Path Integrals, Quantum Field Theory, Topology and Geometry

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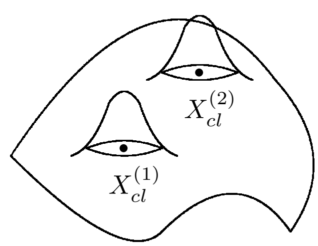
Abstract

Notes for a mini-course on some aspects of the connections between path integrals, quantum field theory, topology and geometry, delivered at the XII Fall Workshop on Geometry and Physics, Universidade de Coimbra, September 8–10, 2003. These notes are mostly targeted at geometers with very little or no experience with the Feynman path integral.



$\mathcal{P} = \{X(\cdot)\}$

MSC 2000: 81S40, 58D30, 28C20, 81Q30, 81T45, 81T40, 57R56

Contents

1 Introduction

These notes address some aspects of the growing influence of a (mostly) non existing mathematical object, the Feynman path integral (PI), on geometry, topology and quantum field theory.

Let me first make a very brief and (even more) incomplete historical introduction. It is natural to select two periods in the life of this 55 year old exotic tool invented (discovered?) by Feynman in his 1948 paper [Fe].

1948–1982

- In physics (quantum electrodynamics, electroweak model, and quantum chromodynamics) leads to predictions that are in extraordinary agreement with experiment.
- Mathematically the path integral is well defined for a large (but not large enough!) class of systems as a nice σ-additive measure on ∞-dimensional spaces of paths and fields.
- Failure of the attempts to extend the rigorous definition to the physically most interesting examples of nonabelian Yang-Mills theories in $\mathbb{R}^4$ or general relativity in $d \geq 4$.

1982–2003

- A significative part of the above mathematical difficulties persist today. Until the eighties mostly mathematical physicists specialized in functional analysis, functional analysts and stochastic analysts were trying to overcome these difficulties (see [AK, AM, CZ, GJ, GeV, JL, Ri, Si1, Si2] and references therein) using techniques of measure theory in infinite dimensions, operator theory and stochastic analysis.
- At approximately the same time that the constructive quantum field theory school was largely giving up on defining non-abelian Yang-Mills theory on $\mathbb{R}^4$ via rigorous PI methods[1] (not to speak about quantum gravity), a growing spiral of new applications of PI techniques, string theory and quantum field theory to geometry and topology started. A far from exaustive list is:

1982: Supersymmetry, Morse theory and index theorems [Wi1, Al]

1988: Verlinde formula [Ver]

1988: Field theory description of Donaldson invariants of 4-manifolds [Wi2]

1988: Cohomological field theories, intersection theory of moduli spaces and quantum cohomology [Wi3, Wi4, Kon1, KM]

1989: Chern-Simons theory and knot theory [Wi5]

1991: Mirror symmetry and enumerative geometry [CDGP]

1992: Moonshine and the monster [Bo]

[1]see the very elucidative introduction of [Ri].

1994: Seiberg-Witten theory [SW1, SW2]

1997: Deformation quantization of Poisson manifolds[Kon2, CF]

1998: Dualities between large N Chern-Simons theory and string theory linking knot invariants of real three manifolds with enumerative invariants of non-compact complex Calabi-Yau 3-folds [GoV, OV, Vaf, MV, OP, LLZ]

Many excellent textbooks and reviews are available on different aspects of the above topics (see for example [AJPS, At1, At2, BK, Bl, BT1, CMR, Di1, Di2, Di3, Gan1, Gan2, Gaw, Gre, Ho, Koh, LL, Marc, Mari1, Mari2, Mari3, PS, Seg1, Seg2, Vaf, Vo, Wi6]).

All these very deep but somewhat mystic and (largely) mathematically ill defined applications of PI to very important questions in geometry and topology, in often physically trivial or unrealistic theories, brought new ideas to the issue of finding a rigorous definition of the PI. Notably, the axiomatic approaches of Atiyah and Segal [At1, At2, Seg1, Seg2] and also extensions of stochastic analysis methods to supersymmetric models [Rog1, Rog2]. However a *complete*, applicable to all (or at least to the most important) cases, satisfactory answer, to what is in general mathematically the PI is yet to be found. What we do know is that it must be the generalization of an integral over infinite dimensional spaces of fields and/or of maps which in some cases does reduce to usual integration with respect to a σ-additive measure. In the following we will try to motivate this and to describe briefly some examples of the recent applications of PI to geometry, topology and quantum field theory.

2 Path integral in quantum mechanics

2.1 Kinematics

Let the configuration space of a physical system be a Riemannian manifold (Q, g). The manifold Q can be finite or infinite dimensional, e.g.

Q	**(one possible) Designation of the System**
Q	Particle propagating on Q
$LM = \mathrm{Map}(S^1, M)/Diff(S^1)$	String propagating in M
$\mathrm{Map}(X, M)/Diff_0(X)$	Brane propagating on M
$\mathrm{Map}(X, \mathbb{R}^D)$	D scalar fields on X
$\mathrm{Map}(X, M)$	Sigma model on X with target space M
$\mathcal{A}^G(P, X)/\mathcal{G}(P)$	Gauge theory on $\mathbb{R} \times X$
$Met(X)/Diff_0(X)$	Gravity on $\mathbb{R} \times X$

Table 1

In table 1, $Diff_0(X)$ denotes the connected component of the identity in the group $Diff(X)$ of orientation preserving diffeomorphisms of X, $\mathcal{A}^G(P, X)$ and $\mathcal{G}(P)$ denote the space of connections on the principal G-bundle P over X and the corresponding group of gauge transformations, respectively.

For simplicity let us restrict ourselves first to the simplest case of $Q = \mathbb{R}^D$. Specifying the configuration space corresponds to specifying the kinematics of the physical theory. The path space is then the space of paths in $\mathbb{R}^D$,

$$\begin{aligned} \mathcal{P} &= \mathrm{Map}([0, T], \mathbb{R}^D)_{x_{in}, x_{out}} \\ &= \left\{ X(\cdot) \in \mathrm{Map}([0, T], \mathbb{R}^D) \mid X(0) = x_{in}; X(T) = x_{out} \right\}. \end{aligned}$$

2.2 Classical dynamics – $S(X)$

The classical dynamics of the system is fixed by choosing a function S on the path space $\mathcal{P}$ called action function(al) and postulating that, classically, from all paths between x_{in} and x_{out} in time T only those corresponding to extrema of the action S are realized. In our case a standard choice is

$$S(X(\cdot)) = \int_0^T dt \left(\frac{\dot{X}^2}{2} - V(X) \right), \tag{1}$$

where we have chosen the mass, $m = 1$, and V denotes the potential (we will always assume that $V \geq 0$). For this to make sense we have of course to consider sufficiently smooth paths e.g.

$$\begin{aligned} \mathcal{P} &= C^2([0, T], \mathbb{R}^D)_{x_{in}, x_{out}} \\ &= \left\{ X(\cdot) \in C^2([0, T], \mathbb{R}^D) \mid X(0) = x_{in}; X(T) = x_{out} \right\}. \end{aligned}$$

The Euler-Lagrange equations for this variational problem coincide with the Newton equations,

$$\ddot{X}^i = -\frac{\partial V}{\partial X^i}.$$

2.3 Quantum dynamics: from the hamiltonian to the path integral formulation – $\sum_{X(\cdot)\in\mathcal{P}} e^{-\frac{1}{\hbar}S(X)}$

The quantum dynamics is more subtle but, thanks to Feynman,“can” be reformulated as a nice generalization of the above simple classical dynamical principle. Let us “deduce” the path integral formulation from the hamiltonian approach. As we will see later in field theory the path integral approach goes much further than the hamiltonian approach. In the Schrödinger representation quantum states are given by vectors (rays) in the Hilbert space

$$\mathcal{H} = L^2(Q, d_g v)$$

and the dynamics is specified by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t) = \widehat{H}\psi(t), \tag{2}$$

where the self-adjoint hamiltonian operator is determined from the classical action and in our example is

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta + V. \tag{3}$$

The solution of the Schrödinger equation with initial value ψ_0 is then given by

$$\psi(t) = e^{-i\frac{\widehat{H}t}{\hbar}}\psi_0.$$

This one parameter group, $e^{-i\frac{\widehat{H}t}{\hbar}}$, of unitary transformations of $L^2(\mathbb{R}^D, d^D v)$ is called the evolution operator and has a central role in the path integral formulation.[2]

For technical reasons and before we go to the path integral representation of the evolution operator we will follow Kac [Ka] and take t to be imaginary, $t = -it'$, $t' \geq 0$. The Schrödinger equation (2) then becomes a heat equation

$$\hbar\frac{\partial}{\partial t'}\psi = \frac{\hbar^2}{2}\Delta\psi - V\psi,$$

[2]Systems for which the phase space P is not the cotangent bundle of a manifold Q (e.g. Chern-Simons theories) do not have a Schrödinger representation. In this case the Hilbert space can, in principle, be obtained by the method of geometric quantization. Typically, instead of functions on Q the quantum states will be polarized sections of an appropriate line bundle on P [Wo].

with much better analytic properties and with evolution operator given by the very nice self adjoint contraction semigroup

$$\widehat{K}_{t'} = e^{-\frac{\widehat{H}t'}{\hbar}}, \quad t' \geq 0.$$

This, also rather mysterious, “trick” will allow us to interpret the Feynman PI in the present case as a nice σ-additive measure in the space of paths in $\mathbb{R}^D$. From now on we will mostly work in imaginary time (Euclidean or Wick rotated formalism in physics terminology) and will omit the $'$ in t. The results obtained in this imaginary time formalism are usually analytic in time so that in the end we can (hope to be able to) continue analytically the results back to real time. Note that formally the PI can be defined for real t and this was the way Feynman introduced it in 1948 [Fe]. But to get a nice σ-additive measure interpretation we need to go to imaginary time. The whole secret comes from the simple semigroup property of the heat evolution operator,

$$\widehat{K}_{t_1} \circ \widehat{K}_{t_2} = \widehat{K}_{t_1+t_2}, \qquad \widehat{K}_t = e^{-\frac{\widehat{H}t}{\hbar}}, \tag{4}$$

which is telling us that the quantum mechanics of a particle in $\mathbb{R}^D$ (or more generally in (Q,g))) defines, in imaginary time, a representation of the semigroup $(\mathbb{R}_+,+)$ in the Hilbert space $L^2(\mathbb{R}^D, d^Dv)$ $(L^2(Q, d_gv))$.

Let us now see how to go from (4) to the PI representation of the kernel of $\widehat{K}_t$ considered as integral operator. For very general classes of V in (3) the evolution operator $\widehat{K}_t, t > 0$ is an integral operator

$$\left(\widehat{K}_t\psi\right)(x) = \int_{\mathbb{R}^D} d^D y K_t(x,y)\psi(y)$$

with C^∞ kernel, which in fact is in the Schwarz space of rapidly decreasing C^∞ functions

$$K_t \in \mathcal{S}(\mathbb{R}^D \times \mathbb{R}^D) \subset C^\infty(\mathbb{R}^D \times \mathbb{R}^D).$$

For $V = 0$,

$$K_t(x,y) = \frac{1}{(2\pi t\hbar)^{D/2}} e^{-\frac{(x-y)^2}{2\hbar t}}.$$

The innocent looking semigroup property (4) written for the kernels takes the form

$$K_{t_1+t_2}(z,y) = \int_{\mathbb{R}^D} dx K_{t_1}(z,x) K_{t_2}(x,y) \tag{5}$$

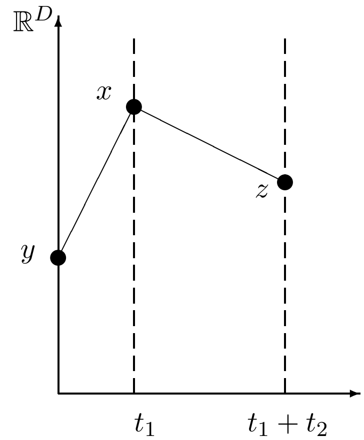


Figure 2.1

or, as Feynman would say it, the amplitude of probability of going from y to z in time t_1+t_2 is the product of the amplitude of going from y to x in time t_1 with the amplitude of going from x to z in time t_2 "summed" over all intermediate x's.

The next step is then

$$\widehat{K}_t = \widehat{K}_{t-t_{n-1}} \circ \cdots \circ \widehat{K}_{t_2-t_1} \circ \widehat{K}_{t_1}$$

and therefore

$$\begin{aligned} K_t(z,y) &= \int_{\mathbb{R}^D} dx_1 K_{t-t_1}(z,x_1) K_{t_1}(x_1,y) = \\ &= \int_{\mathbb{R}^{(n-1)D}} dx_1 \cdots dx_{n-1} K_{t-t_{n-1}}(z,x_{n-1}) \cdots K_{t_1}(x_1,y). \end{aligned} \tag{6}$$

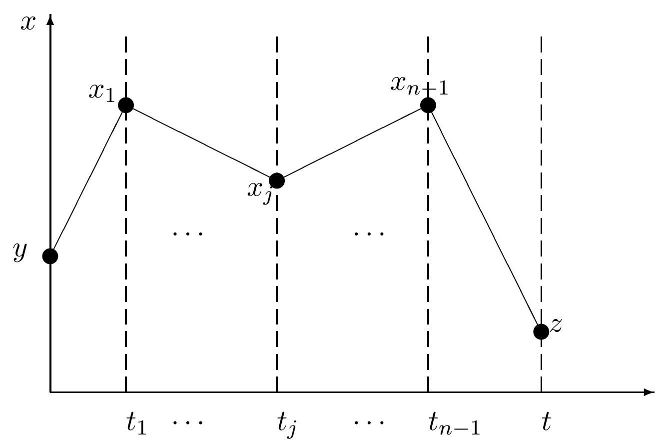


Figure 2.2

Let us now continue for a moment just with the free particle

$$K_t(z,y) = \frac{1}{(2\pi t\hbar)^{D/2}} e^{-\frac{(y-z)^2}{2\hbar t}} = \tag{7}$$

$$= \frac{1}{c_n} \int_{\mathbb{R}^{(n-1)D}} dx_1 \cdots dx_{n-1} e^{-\sum_{j=1}^{n} \frac{1}{2\hbar} \frac{(x_j - x_{j-1})^2}{(t_j - t_{j-1})^2} (t_j - t_{j-1})},$$

where $x_0 = y$, $x_n = z$ and c_n is a normalization constant,

$$c_n = \prod_{j=1}^{n-1} (2\pi(t_j - t_{j-1})\hbar)^{D/2}.$$

This identity is (of course fully rigorously) valid for all finite n. Amazingly, as Wiener taught us in 1923, we can make sense (still rigorously) of the limit $n \to \infty$ of (7) [Wie].

Before going to that let us, following again Feynman, think what would we expect to obtain in such a limit. The x_j's in (7) are the values of a path $X(\cdot)$ from y to z at times t_j, $t_j < t_{j+1}, x_j = X(t_j), X(0) = y, X(t) = z$ (see Figure 2.2) and if $X(\cdot)$ has piecewise constant velocity then

$$\frac{(X(t_j) - X(t_{j-1}))^2}{(t_j - t_{j-1})^2}$$

is the velocity square in the interval $[t_{j-1}, t_j]$. We then expect in the limit $n \to \infty$, eq. (7) to become

$$K_t(y, z) = \text{“} \frac{1}{c_\infty} \int_{\mathcal{P}_{yz}^t} DX(\cdot) e^{-\frac{1}{\hbar} \int_0^t dt \frac{\dot{X}^2}{2}} \text{”}$$

or, in general ($V \neq 0$),

$$K_t(y, z) = \text{“} \frac{1}{c_\infty} \int_{\mathcal{P}_{yz}^t} DX(\cdot) e^{-\frac{1}{\hbar} S(X(\cdot))} \text{”},$$

with S given by (1).

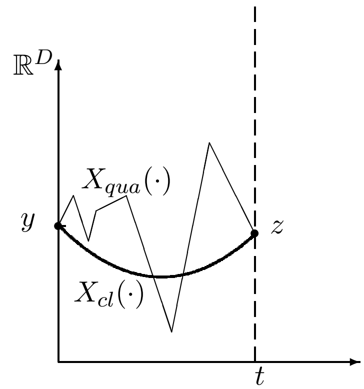


Figure 2.3

This is the insight of Feynman. The integral kernel of the evolution operator, $e^{-t\widehat{H}/\hbar}$, i.e. the amplitude of going from y to z in time t is obtained by summing over *all* paths from y to z in time t weighted by $e^{-\frac{S(X(\cdot))}{\hbar}}$,

$$K_t(y,z) \propto \sum_{\mathcal{P}^t_{yz}} e^{-\frac{S(X(\cdot))}{\hbar}},$$

or in real time

$$K_t(y,z) \propto \sum_{\mathcal{P}^t_{yz}} e^{i\frac{S(X(\cdot))}{\hbar}}.$$

An amazing characteristic of this heuristic formalism, that we will encounter repeatedly, is that it facilitates all kinds of generalizations. Besides the case of $V \neq 0$ in $\mathbb{R}^D$ that we mentioned we can consider a particle moving in a manifold Q with a metric g and a potential V. The classical action in that case is (in imaginary time)

$$S(X(\cdot); g, V, dt^2) = \int_0^t dt \left[\frac{1}{2} g(\dot{X}, \dot{X}) + V(X)\right] \tag{8}$$

and "therefore"

$$K_t(x,y) = \text{``}\ \frac{1}{c}\int_{\mathcal{P}^t_{xy}} DX(\cdot)\ e^{-\frac{S(X(\cdot);g,V,dt^2)}{\hbar}}\ \text{''}, \tag{9}$$

where dt^2, in the argument of S, is the metric which we are using in the interval $[0,t]$. Changing the metric is equivalent to changing the length t of the interval. We can then use an appropriate definition of the path integral to obtain properties of K_t or of other relevant objects.

So, within this formalism, we can say that the quantum theory of a system with configuration space Q requires, initially, the same data as the classical theory (Q, S)

$$S\ :\ \mathcal{P}^t_{xy} \longrightarrow \mathbb{R}.$$

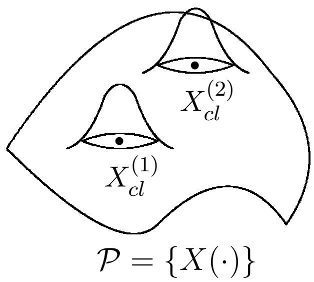


Figure 2.4

However, in the quantum theory, instead of looking for the paths that are minima (or, more generally, stationary points) of S one constructs a measure on $\mathcal{P}_{xy}^t$ that should correspond to the heuristic formula

$$D\mu_S^\hbar = \text{``}\ \frac{1}{c}\ DX(\cdot)e^{-\frac{S(X;g,V,dt^2)}{\hbar}}\ \text{''}. \tag{10}$$

The link between the quantum and the classical theories being that for small $\hbar$ (semiclassical limit) the measure concentrates around the classical solutions.

2.4 Rigorous path integral measures

Let us now go back to (6) and (7) and see how the limit $n \to \infty$ can be taken in a rigorous way following the approach of Kolmogorov. In fact we can describe the general case of a particle on a Riemannian manifold (Q, g), potential V and classical action (8). Under reasonable assumptions on g and V (including geodesic completeness and V smooth and bounded from below) the hamiltonian can be chosen, in the Schrödinger representation, to be the self adjoint operator

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta_g + V.$$

Its evolution operator $\widehat{K}_t = e^{-\frac{t\widehat{H}}{\hbar}}$ has a smooth nonnegative integrable kernel

$$0 \leq K_t \in C^\infty(Q \times Q) \cap L^1(Q \times Q, d_g v \otimes d_g v), \quad t > 0,$$

where $d_g v$ denotes the measure on Q associated with the metric g.

Consider on $\widehat{\mathcal{P}} = \mathrm{Map}_{xy}([0,t], Q) = Q^{]0,t[} \ni X(\cdot)$ the Tikhonoff topology[3] and the continuous maps

$$\begin{array}{rcl} p_{t_1,\ldots,t_n} \ :\ \widehat{\mathcal{P}} & \longrightarrow & Q^n = Q \times \cdots \times Q \\ X(\cdot) & \mapsto & (X(t_1), \ldots, X(t_n)). \end{array}$$

Definition 1. *(i) The subset $\mathcal{C} \subset \widehat{\mathcal{P}}$ is called (measurable) cylindrical if there exist $0 < t_1 < \cdots < t_n < t$ and a Borel set $B \subset Q^n$ such that*

$$\mathcal{C} = p_{t_1,\ldots,t_n}^{-1}(B) = \left\{X(\cdot) \in \widehat{\mathcal{P}}\ :\ (X(t_1), \ldots, X(t_n)) \in B\right\}. \tag{11}$$

[3] i.e. the weakest topology on $\widehat{\mathcal{P}}$ for which all the maps $p_{t_1,\ldots,t_n}$ are continuous.

(ii) *The function* f *on* $\widehat{\mathcal{P}}$ *is called cylindrical if there exist* $0 < t_1 < \cdots < t_n < t$ *and a measurable function* F *on* Q^n *such that* $f = p^*_{t_1,\ldots,t_n}(F)$, *or, equivalently,* $f(X(\cdot)) = F(X(t_1), \ldots, X(t_n))$.

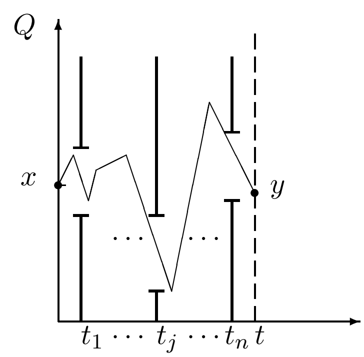


Figure 2.5

The cylindrical sets of course include the Feynman sets, i.e. sets of the form (11) with

$$B = I_1 \times \cdots \times I_n,$$

where the I_j are Borel sets on Q. The cylindrical sets form an algebra of sets which is not a σ-algebra. From (6) (with Q instead of $\mathbb{R}^D$) it follows that if the Feynman PI measure on $\widehat{\mathcal{P}}$, $D\mu^{\hbar}_S$, exists then cylindrical sets (11) should have the measure

$$\begin{aligned} \mu^{\hbar}_S(\mathcal{C}) &= \int_{\widehat{\mathcal{P}}^t_{xy}} D\mu^{\hbar}_S(X(\cdot))\ \chi_{\mathcal{C}} \\ &= \int_B d_g v(z_1)\ldots d_g v(z_n) K_{t-t_n}(y, z_n)\ldots K_{t_1}(z_1, x), \qquad (12) \end{aligned}$$

where $\chi_{\mathcal{C}}$ denotes the characteristic function of the set $\mathcal{C}$. In particular, if $B = Q^n$ then $\mathcal{C} = \widehat{\mathcal{P}}^t_{xy}$ and the semigroup property gives us the simplest consistency conditions on (12): $\mu^{\hbar}_S(\widehat{\mathcal{P}}^t_{xy})$ does not depend on the choice of n and of $0 < t_1 < \cdots < t_n < t$ and

$$\begin{aligned} \mu^{\hbar}_S(\widehat{\mathcal{P}}^t_{xy}) &= K_t(y, x) = \int_{\widehat{\mathcal{P}}^t_{xy}} D\mu^{\hbar}_S\ \ 1 = \\ &= \int_{Q^n} d_g v(z_1)\ldots d_g v(z_n) K_{t-t_n}(y, z_n)\ldots K_{t_1}(z_1, x). \qquad (13) \end{aligned}$$

Having $\mu^{\hbar}_S$ defined on cylindrical sets we define integrability of cylindrical functions, which include correlators or moments of the measure, reducing it to integrability of functions on the finite dimensional spaces Q^n. For example, if

$Q = \mathbb{R}$, the moments of the measure, if finite, are given by

$$\begin{aligned} F_{a_1,\ldots,a_n}(t_1,\ldots,t_n) &= \langle (X(t_1))^{a_1}\ldots(X(t_n))^{a_n}\rangle = \\ &= \int_{\widehat{\mathcal{P}}} D\mu_S^{\hbar}(X(\cdot))X^{a_1}(t_1)\ldots X^{a_n}(t_n) = \\ &= \int_{\mathbb{R}^n} d\mu_{t_1\ldots t_n} z_1^{a_1}\ldots z_n^{a_n}, \end{aligned}$$

where $d\mu_{t_1\ldots t_n}$ is the measure on the r.h.s. of (12),

$$d\mu_{t_1\ldots t_n}(z_1,\ldots,z_n) = dz_1\ldots dz_n K_{t-t_n}(y,z_n)\ldots K_{t_1}(z_1,x), \tag{14}$$

and coincides with the pushforward of $\mu_S^{\hbar}$ to $\mathbb{R}^n$ with respect to $p_{t_1,\ldots,t_n}$.

At this point the following three natural questions arise:

Q1. Are the cylindrical measures on $\widehat{\mathcal{P}}_{xy}^t$ extendable to a σ-additive measure on the σ-algebra generated by the cylindrical sets?

Q2. When does a family of (finite) measures $\{\mu_{t_1,\ldots t_n}\}_{0<t_1<\cdots<t_n<t}$ on Q^n define a cylindrical measure on $\widehat{\mathcal{P}}_{xy}^t$?

Q3. The space $\widehat{\mathcal{P}}_{xy}^t$ is too big and not adequate to usual physical applications. What can we say about the support of relevant PI measures?

The first two questions were answered by the theory developed by Kolmogorov.

Theorem 1 (Kolmogorov). *[Ya]*

1. *Every cylindrical measure on $\widehat{\mathcal{P}}_{xy}^t$ is extendable to a σ-additive measure on the σ-algebra generated by the cylindrical sets.*
2. *A family of finite measures $\{\mu_{t_1,\ldots t_n}\}_{0<t_1<\cdots<t_n<t}$ on Q^n (for all n and all choices $0<t_1<\cdots<t_n<t$) defines a (σ-additive) measure on $\widehat{\mathcal{P}}_{xy}^t$ if and only if these measures satisfy natural consistency conditions.*

□

The consistency conditions correspond to representing the cylindrical sets as cylindrical with respect to different projections. In (13) we considered the case of $\widehat{\mathcal{P}}_{xy}^t$ which is cylindrical with respect to all projections. The consistency (independence of the projection) followed from the semigroup property. It is easy to show that for families of measures of the type (14) the consistency conditions are equivalent to the semigroup property on $K_t(x,y)$.

The question on the support of the measures is more delicate but the answer is also positive.

Theorem 2. *[GJ, JL]*

1. *The measure $\mu_S^\hbar$ on $\widehat{\mathcal{P}}_{xy}^t$ is supported on the subset $\mathcal{P}_{xy}^t$ of continuous paths (in fact have Hölder class $\frac{1}{2}-\epsilon, \epsilon > 0$) from x to y in time t.*
2. *The subset of $\widehat{\mathcal{P}}_{xy}^t$ of paths that are not differentiable at any point is a full measure subset.*

□

Of importance is also the following

Theorem 3. *(Inexistence of Lebesgue measure) [JL] If a measure on $\widehat{\mathcal{P}}_{00}^t$ is invariant under translations and there is a set with positive measure then the measure of that set is infinite.*

□

So we have bad news and good news:

BN. In the heuristic formula (10) for the PI measure nothing seems to make sense. From (7) we see that the normalization constant $c = \lim_{n\to\infty} c_n = 0$, from theorem 3 it follows that a Lebesgue-like measure $DX(\cdot)$ is infinite and from theorem 2 we conclude that even if such a measure existed than the weight $e^{-S(X)/\hbar}$ would be zero almost everywhere.

GN. Nevertheless, in the present subsection, we were able to make rigorous sense of the above

$$\infty \times 0$$

in the limit

$$\prod_{j=1}^{n-1} \frac{1}{(2\pi(t_j-t_{j-1})\hbar)^{D/2}} \int_{\mathbb{R}^{(n-1)D}} dx_1 \cdots dx_{n-1} e^{-\sum_{j=1}^{n-1} \frac{1}{2\hbar} \frac{(x_j-x_{j-1})^2}{(t_j-t_{j-1})^2}(t_j-t_{j-1})}$$
$$\downarrow \ n \to \infty$$
$$\text{“}\ \frac{1}{c}\int_{\mathcal{P}} DX(\cdot) e^{-\frac{S(X)}{\hbar}}\ \text{''},$$

by interpreting (7) as defining joint distributions of a finite number of random variables. These joint distributions define a cylindrical measure, which by theorem 1, can be extended to a unique σ-additive measure on $\widehat{\mathcal{P}}_{xy}^t$.

Which lesson should we learn from this?

L. The most reasonable thing to do when one gets to this point seems to be:
Discard the appealing but clearly meaningless expressions (9) and (10).

L'. One of the main goals of the present lectures consists in arguing in favor of a totally different answer:
Keep (9) and (10) as a secret device (until a full justification comes). The heuristic expression must have a deep truth in it since it has opened (or at least significantly facilitated) the way to all kinds of generalizations and "gadgets" that have proved to be extremely proficuous both from the physics and the mathematics points of view. For instance, the extension $\mathbb{R}^D \to Q$*, the huge generalizations that are possible in field theories* [4]*, dualities, the Kontsevich* $\star$*-formula for the deformation quantization of Poisson manifolds and the semiclassical, perturbative and large* N *expansions.*

3 Quantum field theories (QFT): axiomatics and examples

3.1 From paths in infinite dimensional spaces of fields in d dimensions to fields in $d+1$ dimensions

Field theories are physical models for which the configuration spaces Q are spaces of fields on a d-dimensional manifold Σ. These can be global sections of vector bundles or of more general sheaves, connections on principal bundles, maps to other manifolds, $\mathrm{Map}(\Sigma, M)$, etc. The example of a particle in Q considered in the previous section corresponds to $\Sigma = \{*\}$, so that $\mathrm{Map}(\{*\}, Q) \cong Q$. The manifold Σ may have fixed extra structure(s) like a metric, a symplectic structure, a complex structure, etc.

[4] some of which we will address in the following and which are behind the spectacular recent applications to geometry, topology and algebra

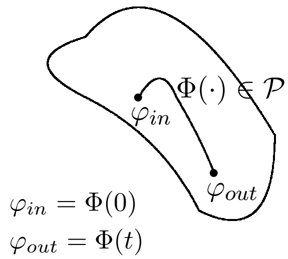


Figure 3.1

The simplest example of a field theory on Σ is given by a scalar field on Σ for which

$$Q \overset{(\subseteq)}{=} \mathrm{Map}(\Sigma, \mathbb{R}).$$

Like in finite dimensions, classical (and quantum) dynamics is specified by choosing a function S on the (a) space of paths $\mathcal{P} = \mathcal{P}(Q)$ in Q

$$\begin{array}{rcl} C([0,t],Q) \supseteq \mathcal{P} & \xrightarrow{S} & \mathbb{R} \\ \Phi = (\varphi(\cdot)) & \longmapsto & S(\Phi). \end{array}$$

If Σ has a (Riemannian) metric $\widetilde{\gamma}$, classically one usually chooses,

$$\mathcal{P} = C^{\infty}([0,t] \times \Sigma, \mathbb{R}) \quad (\subset C([0,t],Q)) \tag{15}$$

and

$$\begin{aligned} S(\Phi; \widetilde{\gamma}, V) &= \int_0^t dt \int_{\Sigma} d_{\widetilde{\gamma}} v \left[\frac{\dot{\Phi}^2}{2} + \frac{||\nabla_{\Sigma}\Phi||_{\widetilde{\gamma}}{}^2}{2} + V(\Phi) \right] \\ &= \int_{[0,t]\times\Sigma} d_{\gamma} v \left[\frac{||\nabla\Phi||_{\gamma}{}^2}{2} + V(\Phi) \right], \end{aligned}$$

where the potential V is a (non-negative) function on the target $\mathbb{R}$ and $\gamma = dt^2 \oplus \widetilde{\gamma}$. Then, in classical field theory, we are interested in the stationary points of this function(al).

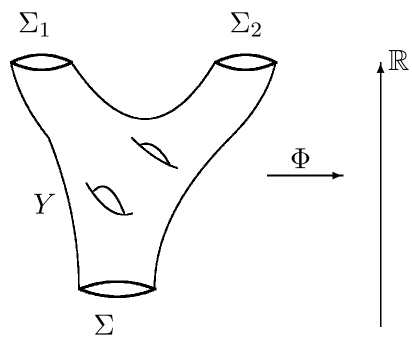


Figure 3.2

Note that the paths Φ in the space of fields Q which we chose in (15) are themselves smooth fields but in the $d+1$-dimensional manifold $[0,t]\times\Sigma$, $\widetilde{\Phi}(t,x) = \Phi(t)(x)$ (we will drop this tilde over Φ). Again, this innocently looking step will open very important new possibilities which is to consider $d+1$-dimensional manifolds Y more general than $[0,t]\times\Sigma$ as in figure 3.2.

Given a metric γ on Y we can define the action

$$S(\Phi;\gamma,V) = \int_Y d_\gamma v \left[\frac{||\nabla\Phi||_\gamma{}^2}{2} + V(\Phi)\right] \tag{16}$$

as a function in the path space

$$\mathcal{P} = C^\infty(Y,\mathbb{R}) \equiv C^\infty(Y).$$

Example 1. *If $d=1$ ($\Rightarrow \dim Y = 2$) and $V=0$ in (16) then S is invariant under conformal rescalings of the metric γ*

$$S(\Phi;e^f\gamma) = S(\Phi;\gamma) \tag{17}$$

for any f, $f\in C^\infty(Y)$, which implies that S depends only on the complex structure j of the Riemann surface (Y,j). This is the simplest example of a conformal field theory (CFT) in 2 dimensions. If, instead of real valued, Φ is complex valued (complex scalar field) one has

$$S(\Phi;\gamma) = \int_Y d_\gamma v\ \frac{||\nabla\Phi||_\gamma{}^2}{2} = -i\int_Y \overline{\partial}\Phi\wedge\partial\overline{\Phi} + \int_Y \Phi^*\omega \tag{18}$$

where ω is the standard (Kähler) two-form for the flat metric on $\mathbb{C}$, $\omega = dx\wedge dy$, $z = x+iy\in\mathbb{C}$. Since the second term in the last equality does not change under continuous (C^1-) variations of Φ which leave the boundary fixed we see that holomorphic and antiholomorphic maps from (Y,j) to $\mathbb{C}$ are (in fact the only) local minima of S (classical configurations or instantons of this system).

A very important, both from the mathematics and the physics point of view, generalization of this example corresponds to considering non-linear sigma models, i.e. models in which the target $\mathbb{C}$ is replaced by a manifold M so that the path space is

$$\mathcal{P} = C^\infty(Y, M).$$

Of particular interest is the case when M is a Kähler manifold with complex structure, J, Kähler two-form, ω, and metric, $g = \omega(J\cdot, \cdot)$. Then the action can be written in a form generalizing (18)

$$\begin{aligned} S(\Phi; \gamma, \omega, J) &= \int_Y d_\gamma v \, \frac{||\nabla\Phi||^2_{\gamma,g}}{2} = \int_Y ||\overline{\partial}\Phi||^2_g + \int_Y \Phi^*\omega \equiv \\ &\equiv -i \int_Y \sum_{a,b=1}^{\mathrm{dimM}} \overline{\partial}\Phi^a \wedge \partial\overline{\Phi}^b g_{a\bar{b}}(\Phi) + \frac{i}{2} \int_Y \sum_{a,b=1}^{\mathrm{dimM}} d\Phi^a \wedge d\overline{\Phi}^b g_{a\bar{b}}(\Phi). \end{aligned}$$

The classical configurations are again given by holomorphic and anti-holomorphic maps from (Y, j) to M,

$$\Phi \; : \; \overline{\partial}\Phi^a = 0,$$

and

$$\Phi \; : \; \partial\Phi^a = 0.$$

Example 2. *Let $d = \mathrm{dim}\Sigma = 2$, G be a connected Lie group and $Q = \mathcal{A}^G(\Sigma)$ be the space of G connections on the trivial G bundle over Σ. Again we consider paths $A(\cdot)$ in Q that can be considered G connections on the trivial bundle over $Y = [0, t] \times \Sigma$*

$$A(t) = \sum_{k=1}^{2} A_k(t, x) dx^k.$$

We can construct an action which does not depend on any extra structure on $Y = [0, t] \times \Sigma$ and is therefore $Diff(Y)$-invariant

$$S(A) = \frac{1}{4\pi} \int_Y \mathrm{tr} \left(A \wedge dA - \frac{2}{3} A \wedge A \wedge A \right). \tag{19}$$

This is the celebrated Chern-Simons action, which remains valid for a connection A on a more general 3 manifold Y. Note that the action (19) is not invariant under large gauge transformations (not connected to the identity of the group of gauge transformations $C^\infty(Y, G)$) but changes by integer values (provided that the invariant form tr *on the Lie algebra of G is adequately chosen). The classical configurations or stationary points of (19) are the flat connections, i.e. those satisfying*

$$F_A = dA + A \wedge A = 0. \tag{20}$$

3.2 Axiomatic approaches to quantum field theory

The analogue of (9), for a $d+1$-dimensional QFT on e.g. a (compact) oriented manifold like Y on figure 3.2, is

$$K_Y(\varphi_1, \varphi_2; \varphi) = \frac{1}{c} \int_{\mathcal{P}^Y_{\varphi_1,\varphi_2;\varphi}} D\Phi \; e^{-S(\Phi)}, \tag{21}$$

where we "sum" over all fields on Y with specified boundary values,

$$\Phi_{|\Sigma} = \varphi; \; \Phi_{|\Sigma_1} = \varphi_1; \; \Phi_{|\Sigma_2} = \varphi_2.$$

The result is (to be thought as) the integral kernel for an operator

$$\widehat{K}_Y \; \in \mathrm{Hom}(\mathcal{H}_\Sigma, \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}),$$

where we are assuming that to the closed d-dimensional oriented manifolds Σ, Σ_1 and Σ_2 we have associated Hilbert spaces $\mathcal{H}_\Sigma, \mathcal{H}_{\Sigma_1}$ and $\mathcal{H}_{\Sigma_2}$ of functions (sections) on spaces of fields. So, by varying the $d+1$-dimensional manifolds Y (possibly with given extra structure γ – see for example section 3.2.1) with boundary and choosing sets of fields (paths to be summed) and action functionals (to weight the sum), the path integral (21) is a gadget that makes correspond an operator to every Y

$$Y \longmapsto \widehat{K}_Y. \tag{22}$$

This correspondence has to satisfy natural properties, which depend on the theory. First of all we have to specify the vector spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ between which $\widehat{K}_Y$ is a morphism (operator)

$$\widehat{K}_Y \; \in \mathrm{Hom}(\mathcal{H}_{in}, \mathcal{H}_{out}).$$

The boundary values of fields on $\Sigma = \partial Y$ define spaces $Q(\Sigma)$ for every boundary Σ (possibly also with extra structure h. We will however, for simplicity, omit the dependence on γ and h). This correspondence

$$\Sigma \mapsto Q(\Sigma) \tag{23}$$

satisfies the obvious property

$$Q(\Sigma_1 \coprod \Sigma_2) = Q(\Sigma_1) \times Q(\Sigma_2). \tag{24}$$

Then, to these spaces, we make correspond Hilbert spaces of functions (sections) on $Q(\Sigma)$ so that we have the correspondence

$$\Sigma \mapsto \mathcal{H}_\Sigma, \tag{25}$$

which, in the simplest cases, is

$$\mathcal{H}_\Sigma = L^2(Q(\Sigma), d\mu_{Q(\Sigma)}),$$

where $d\mu_{Q(\Sigma)}$ is a measure on $Q(\Sigma)$. From (24) we obtain the properties

$$\begin{array}{rcll} \Sigma = \Sigma_1 \coprod \Sigma_2 & \mapsto & \mathcal{H}_\Sigma & (= \; L^2(Q(\Sigma_1) \times Q(\Sigma_2), d\mu_{Q(\Sigma_1)\times Q(\Sigma_2)})) \\ & & & \cong \; \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \\ \emptyset & \mapsto & \mathbb{C}. & \end{array} \tag{26}$$

The property

$$\mathcal{H}_{\Sigma^*} \cong \mathcal{H}^*_\Sigma$$

is also required for consistency, where Σ^* denotes the manifold Σ with opposite orientation. In the axiomatic approaches initiated by Segal [Seg1, Seg2] and Atiyah [At1] the difficulties in defining the PI (21) are circumvented by promoting these and other "properties" (e.g. the very important sewing axiom (28) below) into the defining axioms of a QFT.

The remaining obvious properties that (25) and (21), (22),

$$\begin{array}{rcl} \Sigma & \mapsto & \mathcal{H}_\Sigma \\ \mathrm{mor}(\Sigma_{in}, \Sigma_{out}) \ni Y & \mapsto & \widehat{K}_Y \in \mathrm{Hom}(\mathcal{H}_{\Sigma_{in}}, \mathcal{H}_{\Sigma_{out}}) \cong \mathcal{H}^*_{\Sigma_{in}} \otimes \mathcal{H}_{\Sigma_{out}}, \end{array} \tag{27}$$

have to satisfy may, in the simpler cases (see a more detailed comment in the beginning of section 3.2.4), be summarized by saying that a QFT defines a functor from the (tensor) category of $d+1$ cobordisms with extra structures to the (tensor) category of (Hilbert) vector spaces. In particular let $\Sigma \subset \partial Y_1, \Sigma^* \subset \partial Y_2$. Then the sewing axiom reads

$$Y = Y_1 \cup_\Sigma Y_2 \Rightarrow \widehat{K}_Y = \mathrm{tr}_{\mathcal{H}_\Sigma} \widehat{K}_{Y_2} \otimes \widehat{K}_{Y_1}. \tag{28}$$

Note that if $\Sigma = \partial Y_{1_{out}}$ and $\Sigma^* = \partial Y_{2_{in}}$, then the trace in (28) coincides with the composition of morphisms,

$$\widehat{K}_Y = \mathrm{tr}_{\mathcal{H}_\Sigma} \widehat{K}_{Y_2} \otimes \widehat{K}_{Y_1} = \widehat{K}_{Y_2} \circ \widehat{K}_{Y_1}. \tag{29}$$

This axiom includes the fundamental quantum mechanical semigroup property (4) and (5) (in fact one such property for each closed d-dimensional manifold Σ). Choosing $Y_1 = [0, t_1] \times \Sigma, Y_2 = [t_1, t_1 + t_2] \times \Sigma$ we have

$$Y = Y_1 \cup_\Sigma Y_2 = [0, t_1 + t_2] \times \Sigma, \tag{30}$$

which implies that the evolution operator (one for each Σ),

$$\widehat{K}^\Sigma_t := \widehat{K}_{[0,t]\times\Sigma} \in \mathrm{End}(\mathcal{H}_\Sigma),$$

satisfies the semigroup property

$$\widehat{K}^{\Sigma}_{t_1+t_2} = \widehat{K}^{\Sigma}_{t_1} \circ \widehat{K}^{\Sigma}_{t_2}. \tag{31}$$

If the generators, $\widehat{H}^{\Sigma}$, of these semigroups exist,

$$\widehat{K}^{\Sigma}_{t} = e^{-\frac{t\widehat{H}}{\hbar}^{\Sigma}}, \tag{32}$$

they play the role of hamiltonians in the QFT.

However, since for $d > 0$ there are many more manifolds than just those with product structure (30), the sewing axiom contains much more than the semigroup properties (31).

If Y is closed then $\Sigma_{in} = \Sigma_{out} = \emptyset$ and from (26) and (27) we conclude that $\widehat{K}_Y \in \mathbb{C}$. This number is called the partition function of the given QFT for the manifold (in general with extra structure) Y

$$Z(Y) = \widehat{K}_Y \in \mathbb{C} \text{ if } \partial Y = \emptyset. \tag{33}$$

On the other hand, if Y is such that it has only *out* (or *in*) boundary $\Sigma_{in} = \emptyset$ ($\Sigma_{out} = \emptyset$), then from (27), $\widehat{K}_Y$ is a vector $\widehat{K}_Y \in \mathcal{H}_\Sigma$ ($\widehat{K}_Y$ is a covector $\widehat{K}_Y \in \mathcal{H}^*_\Sigma$) and we will use the Dirac notation

$$|\psi_Y> := \widehat{K}_Y \qquad (<\psi_Y| := \widehat{K}_Y). \tag{34}$$

3.2.1 Two-dimensional CFT: Semigroup of annuli and Virasoro semigroup

Significative examples of the complexity of the sewing axiom are given by conformal field theories in 2 dimensions ($2d$) for which the required extra structure on the $2d$ manifolds with boundary is a complex structure. The $2d$ manifolds are assumed to be complements of non overlapping disks in closed Riemann surfaces. These disks correspond to choices of local coordinates in neighborhoods of points in the closed Riemann surface $\widetilde{Y}$. Then, one takes out the interior of the corresponding unit disks and chooses an orientation on the boundary circles to obtain the Riemann surface with parametrized boundary Y. If the orientation of the boundary component Σ_1 coincides (does not coincide) with the orientation induced from Y then $\Sigma_1 \subset \Sigma_{out}$ ($\Sigma_1 \subset \Sigma_{in}$).

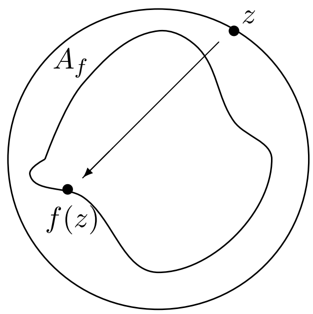


Figure 3.4

A Riemann surface, which is topologically a cylinder $[0,t]\times S^1$, has an infinite dimensional moduli space of inequivalent complex structures parametrized by annuli A_f in the complex plane $\mathbb{C}$, as represented in figure 3.4, where f denotes a diffeomorphism, $z \mapsto f(z), |z| = 1$, from the unit circle in $\mathbb{C}$ to the "*out*" boundary of the annulus A_f, subject to the obvious restriction $|f(z)| < 1$ or $|f(z)| \equiv 1$, in which case $f \in Diff(S^1)$ [Seg1, Seg2]. These annuli are particular morphisms of the category of cobordisms of Riemann surfaces $A_f \in \mathrm{mor}(S^1, S^1)$. Their composition is given by

$$A_{f_2} \circ A_{f_1} = A_{f_1 \circ f_2} \tag{35}$$

and endows the set $\mathcal{A}nn$ of all annuli, with the structure of a subsemigroup of the category. This is the Segal semigroup of annuli, which is the complexification of $Diff(S^1)$ ($Diff(S^1)$ plays the role of the Shilov boundary of $\mathcal{A}nn$). A_f tends to the Shilov boundary when the "thickness" of the annulus tends to zero, approaching an element of $Diff(S^1)$. For an algebro-geometric study of $\mathcal{A}nn$ and of its universal central extension see [MP].

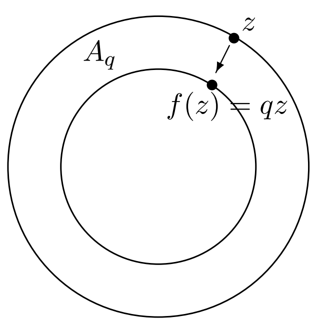


Figure 3.5

The direct product cylinder case (30) (for Riemann surfaces with parametrized boundaries) corresponds to $f(z) = const = q \in \mathbb{R}, 0 < q \leq 1$ (see figure 3.5). Of importance is also the complexification of this subsemigroup obtained by letting $q \in \mathbb{C}, 0 < |q| \leq 1$.

According to the axiomatic description above, a 2-dimensional ($2d$) conformal QFT (CFT) makes correspond to S^1 an Hilbert space $\mathcal{H}$ and to each annulus

A_f, $A_f \in \mathrm{mor}(S^1, S^1)$, an operator

$$A_f \;\mapsto\; \widehat{K}_{A_f} \;:\; \mathcal{H} \to \mathcal{H}\;.$$

The sewing axiom (29) means that this map defines a representation of $\mathcal{A}nn$[5]. The two generators of the subsemigroup $\widehat{K}_{A_q}$ corresponding to the direct product annuli in figure 3.5 (and their twisted versions) are the hamiltonians and are usually represented by L_0 and $\overline{L}_0$ so that

$$\widehat{K}_{A_q} = q^{L_0}\overline{q}^{\overline{L}_0}.$$

This explains why projective representations of $\mathcal{A}nn$, or equivalently representations of its central extension, the Virasoro (semi)group Vir, play such a central role in CFT: the Hilbert space $\mathcal{H}$ carries always a representation of Vir.

3.2.2 Two-dimensional CFT: vertex operator algebras

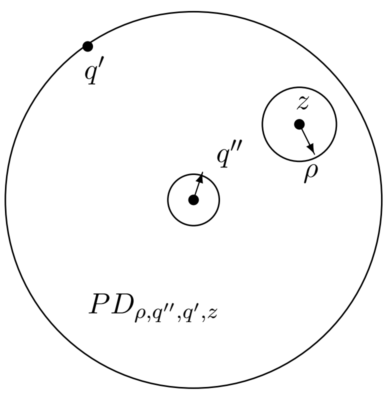


Figure 3.6

Besides the annuli A_f, the pants diagram Riemann surfaces $PD_{\rho,q'',q',z}$ of the type shown in figure 3.6 is also of particular importance in $2d$ CFT. The operators corresponding to these Riemann surfaces,

$$\widehat{K}_{PD_{\rho,q'',q',z}} \;:\; \quad \mathrm{Hom}(\mathcal{H}\otimes\mathcal{H},\mathcal{H}) \cong \cong \mathrm{Hom}(\mathcal{H},\mathcal{H}^*\otimes\mathcal{H}),$$

define a map from vectors to operators which corresponds (Theorem 6.19 in [Ts] for σ models with torii as targets) to the structure of vertex operator algebras that were axiomatized by Borcherds and were crucial for progress on the Moonshine conjecture [Bo, Gan1, Gan2]. The precise relation is

$$\widehat{K}_{PD_{\rho,q'',q',z}} = q'^{-L_0}\overline{q'}^{-\overline{L}_0}\; Y(\rho^{L_0}\overline{\rho}^{\overline{L}_0}\;\cdot\;,\; z)\; q''^{L_0}\overline{q''}^{\overline{L}_0}, \tag{36}$$

where Y is the map from vectors to operators that satisfies the Borcherds vertex operator algebra axioms. This is a very complicated set of axioms (as should be expected from (36)), which were abstracted from the PI realization [Ts].

[5]In general, due to the Weyl anomaly, this representation is only projective. We are describing here the situation with vanishing central charge, $c = 0$.

3.2.3 Topological quantum field theories and representations of mapping class groups

If a $d+1$ QFT has no extra structure besides the differentiable structure on the $d+1$-dimensional manifolds Y, then the map $Y \mapsto \widehat{K}_Y$ (27) is invariant under $Diff(Y)$. In particular, if Y is closed then the partition function, $Z(Y)$, is a diffeomorphism invariant of the manifold. Diffeomorphism invariant theories for which the correspondence (27) is a functor so that, in particular, the sewing axiom (28) is valid, are called topological quantum field theories (TQFT) – usually the Hilbert spaces $\mathcal{H}_\Sigma$ are also required to be finite dimensional. Notice that this is not the case in theories of "quantum gravity" or in parametrized field theories, in which, in the path integral (21), one sums also over the metric tensors γ on Y (see also comments in the beginning of section 3.2.4). In a TQFT, the diffeomorphism invariance implies that the operators $\widehat{K}_{[0,t]\times\Sigma}$, corresponding to manifolds that are topologically cylinders, are all equal to the identity in $\mathrm{End}(\mathcal{H}_\Sigma)$. This means that, in these theories, not only the hamiltonians $\widehat{H}^\Sigma$ (32) vanish but also the analogues of the semigroup of annuli (35) for a $2d$ CFT are trivial.

We will review the calculation of the topological invariant for $Y = S^3$ in Chern-Simons topological quantum field theory in section 4.2. By decomposing Y as a union of manifolds with boundary, e.g.

$$Y = Y_1 \cup_{\Sigma_a} Y_3 \cup_{\Sigma_b} Y_2,$$

the partition function can be reduced, with the help of the sewing axiom (27), to the calculation of inner products and compositions of operators for simpler manifolds. In the notation of (34),

$$Z(Y) =< \psi_{Y_2} | \widehat{K}_{Y_3} \; \psi_{Y_1} > \in \mathbb{C}. \tag{37}$$

Diffeomorphisms of d-dimensional manifolds that are not homotopic to the identity (or mapping class transformations), play an important role in TQFT (see e.g. [BK, Koh]). Indeed, let in (37) $Y_3 = \emptyset, \Sigma = \Sigma_a = \Sigma_b$, so that we are dividing Y in the union of two manifolds with boundary by cutting along Σ. We can do surgery on Y along Σ by using a diffeomorphism f, $f \in Diff(\Sigma)$, not homotopic to the identity to glue back $\Sigma \subset Y_1$ with $\Sigma^* \subset Y_2$. The result is a $d+1$-dimensional manifold Y_f that is, in general, not diffeomorphic to Y. If the vectors $|\psi_Y >$ (possibly with extra observables) span $\mathcal{H}_\Sigma$ (see the next subsection) then the topological invariants $Z(Y_f)$, for all Y with the above decomposition, define an operator $U_f \in \mathrm{End}(\mathcal{H}_\Sigma)$, through

$$Z(Y_f) =< \psi_{Y_2} | U_f \; \psi_{Y_1} > . \tag{38}$$

This defines (in general projective) homomorphisms (one for every closed d-dimensional manifold Σ)

$$\begin{array}{ccc} Diff(\Sigma) & \longrightarrow & \mathrm{End}(\mathcal{H}_\Sigma) \\ f & \mapsto & U_f \end{array}$$

which are trivial on the components of the identity. Therefore a $d+1$-dimensional TQFT defines (projective) representations of the mapping class groups $\Gamma(\Sigma) = Diff(\Sigma)/Diff_0(\Sigma)$ of closed d-dimensional manifolds on the corresponding vector spaces $\mathcal{H}_\Sigma$

$$\begin{array}{ccc} \Gamma(\Sigma) & \longrightarrow & \mathrm{End}(\mathcal{H}_\Sigma) \\ [f] & \mapsto & U_f. \end{array}$$

3.2.4 Gravitational and parametrized theories, Hartle-Hawking wave function of the universe and observable to state map

Consider a again a general QFT, not necessarily a CFT or in $2d$.

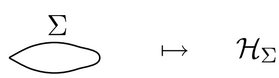


Figure 3.7

As we saw in (33) a QFT in more than one dimension (therefore a possibility we did not have in quantum mechanics) gives a purely geometric way of constructing states (vectors) in the vector spaces $\mathcal{H}_\Sigma$. For every $d+1$-dimensional Y manifold with boundary $\partial Y = \Sigma$ (and in general with extra structure) we get a state $|\psi_Y>$.

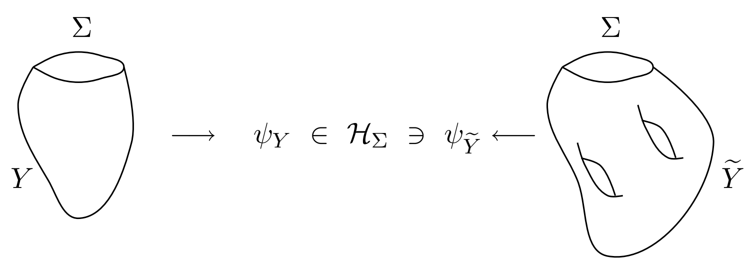


Figure 3.8

In theories (gauge) invariant under diffeomorphisms, i.e. such that $\widehat{K}_{fY} = \widehat{K}_Y$ for all $d+1$-dimensional manifolds Y and all $f \in Diff(Y)$, the map

$Y \mapsto \widehat{K}_Y$ in (27) may not be a functor from the category of $d+1$ cobordisms to the category of Hilbert spaces. This is indeed the case in theories for which the metric on (compact) Y's is one of the dynamical variables (theories of $d+1$ quantum gravity) and also in the so called parametrized field theories [HK]. Let us, to illustrate, consider the simplest parametrized system: the parametrized particle on the manifold M. Since $[0,t] \overset{diff}{\cong} [0,t']$, for all $t' > 0$, we have

$$\underset{\leftrightarrow}{\widehat{K}}^{par} := \widehat{K}^{par}_{[0,t]} = \widehat{K}^{par}_{[0,t']}.$$

Being a functor would imply that $\underset{\leftrightarrow}{\widehat{K}}^{par} = \mathrm{Id}_{\mathcal{H}}$. In fact, this operator for the (Euclidean) parametrized particle can be obtained from the operator $\widehat{K}_{[0,t]}$ for an unparametrized particle with fixed metric $\gamma_\tau = \gamma(\tau)d\tau^2$ on $[0,t]$ as in (8), (9), by integrating over the diffeomorphism classes of metrics on $[0,t]$. Since,

$$\begin{array}{rcl} Met([0,t])/Diff([0,t]) & \cong & \mathbb{R}_+ \\ \gamma & \longmapsto & T = \int_0^t \sqrt{\gamma_{00}}\, d\tau, \end{array}$$

we can choose $\gamma = dt^2$ on the interval $[0,t]$ and integrate over its length t,

$$\underset{\leftrightarrow}{\widehat{K}}^{par} = \int_0^\infty dt\ \widehat{K}_{[0,t]} = \int_0^\infty dt\ e^{-t\widehat{H}} = \widehat{H}^{-1} =: \widehat{D}.$$

The resulting integral kernel

$$\underset{\leftrightarrow}{\widehat{K}}^{par}(x,y) = D(x,y),$$

is called the propagator (from y to x in M). The above idea of representing the propagator of a parametrized system as an integral over diffeomorphism classes of metrics on the line of the evolution operator (thus, corresponding to a functor, satisfying the semigroup law (4)) for an unparametrized system, is used with great success in (super)string theory. The string propagators, and other amplitudes, are represented as sums over genera of integrals over the moduli space of curves $\mathcal{M}_{g,n}$, of a given genus g and number of punctures n, of amplitudes calculated with the help of an associated (super)conformal field theory,

$$\widehat{K}^{string} = \sum_{g=0}^{\infty} \int_{\mathcal{M}_{g,n}} dm_{g,n}\ \widehat{K}^{CFT}_{Y,m_{g,n}} \quad . \tag{39}$$

As we have seen, the latter corresponds to a functor from the category of two-dimensional cobordisms with complex structures and parametrized boundaries to the category of Hilbert spaces.

For theories of quantum gravity on non compact manifolds with appropriate asymptotic conditions on the fields (e.g. asymptotically flat or asymptotically anti-De Sitter geometries) it is possible to formulate a set of axioms for the PI measure, which mimic the Osterwalder-Schrader axioms [OS1, OS2] for quantum field theories in flat space-time [AMMT]. These axioms do not refer to any background structure on Y,but the asymptotic conditions. The analogue of the Osterwalder-Schrader reconstruction theorem is valid, which allows to reconstruct the hamiltonian formalism from the covariant PI. In this framework, non trivial hamiltonian operators are associated with diffeomorphisms of Y which are asymptotic to time translations at infinity.

The possibility of constructing states in the Hilbert spaces $\mathcal{H}_\Sigma$ in a purely geometric way is considered of interest for possible cosmological implications of quantum gravity. Hartle and Hawking [HH] suggested the following answer to a long standing problem of cosmology:

Q. Which were the initial conditions of the Universe after the big bang and why did the universe decide to choose them?

A. There is no doubt that the most economic way for nature to resolve the problem of initial conditions for our Universe would be to choose a framework in which no initial conditions are needed. This is (in principle) made possible by the extension to quantum gravity of the Euclidean PI formalism of QFT [Ha], due to the fact that states like those in figure 3.8 do not need initial conditions as they have an empty *in* boundary. This is the essence of the Hartle-Hawking proposal for the wave function of the Universe, called noboundary (or Hartle-Hawking) wave function of the Universe [HH].

Let us describe briefly the Hartle-Hawking proposal. Pure quantum gravity in $4d$ is a QFT in which amplitudes like (21) (with metric tensors g as the fields Φ) are obtained by summing over all metrics on $4d$ manifolds Y (in fact diffeomorphism classes of metrics, called geometries), weighted by exponential of minus the Einstein-Hilbert action (or variations of it) with a fixed induced metric h on the $3d$ boundary Σ. Similarly to (39), one has then to sum over all (diffeomorphism classes of) $4d$ manifolds Y with fixed boundary Σ. In particular, if the boundary Σ has only an *out* part (possibly disconnected), the resulting vector $|\psi_0>$ is a (hopefully square integrable with respect to some measure) function of the 3-metrics (3-geometries) h on Σ. Such vector would give the probability amplitudes for the different 3-metrics of the spatial manifold Σ (our Universe),

$$\psi_0(h;\Sigma) = \sum_{\{Y \ : \ \partial Y = \Sigma\}} \int_{\{g \in \mathrm{Met}(Y): \ i_\Sigma^* g = h\}} Dg \ e^{-S_{\mathrm{grav}}(g)}, \tag{40}$$

where S_{grav} is the classical gravitational action,

$$S_{\mathrm{grav}}(g) = -\frac{1}{16\pi G}\int_Y d_g v\ (R[g] - 2\Lambda) - \frac{1}{8\pi G}\int_\Sigma d_h v K, \tag{41}$$

K is the trace of the extrinsic curvature of Σ, G is the Newton constant and Λ is the cosmological constant. The state ψ_0 is the Hartle-Hawking wave function for an Universe without matter. In more realistic models, one has to add matter (i.e. nongravitational) degrees of freedom, Φ, and change the action in (41) appropriately,

$$S(g,\Phi) = S_{\mathrm{grav}}(g) + S_{\mathrm{matter}}(\Phi, g).$$

The Hartle-Hawking wave function of (hopefully) our Universe was then, in the beginning of times,

$$\psi_0(h,\phi;\Sigma) = \sum_{\{Y\ :\ \partial Y = \Sigma\}} \int_{\{(g,\Phi)\ :\ i^*_\Sigma g = h,\ \Phi_{|\Sigma} = \varphi\}} DgD\Phi\ e^{-S(g)}. \tag{42}$$

Even though the PIs (40) and (42) are written in imaginary time, they are worse defined than in other theories with infinite dimensional symmetry groups, due to the complicated nature of the quotient $\mathrm{Met}(Y)/Diff(Y)$ and the fact that the (euclidean) Einstein-Hilbert action is unbounded from below. Unfortunately the difficulties in making sense of the PI for quantum gravity, even for "minisuperspace" models, have not allowed, so far, to give a firm mathematical basis to this conjecture. See however [Am, An, BC, Lo, RR] and references therein for very interesting recent developments.

In a QFT, one can produce more states in the Hilbert spaces $\mathcal{H}_\Sigma$ by integrating in (21) not the function $\mathcal{O}(\Phi) = 1$ (with respect to the measure "$\frac{1}{c}e^{-S}D\Phi$") but other functions (also called observables or operators[6]), $\mathcal{O}$, depending on the values of the fields inside Y

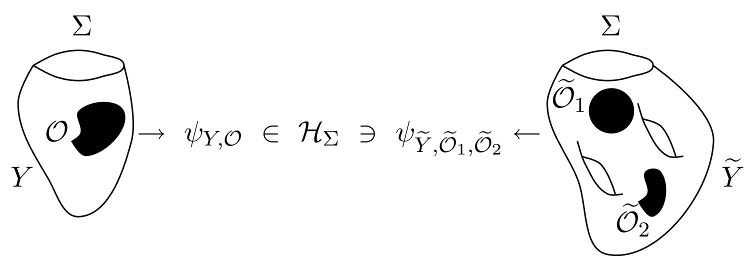


Figure 3.9

[6]Due to the fact that, in the hamiltonian formalism, these functions correspond to operators.

$$Y, \mathcal{O} \longmapsto \psi_{Y,\mathcal{O}}(\varphi) = \frac{1}{c} \int_{\mathcal{P}_\varphi^Y} \mathcal{O}(\Phi) e^{-S(\Phi)} D\Phi. \tag{43}$$

In QFT, this map is called the observables (or operators) to state map.

We can, of course, also take more general $d+1$-manifolds Y with both *in* and *out* boundaries (or just *in* or none) in which case the functor (27) is modified to the new functor

$$\begin{array}{rcl} \Sigma & \mapsto & \mathcal{H}_\Sigma \\ \mathrm{mor}(\Sigma_{in}, \Sigma_{out}) \ni (Y, \mathcal{O}) & \mapsto & \widehat{K}_{Y,\mathcal{O}} \in \mathrm{Hom}(\mathcal{H}_{\Sigma_{in}}, \mathcal{H}_{\Sigma_{out}}), \end{array} \tag{44}$$

satisfying a similar set of axioms.

The axioms have to be significantly complicated if the observables are allowed to depend on values of the fields on the boundary. This possibility is important for the calculation of the Jones polynomial with the help of Chern-Simons theory.

3.3 Symmetries of the action

One of the central points in the modern theory of path integrals, that we will not address in these lectures (see section 4.1 for a "way out" in gauge theories and for few more comments), consists in the fact that, in the most interesting situations, the action S has a noncompact infinite dimensional Lie group, $\mathcal{G}$, of symmetries. This makes the case of turning (21) (or (43), (44)) into a meaningful object, from the measure theoretical point of view, even more difficult (all examples in the present section are instances of this situation). As often before (hinting once again to the fundamental, unrevealed, (mathematical) meaning of PI), the "escape forward" attitude of theoretical and mathematical physicists opens the doors to new and very beautiful interactions with mathematics.

In an oversimplified version the "folklore" goes as follows. If the integrand in say (43) is invariant under $\mathcal{G}$ then, even if we can construct $D\Phi$, the integral will (for sufficiently nice $\mathcal{G}$-actions) factorize into an integral over the orbits (giving the infinite volume of the orbit $\mathcal{F}$) and an integral over the quotient

$$\psi_{Y,\mathcal{O}}(\varphi) = \mathrm{Vol}(\mathcal{F}) \cdot \frac{1}{c} \int_{\mathcal{P}_\varphi^Y/\mathcal{G}} \mathcal{O}([\Phi])\; e^{-S([\Phi])} D[\Phi] \;=\; \infty. \tag{45}$$

In a sense, this integral diverges even before we can start defining it. So we say that the "correct" object is

$$\psi_{Y,\mathcal{O}}(\varphi) = \frac{1}{c} \int_{\mathcal{P}_\varphi^Y/\mathcal{G}} \mathcal{O}([\Phi])\; e^{-S([\Phi])} D[\Phi]. \tag{46}$$

However, since the quotient spaces $\mathcal{P}/\mathcal{G}$ are typically very complicated, we usually know even less what is $D[\Phi]$ on $\mathcal{P}/\mathcal{G}$ than what is $D\Phi$ on $\mathcal{P}$. No problem. We go back again to the heuristic expression (43) and "before" it leads to (45) we "use" it to define a measure on the quotient and therefore to give a meaning to (46). This is "done" by *fixing a gauge* or, in other words, by choosing a section σ of

$$\mathcal{P} \overset{\overset{\sigma}{\curvearrowleft}}{\longrightarrow} \mathcal{P}/\mathcal{G} \tag{47}$$

and defining a δ-function, $\delta_{\mathcal{C}}$, to restrict integration to the image of σ, $\mathcal{C} := \sigma(\mathcal{P}/\mathcal{G})$. Of course, a global section σ may not be defined, leading to the so called Gribbov ambiguity (related issues are beautifully discussed by Shabanov [Sh]). The gauge fixing has to be done in such a way that the integrals of gauge-invariant observables will be independent on the choice of the gauge and this is achieved by introducing an appropriate determinant. In the PI formalism the contribution of this determinant is best described and studied by representing it as an integral over Grassman-algebra valued fields, called ghosts.

This leads to very nice and deep relations between the PI formalism and equivariant cohomology and localization and ultimately to the relations between cohomological QFT and intersection numbers on different moduli spaces [Bl, BT2, CMR, Di2, Ho, LL, LM, Marc, Mari1, Mari2, Wi4].

4 Further examples of the PI $\Leftrightarrow$ Topology and Geometry correspondence

4.1 Knots, loops and connections. Rigorous measures

An important problem in $3d$ topology consists in distinguishing knots and links in 3 manifolds Y by functions on the space of all links that are invariant under $Diff_0(Y)$.

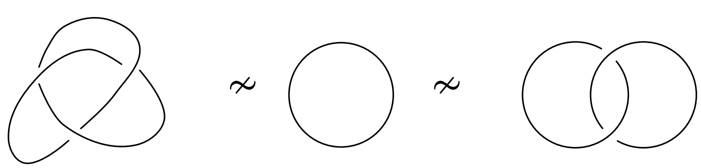


Figure 4.1

In 1985 V. Jones introduced a function (the Jones polynomial) that could distinguish many knots which had not been distinguished by any other invariant function until then [Jo]. The Jones polynomial

$$V \ : \ \text{Links in } S^3 \to \mathbb{Z}[t, t^{-1}] \tag{48}$$

is defined by

$$\begin{array}{ll} \text{(i)} & V(\cdot) \text{ is } Diff_0(S^3) - \text{invariant;} \\ \text{(ii)} & V(U) = 1, \text{ where } U \text{ is the unknot;} \end{array}$$

and the skein relations (see figure 4.2)

$$\text{(iii) } t^2 V(L_+) - t^{-2} V(L_-) = (t - t^{-1}) V(L_0). \tag{49}$$

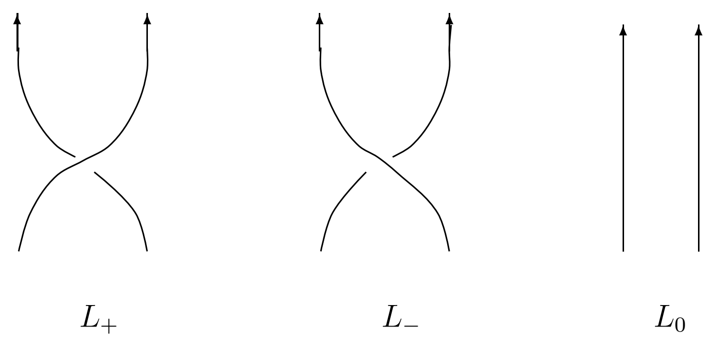


Figure 4.2

Similarly to previous polynomial invariants, the Jones polynomial was originally defined through properties of projections to a $2d$ plane reflected in these skein relations. In 1987 Atiyah proposed the challenge of finding an intrinsic $3d$ definition of the Jones polynomial using a $3d$ TQFT. This challenge was beautifully solved by Witten in 1988 [Wi5].

We will return to knot and link invariants in the next subsection. Below, we will describe briefly a way of constructing rigorous measures on (compactifications of) spaces of connections, which has been very usefull in Loop Quantum Gravity [Th1, As, Th2, AL3]. Let Y be a manifold and $\mathcal{A}^{SU(n)}(Y)$ be the space of $SU(n)$-connections on the trivial $SU(n)$-bundle over Y so that

$$\mathcal{A}^{SU(n)}(Y) \cong \Omega^1(Y) \otimes su(n).$$

The infinite dimensional Lie group $\mathcal{G}(Y) = C^\infty(Y, SU(n))$ of gauge transformations (vertical automorphisms of the bundle) acts on $\mathcal{A}^{SU(n)}(Y)$ by

$$A \mapsto gAg^{-1} + gdg^{-1}.$$

Geometrically and physically we are really interested in the quotient

$$\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y). \tag{50}$$

A very large class of $\mathcal{G}(Y)$-invariant functions on $\mathcal{A}^{SU(n)}(Y)$ is obtained by taking the vector space spanned by the so called Wilson loop variables (or functions)

$$\begin{aligned} W(\alpha,\cdot)\ :\ \mathcal{A}^{SU(n)}(Y) &\longrightarrow \mathbb{C} \\ A &\longmapsto W(\alpha,A) = \mathrm{tr}(U_\alpha(A)), \end{aligned} \tag{51}$$

where α is a C^∞-loop in Y, $U_\alpha(A)$ is the holonomy of A along the loop α and the trace is taken with respect to the fundamental representation. Being gauge invariant they define functions on the quotient (50). Let $\mathcal{B}$ denote the commutative C^*-algebra (with norm of supremum), called holonomy algebra, generated by the Wilson variables

$$\mathcal{B} = \overline{\left\{\sum_i c_i W(\alpha_i,\cdot)\right\}} \subset C_b(\mathcal{A}^{SU(n)}(Y)). \tag{52}$$

The following result is important [Bar, AI, AL1, AL2]

Theorem 4 (Barret, Ashtekar-Isham-Lewandowski). *The Wilson functions separate the points in $\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)$ and the set of maximal ideals of $\mathcal{B}$,*

$$\mathrm{Spec}_m(\mathcal{B}) = \overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}, \tag{53}$$

is (for the Gel'fand topology) a compactification of $\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)$. The image of the natural inclusion

$$\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y) \hookrightarrow \overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)} \tag{54}$$

is topologically dense and

$$\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)} \cong \overline{\mathcal{A}^{SU(n)}(Y)}/\overline{\mathcal{G}(Y)},$$

where $\overline{\mathcal{A}^{SU(n)}(Y)}$ is a suitable compactification of $\mathcal{A}^{SU(n)}(Y)$ and $\overline{\mathcal{G}(Y)}$ is the natural compactification of $\mathcal{G}(Y)$ given by all, not necessarily continuous, maps from Y to $SU(n)$,

$$\overline{\mathcal{G}(Y)} = \mathrm{Map}(Y, SU(n)) = SU(n)^Y.$$

□

The compacteness of $\overline{\mathcal{G}(Y)}$ is very important for the construction of rigorous (not necessarily PI) measures. Indeed, we can integrate $\overline{\mathcal{G}(Y)}$-invariant functions in (43) since the factor $\mathrm{Vol}(\mathcal{F})$ in (45) is now finite. This of course resolves problems with Gribbov ambiguities and alike since in this approach we do not need to choose a gauge, i.e. a section σ in (47). Technically, we define a finite measure on $\overline{\mathcal{A}^{SU(n)}(Y)}$ and then define a measure on $\overline{\mathcal{A}^{SU(n)}(Y)}/\overline{\mathcal{G}(Y)}$ by pushforward with respect to the measurable projection

$$\pi \ : \ \overline{\mathcal{A}^{SU(n)}(Y)} \ \longrightarrow \ \overline{\mathcal{A}^{SU(n)}(Y)}/\overline{\mathcal{G}(Y)}.$$

The measure can also be defined directly on the quotient (53). In fact, to define a measure on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ is equivalent to finding a positive (and therefore automatically continuous) functional on the holonomy C^* algebra $\mathcal{B}$. This strategy was proposed, in the framework of Ashtekar or loop nonperturbative hamiltonian quantum gravity, in [AI, AL1][7] (see the recent reviews [Th1, As, Th2, Vel1, AL3] and references therein. Complementary viewpoints are described in [GP, Rov, Sm]). Applications to Yang-Mills theories were studied e.g. in [As1, Th3, GP, As2, Fl, FL, Vi].

Ashtekar and Lewandowski defined in [AL1] a $Diff(Y)$-invariant probability measure on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ that has played a very important role in rigorous approaches to loop quantum gravity. Let us briefly recall its construction. Perhaps the easiest way to define the Ashtekar-Lewandowski (AL) measure is the following one proposed in [AL2] within the context of piecewise analytic loops and edges (see also [AMM, Bae2] for other pedagogical introductions to measure theory in this context). Consider on Y the set Λ of graphs with analytic oriented edges. Let $E(\gamma)$ and $V(\gamma)$ denote, respectively, the number of edges and the number of vertices of the graph γ. The set Λ is naturally a directed set with respect to the relation

$$\gamma \leq \gamma' \tag{55}$$

whenever each edge of γ can be represented as a composition of edges of γ' and each vertex in γ is a vertex in γ'. We will use (55) to define the projective limit $\overline{\mathcal{A}^{SU(n)}(Y)}$. Consider the surjective projections

$$\begin{aligned} p_\gamma \ : \mathcal{A}^{SU(n)}(Y) \ &\longrightarrow \ SU(n)^{E(\gamma)} \\ A \ &\longmapsto \ (U_{e_1}(A), \ldots, U_{e_{E(\gamma)}}(A)), \end{aligned}$$

[7] In the framework of loop quantum gravity the above measures are defined within the hamiltonian formalism and therefore these are not PI measures but measures on $\overline{\mathcal{A}^{SU(n)}(\Sigma)/\mathcal{G}(\Sigma)}$. The same formalism can in principle be used to study the covariant PI of general relativity with Ashtekar variables.

where we assumed, for simplicity, that the edges of the graphs γ were ordered, $\{e_1, \ldots, e_{E(\gamma)}\}$, to obtain explicit maps to $SU(n)^{E(\gamma)}$ (these orderings play no other role then to make these maps explicit). For $\gamma \leq \gamma'$ there are unique projections $p_{\gamma\gamma'}$ that make the following diagrams commutative

$$\begin{array}{ccc} \mathcal{A}^{SU(n)}(Y) & & \\ {\scriptstyle p_\gamma}\downarrow & \searrow^{p_{\gamma'}} & \\ SU(n)^{E(\gamma)} & \xleftarrow{p_{\gamma\gamma'}} & SU(n)^{E(\gamma')}. \end{array}$$

The family $\{G^{E(\gamma)}, p_{\gamma\gamma'}\}_{\gamma\gamma'\in\Lambda}$ is a projective family of compact Hausdorff spaces with continuous projections. Then the projective limit $\overline{\mathcal{A}^{SU(n)}(Y)}$ of this family is also a compact Hausdorff space that has the following neat algebraic characterization: it coincides with the set of all (not necessarily continuous) morphisms from the grupoid of piecewise analytic edges in Y to $SU(n)$ [Bae1, Vel2]. In this framework $\mathcal{A}^{SU(n)}(Y)$ corresponds to the smooth morphisms (see [CP] for the case of $\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)$). The natural map from $\mathcal{A}^{SU(n)}(Y)$ to $\overline{\mathcal{A}^{SU(n)}(Y)}$ is injective and its image is topologically dense. By introducing on each $SU(n)^{E(\gamma)}$ a measure, μ^H_γ, equal to the product of Haar measures on each copy of $SU(n)$ we obtain a projective family of measure spaces

$$\{G^{E(\gamma)}, p_{\gamma\gamma'}, \mu^H_\gamma\}_{\gamma\gamma'\in\Lambda} \ ,$$

which satisfies the consistency conditions to define a cylindrical measure on $\overline{\mathcal{A}^{SU(n)}(Y)}$. This family plays a role analogous to the family (14) in quantum mechanics and an analogous result to that case shows that the cylindrical measure has a unique extension to a σ-additive Borel measure on $\overline{\mathcal{A}^{SU(n)}(Y)}$ (see [Ya]) that we denote by $\widetilde{\mu}_{AL}$. From the theorem 4 we see that $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ can be obtained from $\overline{\mathcal{A}^{SU(n)}(Y)}$ by quotienting by the compact topological group of generalized gauge transformations, $\overline{\mathcal{G}(Y)} = \mathrm{Map}(Y, SU(n))$. The AL measure μ_{AL} is the measure on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ obtained by taking the push-forward of $\widetilde{\mu}_{AL}$ with respect to this projection. It is easy to verify that this measure is diffeomorphism invariant. The following result is valid [MM, MTV].

Theorem 5 (Marolf-M,M-Thiemann-Velhinho). *The image of the inclusion (54) has zero μ_{AL}-measure and $Diff(Y)$ acts ergodically on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$.* □

Other diffeomorphism invariant measures on $\overline{\mathcal{A}^{SU(n)}(Y)}$ in this approach were defined in [Bae2].

A nice toy example to study and to compare with the general framework of section 3.2 is that of $2d$ Yang-Mills theory [GKS, Wi7, Wi8, Sen, CMR, KS, NS, Th3, As2, BS]. Recall that the Yang-Mills action can be defined for any Riemannian (Lorentzian) manifold Y of dimension $d+1$ greater or equal to 2

$$S_{YM}(A;\gamma) = -\frac{1}{4g^2}\int_Y \mathrm{tr}\,(F \wedge *F)\,, \tag{56}$$

where $*$ denotes the (metric dependent) Hodge star operator. The action S_{YM} is invariant under $\mathcal{G}(Y)$. In two dimensions the symmetry group is significantly larger as S_{YM} is also invariant under the group of area preserving diffeomorphisms, $Diff_\omega(Y)$,

$$S_{YM}(f^*A;\gamma) = S_{YM}(A;\gamma), \quad \forall f \;:\; f^*\omega_\gamma = \omega_\gamma,$$

where ω_γ denotes the area 2-form. In [Th3, As2] the Yang-Mills PI measure has been constructed in the above formalism (using Wilson random variables) without the need to choose a gauge. It has been shown on a cylinder, $Y = [0,t]\times S^1$, that the analogue of the Osterwalder-Schräder reconstruction theorem is valid and leads to the same Hilbert space that is obtained by reducing first and quantizing after: the space of square integrable class functions on the gauge group (corresponding to holonomy around the homotopically nontrivial loop).

4.2 Quantum Chern-Simons theories. Invariants of 3-manifolds, knots and links

Let us go back to the problem of finding an intrinsic $3d$ definition of the Jones polynomial invariant of knots [Jo]. In the present section I follow mainly [Koh, Mari2, Mari3].

Let Y be a closed $3d$-manifold. The Wilson variables (51) define functions on the product

$$\begin{array}{llllll} W \;:\; \mathcal{L}i(Y) & \times & \mathcal{A}^{SU(n)}(Y)/\mathcal{G} & \longrightarrow & \mathbb{C} & \\ \quad\;\;(\;\alpha & , & [A]\;) & \longmapsto & W(\alpha,A) = \prod_j \mathrm{tr}\, U_{\alpha_j}(A), & (57) \\ & & & & \text{where}\;\; \alpha = (\alpha_1,\ldots,\alpha_m), & \end{array}$$

and $\mathcal{L}i(Y)$ denotes the space of links in Y. It is sometimes convenient to think of W in (57) as the integral kernel of a sort of a generalized Fourier transform from connections to loops (links). This is precisely the sense in which the loop representation was introduced in nonperturbative quantum gravity [RS, Rov]. The mathematical basis for this analogy comes from the fact that the analogue of the Bochner theorem is valid. Namely, there is a bijective correspondence between regular Borel measures on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ and functions on the space singular links (i.e. links with arbitrary self intersections) satisfying an additional property of positivity [AI, AL1].

To get functions of links we can start by choosing particular connections $A \in \mathcal{A}^{SU(n)}(Y)$ and consider the functions

$$W(\cdot, A) \;:\; \mathcal{L}i(Y) \;\longrightarrow\; \mathbb{C}. \tag{58}$$

The resulting function on $\mathcal{L}i(Y)$ is not $Diff(Y)$-invariant unless the connection A is flat, i.e. it satisfies the classical equations of motion (20). Unfortunately, for a flat connection A, the function (58) is not only $Diff(Y)$-invariant but also homotopy invariant and therefore of no use for knot theory.

The second thing we could try in order to construct knot (link) invariants would be to do a weighted sum over the second argument of (57) in a $Diff(Y)$-invariant way. This is how $Diff(Y)$-invariant QFT, or TQFT of connections enters the topological problem of constructing knot invariants. The Ashtekar-Lewandowski measure that we mentioned in section 4.1 is $Diff(Y)$-invariant but it does not lead to nice invariants. The Yang-Mills action (56) is not $Diff(Y)$ invariant so that it does not lead to knot invariants either. But, as we mentioned in section 3.1, there is in $3d$ a natural function on $\mathcal{A}^{SU(n)}(Y)$ that is $Diff(Y)$-invariant: the Chern-Simons action S_{CS} (19), with classical equations of motion (20). We therefore want to give a meaning to

$$\begin{aligned} < W_\alpha >_{Y,R,k} &= \sum_{A \in \mathcal{A}^{SU(n)}(Y)} W_R(\alpha, A)\; e^{ikS_{CS}(A)} \equiv \\ &\equiv \frac{1}{Z^{(k)}(Y)} \int_{\mathcal{A}^{SU(n)}(Y)} W_R(\alpha, A)\; e^{ikS_{CS}(A)} DA, \end{aligned} \tag{59}$$

where α is a simple loop, the subscript R means that the trace is taken with respect to the irreducible representation R of the gauge group, $W_R(\alpha, A) = \mathrm{tr}_R(U_\alpha(A))$, and $Z^{(k)}$ denotes the partition function of Chern-Simons theory at level k (see (33))

$$Z^{(k)}(Y) = \int_{\mathcal{A}^{SU(n)}(Y)} e^{ikS_{CS}(A)} DA\;. \tag{60}$$

The number k in the exponent has to be integer for the weight in (59) and (60) to be $\mathcal{G}(Y)$-invariant. It is called the level of the theory and plays the role of $1/\hbar$ (in particular, the large k limit corresponds to the semiclassical limit). So, if we manage to define a measure $D[A]$ on $\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)$, which preserves the $Diff(Y)$-invariance then the function of loops $W_R^{(k)}(\alpha; Y) = < W_\alpha >_{Y,R,k}$

in (59) is a knot invariant. To obtain an invariant of links we have to take several nonintersecting linked loops $\alpha_1, \ldots, \alpha_m$ in Y and to substitute (59) by

$$< W_\alpha >_{Y,\vec{R},k} = \frac{1}{Z^{(k)}(Y)} \int_{\mathcal{A}^{SU(n)}(Y)} W_{R_1}(\alpha_1, A) \cdots W_{R_m}(\alpha_m, A)\, e^{ikS_{CS}(A)}\, DA, \tag{61}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\vec{R} = (R_1, \cdots, R_m)$. As we mentioned above, the functions on the l.h.s. of (59) and (61) correspond to a (partial, nonlinear) Fourier transform of the measure $DA\, e^{ikS_{CS}}/Z^{(k)}$. So, why not to try to make sense of "$DA\, e^{ikS_{CS}}$" as a (complex) measure on $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$?
First of all, it does not seem to be possible to approximate $e^{ikS_{CS}}$ by Wilson functions so that the space $\overline{\mathcal{A}^{SU(n)}(Y)/\mathcal{G}(Y)}$ may not be "enough". On the other hand, it turns out that the integrals (59) have a framing dependence which means that the map

$$\alpha \longmapsto W_R^{(k)}(\alpha; Y) = < W_\alpha >_{Y,R,k}$$

cannot define a linear functional on the holonomy algebra $\mathcal{B}$ (52). By formal perturbative PI methods one shows that the framing dependence of $W_R^{(k)}(\alpha; Y)$ (which is a soft $Diff(Y)$ breaking) is such that the ratio (by choosing appropriately the framings)

$$NW_R^{(k)}(\alpha; Y) = \frac{W_R^{(k)}(\alpha; Y)}{W_R^{(k)}(U; Y)},$$

where U is the unknot and α is a knot, is indeed $Diff(Y)$-invariant. Again, by formal perturbative PI manipulations one shows that for $G = SU(2)$ and R equal to the fundamental representation, $R = \Box$, this invariant coincides with the Jones polynomial evaluated at the root of unity, $t = e^{\frac{2\pi i}{k+2}}$,

$$NW_\Box^{(k)}(\alpha; S^3) = V(\alpha)(e^{\frac{2\pi i}{k+2}}).$$

This is done by showing that $NW_\Box^{(k)}(\alpha; S^3)$ satisfies the skein relations (49).

Let us see, in some simpler examples, how Witten [Wi5] uses the fact that Chern-Simons theory is a $3d$ TQFT (see section 3.2.3) to obtain exact non-perturbative (to all orders in $1/k$ (!)) results for some of the above quantities.

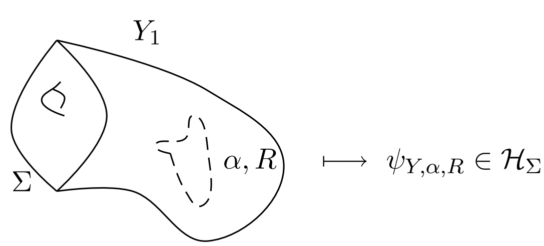


Figure 4.3

As in (43), to the manifold Y_1, in figure 4.3, with observable $W_R(\alpha, \cdot)$ inside, there corresponds the state,

$$\psi_{Y,\alpha,R} \in \mathcal{H}_\Sigma. \tag{62}$$

If we perform an hamiltonian analysis of S_{CS} on $\Sigma \times \mathbb{R}$ we obtain that the physical phase space is

$$\mathcal{A}_{Firr}^{SU(n)}(\Sigma)/\mathcal{G}(\Sigma) = \mathcal{A}_{irr}^{SU(n)}(\Sigma)//\mathcal{G}(\Sigma) \cong \mathcal{M}_n(X),$$

where ($\mathcal{A}_{Firr}^{SU(n)}$) $\mathcal{A}_{irr}^{SU(n)}$ denotes the space of (flat) irreducible connections, $X = (\Sigma, j)$ denotes Σ with a choice of complex structure j and $\mathcal{M}_n(X)$ denotes the moduli space of rank n stable holomorphic vector bundles with trivial determinant on X. The space $\mathcal{M}_n(X)$ carries a natural symplectic structure ω_0. Its cohomology class, $[\omega_0]$, on the so called Deligne-Mumford compactification $\overline{\mathcal{M}_n(X)}$ of $\mathcal{M}_n(X)$ coincides with the first Chern class of the generator, Θ, of the space of isomorphism classes of (holomorphic) line bundles on $\overline{\mathcal{M}_n(X)}$, $Pic(\overline{\mathcal{M}_n(X)}) = \Theta^{\mathbb{Z}}$. The Hilbert spaces $\mathcal{H}_\Sigma^{(k)}$ then coincide with the spaces of level k generalized theta functions,

$$\mathcal{H}_\Sigma^{(k)} = H^0(\overline{\mathcal{M}_n(X)}, \Theta^k),$$

and can be obtained by the method of geometric quantization (the dependence on j requires extra care but will be unimportant in our discussion) [AdPW]. These are finite dimensional vector spaces with dimension obtained by Verlinde with the help of CFT techniques [Ver]. In particular, $\mathcal{H}_{S^2}^{(k)} = \mathbb{C}$ and $\mathcal{H}_{T^2}^{(k)}$ has a basis indexed by integrable representations, $\widehat{\Lambda}_k = \{\widehat{R}\}$, of the Kac-Moody algebra $\widehat{su(n)}$ at level k, which in turn correspond to a (finite) subset, $\Lambda_k = \{R\}$, of the irreducible representations of $SU(n)$,

$$\mathcal{H}_{T^2}^{(k)} = \mathrm{span}_{\mathbb{C}} \{|R> \ , \ R \in \Lambda_k\}.$$

The states $|R>$ can be chosen to be orthonormal [EMSS, LR]

$$< R|S >= \delta_{RS}.$$

More explicitly, let $\{\lambda_1, \cdots, \lambda_{n-1}\}$ denote fundamental weights of $SU(n)$ and ρ denote the Weyl vector $\rho = \sum_{i=1}^{n-1} \lambda_i$. The subset Λ_k consists of the representations R_λ with highest weight, $\lambda = \sum_{i=1}^{n-1} m_i\lambda_i$, satisfying the inequalities $0 \leq m_i \leq k$.

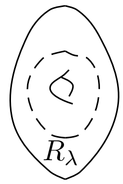


Figure 4.4

From the PI point of view the state $|R_\lambda >\in \mathcal{H}^{(k)}_{T^2}$ corresponds naturally to the state (62) with Y a solid torus, α the loop in figure 4.4 along the noncontractible cycle which generates $\pi_1(Y)$ and $R = R_\lambda$. It is known from CFT that the space $\mathcal{H}^{(k)}_{T^2}$ carries an irreducible representation of $\Gamma(T^2) \equiv Diff(T^2)/Diff_0(T^2) \cong SL(2,\mathbb{Z})$. Furthermore, this representation coincides with the representation defined by Chern-Simons TQFT at level k through (38) [Koh].

Let us use this fact to find the exact (nonperturbative!) expression for the partition function of S^3 and invariants for the Hopf link in S^3. The kernel of the homomorphism corresponding to the induced action of $Diff(T^2)$ on $H_1(T^2,\mathbb{Z}) \cong \mathbb{Z}^2$ is $Diff_0(T^2)$ so that the homotopy type of a diffeomorphism $f \in Diff(T^2)$ is fully characterized by the way f_* acts on one-cycles. The homomorphism $f \mapsto f_*$ is surjective and, since $SL(2,\mathbb{Z})$ is generated by the two elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the representation σ_k of $SL(2,\mathbb{Z})$ on $\mathcal{H}^{(k)}_{T^2}$ is given by defining $T^{(k)} = \sigma_k(T)$, $S^{(k)} = \sigma_k(S)$. We have (see [Koh, Mari2, Mari3])

$$\begin{aligned} < R_\lambda|U_T|R_{\lambda'} > &= T^{(k)}_{\lambda\lambda'} = \delta_{\lambda\lambda'} e^{2\pi i(h_\lambda - c/24)} \\ < R_\lambda|U_S|R_{\lambda'} > &= S^{(k)}_{\lambda\lambda'} = \\ &= \frac{(i)^{n(n-1)/2}}{(k+n)^{\frac{n-1}{2}}} n^{1/2} \sum_{w\in W} \epsilon(w)\; \exp(-\frac{2\pi i}{k+n}(\lambda+\rho)\cdot w(\lambda'+\rho)), \end{aligned}$$

where W is the Weyl group of $SU(n)$, $\epsilon(w)$ denotes the parity of $w \in W$, $h_\lambda = \frac{\lambda\cdot(\lambda+\rho)}{2(k+n)}$ coincides with the conformal weight of the primary field associated with R_λ and c is the Virasoro central charge of the corresponding WZW model. If we now glue two solid torii $Y_a = D_a \times S^1_a$, where $a = 1, 2$, and D_a denote disks, along their boundaries, $\partial Y_a = \widetilde{S}^1_a \times S^1_a$, by gluing S^1_1 with S^1_2 and $\widetilde{S}^1_1$ with $\widetilde{S}^1_2$ we obtain $Y = S^2 \times S^1$. Then, in particular, $Z(S^2 \times S^1) =< R_0|R_0 >= 1$, which is consistent with $\dim(\mathcal{H}^{(k)}_{S^2}) = 1$. Indeed, we have

$$\dim(\mathcal{H}^{(k)}_{S^2}) = \mathrm{tr}\; I_{\mathcal{H}^{(k)}_{S^2}} = \mathrm{tr}\; \widehat{K}_{S^2\times[0,t]} = Z(S^2 \times S^1) = 1.$$

The inner product $< R_\lambda | R_{\lambda'} >$ corresponds to having $S^2 \times S^1$ with two nonlinked, nonintersecting loops, β and β', going once around S^1 with representations λ and λ'. Indeed, if we cut $S^2 \times S^1$ along a torus in such a way that each part is a solid torus with one of the loops as in figure 4.4, we see that

$$W^{(k)}_{R_\lambda R_{\lambda'}}(\beta, \beta'; S^2 \times S^1) =< R_\lambda | R_{\lambda'} >= \delta_{\lambda\lambda'}.$$

If we now glue back differently, namely by identifying S^1_1 with $\widetilde{S}^1_2$ and $\widetilde{S}^1_1$ with S^1_2, then we obtain the three sphere,

$$Y_S = (S^2 \times S^1)_S = S^3. \tag{63}$$

with the two loops now linked with linking number 1. The resulting link is called the Hopf link. To see this, notice that a solid torus in S^3 has also a solid torus as complement but a generator of the fundamental group of each of these solid torii becomes trivial in the other. The subscript in (63) indicates that we glued by using a diffeomorphism of T^2 which exchanges the cycles and therefore corresponds to S. Then, the value of the Chern-Simons invariant for the three sphere corresponds to choosing $\lambda = \lambda' = 0$ and is given by

$$\begin{aligned} Z(S^3) &= Z((S^2 \times S^1)_S) =< R_0 | U_S | R_0 >= S^{(k)}_{00} = \\ &= \frac{1}{(k+n)^{\frac{n-1}{2}}} \; n^{1/2} \; \prod_{\alpha > 0} 2 \sin\left(\frac{\pi(\alpha \cdot \rho)}{k+n}\right), \end{aligned}$$

where the product is over the positive roots of $SU(n)$. The invariant for the Hopf link with labelings $\lambda,\ \lambda'$ becomes

$$W^{(k)}_{R_\lambda R_{\lambda'}}(\text{Hopf}; S^3) =< R_\lambda | U_S | R_{\lambda'} >= S^{(k)}_{\lambda\lambda'}.$$

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