Symplectic Geometry and Geometric Quantization[1]

Matthias Blau[2]

NIKHEF-H
P.O. Box 41882, 1009 DB Amsterdam
The Netherlands

[1]This is a preliminary version not intended for distribution outside this School. Thank You.

[2]e-mail: t75@nikhefh.nikhef.nl, 22747::t75

Contents

1 Introduction **2**

2 Symplectic Geometry and Classical Mechanics **6**

2.1 Symplectic Geometry 7

2.2 Relation with Classical Mechanics 10

3 What is Quantization? **12**

4 Geometric Quantization I: Prequantization **18**

4.1 The Prequantum Hilbert Space 18

4.2 Examples 24

5 Geometric Quantization II: Polarizations **26**

5.1 Polarizations 27

5.2 Real Polarizations 29

5.3 Kähler Polarizations 30

6 Geometric Quantization III: Quantization **33**

6.1 Polarized States and the Construction of Quantum Operators . . 34

6.2 The Vertical Polarization and Half-Densities 37

6.3 Kähler Quantization and Metaplectic Correction 40

6.4 Real Polarizations and Bohr-Sommerfeld Varieties 44

References **46**

1 Introduction

At the macroscopic level our world seems to be pretty much governed by the laws of classical physics, i.e. by Newtonian mechanics on the one hand and by Maxwell theory on the other. The former describes the motion of particles under the influence of forces acting on them and applies to such diverse fields as celestial mechanics and elasticity theory. The latter covers almost the entire spectrum of phenomena occurring in electromagnetism and optics. The dynamics in these two theories is governed by deterministic equations of motion and (in principle) it is possible, given the initial conditions, to predict the results of measurements on the system at any later time.

Classical physics could thus provide us with a very satisfactory description of the world we live in, were it not for the fact that it not only fails to give an explanation for a number of phenomena observed at the microscopic level but is, in fact, in plain contradiction with experimental evidence.

As an illustration of what these phenomena are and what they may be trying to tell us about a theory which will have to replace classical physics, I want to mention just two examples. The first has to do with the stability of atoms. From scattering and other experiments it had been deduced that atoms consist of a tiny positively charged nucleus orbited at some distance by negatively charged point-like particles called electrons. Within the realms of classical physics such structures are highly unstable and would be predicted to collapse within fractions of a second, in glaring contradiction with the relative stability of the world around us: orbital motion is accelerated motion, and according to Maxwell theory accelerated charges emit radiation; the electron would thus radiate away energy and spiral into the nucleus of the atom. It was also observed that simple atoms (like Hydrogen atoms) were able to emit and absorb energy only in certain discrete quantities. Combining these two observations it thus appeared to be necessary to postulate the existence of stable orbits for electrons at certain discrete radii (energy levels). This suggests that at the microscopic level nature allows for a discrete (or *quantized*) structure quite unfamiliar from classical physics.

The second is the famous two-slit (gedanken)experiment or other experiments investigating the diffraction and interference patterns of beams of particles like electrons. These indicated that under certain circumstance particles (like electrons) can show interference patterns (and thus a wave-like nature) and that under certain conditions light (whose wave-like nature had finally been universally accepted) showed behaviour characteristic of particles and not of waves. In short, at the microscopic level nature was found to be mind-bogglingly strange and wonderful. Or, in the words of Dirac (quoted from [1, p. 3], *the* authoritative book on quantum mechanics):

> We have here a very striking and general example of the breakdown of classical mechanics - not merely an inaccurracy of its laws of motion, but *an inadequacy of its concepts to supply us with a description of atomic events.*

To make things worse, the outcome of these experiments appeared to depend on the measuring process itself, i.e. on whether or not one was checking through which slit the electron went.[1] This, for the first time, pointed to the necessity of including the effects of observation of a system into a description of the system itself. In particular, it had to be taken into account that one can not make any observations on a suitably 'small' system without perturbing the system itself (technically this is expressed in the uncertainty principle). In the words of Dirac [1, p. 4]:

> A consequence of the preceding discussion is that we must revise our ideas of causality. Causality applies only to a system which is left undisturbed. If a system is small, we cannot observe it without producing a serious disturbance and hence we cannot expect to find any causal connexion between the results of our observations. Causality will still be assumed to apply to undisturbed systems and the equations which will be set up to describe an undisturbed system will be differential equations expressing a causal connexion between conditions at one time and conditions at a later time. These equations will be in close correspondence with the equations of classical mechanics, but they will be connected only indirectly with the results of observations. There is an unavoidable indeterminacy in the calculation of observable results, the theory enabling us to calculate in general only the probability of our obtaining a particular result when making an observation.

Heisenberg and Schrödinger provided two mathematical models (or recipes), later shown to be equivalent, which were able to reproduce the above results and make many other successfully tested predictions. These models, collectively known as *quantum mechanics*, describe the quantum behaviour of (point) particles in flat space under the influence of external forces. Supplemented by some intepretation of, roughly speaking, the role of the measuring process or observer, they constitute a major step forward in the understanding of quantum physics in general.

At a conceptual level, however, the situation was not very satisfactory. In particular, it was not clear how general the proposed models were, which features were to be regarded as fundamental to any quantum version of a classical theory and which were to be attributed to particular properties of the systems considered so far.

In an attempt to gain some insight into this question, it was in particular Dirac who emphasized the formal similarities between classical and quantum mechanics and the necessity of properly understanding these. Again in the words of the master himself [1, p. 84]:

[1] Recent experiments indicate that interference patterns will disappear whenever there is *in principle* the possibility of detecting through which slit the particle (photon, electron) went, regardless of whether there was actually a detector there (switched on) or not. This highly counter-intuitive result appears to be in agreement with theory.

> The value of classical analogy in the development of quantum mechanics depends on the fact that classical mechanics provides a valid description of dynamical systems under certain conditions, when the particles and bodies composing the system are sufficiently massive for the disturbance accompanying an observation to be negligible. Classical mechanics must therefore be a limiting case of quantum mechanics. We should thus expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics, and, from an understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well-known results in classical mechanics; ...

Abstracting from the analogy found between classical mechanics and Schrödinger and Heisenberg quantum mechanics, Dirac formulated a general *quantum condition*, a guideline for passing from a given classical system to the corresponding quantum theory. This process in general is known as *quantization*. And, roughly speaking, quantization consists in replacing the classical algebra of observables (functions on phase space) by an algebra of operators acting on some Hilbert space, the quantum condition relating the commutator of two operators to the Poisson bracket of their classical counterparts. I will explain these rules of *canonical quantization* in section 3, after having introduced the 'important concepts in classical mechanics' referred to in the above quote. Readers familiar with the mathematical description of classical mechanics may wish to move right on to that section and read it as part of the introduction.

Parenthetically, I want to issue a word of caution at this point. At first sight (and perhaps even at second sight), the very concept of quantization appears to be ill-founded since it attempts to construct a 'correct' theory from a theory which is only approximately correct. After all, our world is quantum, and while it seems a legitimate task to try to extract classical mechanics in some limit from quantum mechanics, their seems to be little reason to believe that the inverse construction can always be performed. Furthermore, there is no reason to believe that such a construction would be unique as there could well be (and, in fact, are) lots of different quantum theories which have the same classical limit. Unfortunately, however, it is conceptually very difficult to describe a quantum theory from scratch, without the help of a reference classical theory. Moreover, there is enough to the analogy between classical and quantum mechanics to make quantization a worthwile approach. Perhaps, ultimately, the study of quantization will tell us enough about quantum theory itself to allow us to do away with the very concept of quantization.

But let us now return to less philosophical matters. Unfortunately, Dirac's quantum condition is not as general as one might have hoped it to be or, at least, not sufficiently unambiguous. Thus, to make some headway, it is desirable to find a more intrinsic and constructive description of this quantization procedure. This is the aim of *Geometric Quantization* which, as the name suggests, attempts to provide a geometric interpretation of quantization within an extension of the mathematical framework of classical mechanics (symplectic geometry). More specifically, geometric quantization (abbreviated to GQ henceforth) refers to a body of ideas pioneered independently by Souriau [2] and Kostant [3] in the late 60's and early 70's.

It is the purpose of these lectures to provide an introduction to the role of symplectic geometry in quantization in general and, as a concrete realization of this general picture, to give an introductory account of GQ. Unfortunately, although the fundamental ideas in GQ are very elegant and simple, things tend to become more complicated and mathematically more demanding rather quickly. In these notes I have therefore tried to emphasize primarily these basic ideas and to hint at or illustrate the more advanced machinery (like half-density and half-form quantization, BKS kernels, Bohr - Sommerfeld varieties and distributional wave functions) in terms of simple examples rather than develop it in any great detail. The reason for proceeding in this way is partly lack of space and time and partly that I do not want to advocate GQ as an efficient calculational tool in quantum mechanics anyway (in fact, at present at least it is far from being that), but rather as a procedure of more conceptual interest.

Section 2.1 is a crash-course on the formalism of symplectic geometry. It serves mainly to introduce the nomenclature and (differential form) notation to be used throughout. Section 2.2 explains why this is a natural framework for classical mechanics.

Section 3, in a sense the heart of these notes, is an introduction to the question 'What is quantization?'. It also deals with the no less important questions 'What is it not?' and 'What should or can it not expected to be?'. My own views on this subject have been heavily influenced by the article [4] of Isham which I warmly recommend to anyone interested in these questions.

Section 4 deals with the first step of GQ, known as prequantization. This is an elegant procedure which associates to any 'quantizable' symplectic manifold a Hilbert space carrying a faithful representation of the classical observable algebra. Section 4.1 describes the construction and contains a discussion of the conditions for existence of a prequantization, and their topological classification. Certain simple and prototypical examples are discussed in section 4.2.

Unfortunately, the Hilbert space provided by prequantization, as elegant as the procedure may be, is not the correct one for quantum mechanics, and one needs some way of 'cutting it in half'. This is achieved via the introduction of a polarization of the phase space. Splitting the quantization process into these two steps is nevertheless useful because it serves to isolate some of the ambiguities inherent in passing from a classical to a quantum system. The ambiguity in the choice of a prequantum Hilbert space corresponds, roughly speaking, to the presence of topological superselection rules, while the choice of polarization corresponds to choosing a particular representation within a given topological sector. I hope that the meaning of this sentence will become clearer in the following sections.

In section 5.1 I try to explain why, in the framework of GQ, polarizations arise naturally at this point. Section 5.2 deals with real polarizations, in particular with the vertical polarization of a cotangent bundle. Section 5.3 describes the second important class of polarizations, namely Kähler polarizations. As background material, it also contains a synopsis of facts about complex and Kähler vector spaces and manifolds.

Section 6, finally, deals with the construction of the quantum Hilbert space associated to a polarization and with the construction of operators acting on it. It is at this crucial point that GQ becomes somewhat murky. In section 6.1 I show how polarization preserving observables give rise directly to operators at the quantum level and how the scheme needs to be modified to quantize other observables (via BKS kernels). We then turn to the construction of the quantum Hilbert space. I will describe three prototypical situations (vertical polarizations positive Kähler polarizations and real polarizations with non-simply conected leaves), point out the complications arising in each of these, and explain briefly how these problems can be overcome.

Here these notes end rather abruptly, but I hope that they will have prepared the ground for further inquiries into the existing literature. In particular, I want to draw attention to the recent investigations into the polarization dependence of GQ in [5], motivated by questions arising in topological and conformal field theory, and to the application of GQ to the quantization of constrained systems [6]-[8] (which is a rather natural thing to attempt as constrained systems are also described most effectively in terms of symplectic geometry).

The most glaring omission of these notes is probably the representation theoretic aspect of GQ, i.e. the quantization of coadjoint orbits of a Lie group, which I only touch upon briefly in section 3 and when discussing the quantization of the two-sphere in sections 4 and 6. This relation between GQ and the representation theory of Lie groups is important both mathematically and historically. See [9] for that part of the story.

The basic references for section 2 are Abraham and Marsden [10] and Arnol'd [11]. A wealth of other information on symplectic geometry can be found in the book [12] by Guillemin and Sternberg. My favourite references for section 3 are the book [1] by Dirac and the lectures [4] by Isham. Most of what I will say about GQ (and much more) can be found in the book by Woodhouse [13]. Its imprints on these notes are rather obvious in sections 5 and 6. Other monographs on GQ include [14] and [15].

2 Symplectic Geometry and Classical Mechanics

Symplectic Geometry is the adequate mathematical framework for describing the Hamiltonian version of classical mechanics. As such it is also the most suitable starting point for a geometrization of the canonical quantization procedure.

The purpose of section 2.1 is to introduce the formalism of symplectic geometry and the coordinate independent differential form notation we will use throughout these lectures. Section 2.2 serves to establish the relation of this formalism with that of classical Hamiltonian mechanics.

2.1 Symplectic Geometry

By a symplectic manifold (M, ω) we will mean a smooth real m-dimensional manifold M without boundary, equipped with a closed non-degenerate two-form ω, the symplectic form.[2] 'Closed' means that

$$d\omega = 0 \tag{2.1}$$

($\Leftrightarrow \partial_{[i}\omega_{jk]} = 0$ in local coordinates), where d is the exterior differential

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \quad , \qquad d^2 = 0 \quad , \tag{2.2}$$

on differential forms on M. And 'non-degenerate' means that at each point $x \in M$ the antisymmetric matrix ω_x is non-degenerate, i.e.

$$\det(\omega_x) \neq 0 \qquad \forall x \in M \quad . \tag{2.3}$$

The most important example of a symplectic manifold is a cotangent bundle $M = T^*Q$. This is nothing but the traditional phase space of classical mechanics, Q being known as the configuration space in that context. A cotangent bundle has a canonical symplectic two-form which is globally exact,

$$\omega = d\theta \tag{2.4}$$

(and hence, in particular, closed). Any local coordinate system $\{q^k\}$ on Q can be extended to a coordinate system $\{q^k, p_k\}$ on T^*Q such that θ and ω are locally given by

$$\theta = p_k dq^k \quad , \qquad \omega = dp_k \wedge dq^k \quad . \tag{2.5}$$

We will return to the specific case of cotangent bundles at the end of this section, when we discuss the relation with classical mechanics.

Other examples of symplectic manifolds are orientable two-dimensional surfaces Σ: choose any volume form ω on Σ; as such it is certainly non-degenerate; as a two-form on a two-dimensional manifold it is also certainly closed ($d\omega$ is a three-form and and there are no anti-symmetric three-tensors in two dimensions); and, although it is a fact (known as Darboux's theorem) that on any symplectic manifold one can choose a local coordinate system such that ω takes the form (2.5), ω cannot be globally exact in this case because otherwise the volume of Σ would be zero by Stoke's theorem: If $\omega = d\theta$ were true, then

$$\mathrm{Vol}_\omega(\Sigma) := \int_\Sigma \omega = \int_\Sigma d\theta = \int_{\partial\Sigma} \theta = 0$$

because $\partial\Sigma = \emptyset$. More generally, all Kähler manifolds are symplectic, and we will come back to them in section 5.3.

[2] Like a Riemannian manifold (M, g) is a manifold equipped with a non-degenerate symmetric two-tensor g, the Riemannian metric.

Condition (2.3) has several important consequences. First of all, it implies that M is even-dimensional, $m = 2n$, as an odd-dimensional antisymmetric matrix has zero determinant. The same argument as above shows that the symplectic form of any compact symplectic manifold is cohomologically non-trivial (i.e. closed but not globally exact), because otherwise the symplectic volume

$$Vol_\omega(M) := \tfrac{1}{n!}\int_M \omega^n \tag{2.6}$$

would be zero.

Moreover, as ω is invertible, at each point $x \in M$ it gives an isomorphism between the tangent and cotangent spaces of M at x,

$$\omega_x : T_xM \overset{\text{iso}}{\sim} T^*_xM \quad , \tag{2.7}$$

expressed in local coordinates as

$$X^i \mapsto X^i\omega_{ik} \quad . \tag{2.8}$$

Crudely speaking, like a metric a symplectic form allows us to raise and lower indices on tensors. This extends to an isomorphism between TM and T^*M and between vector fields and one-forms on M,

$$X \mapsto i(X)\omega = \omega(X, .\,) \in \Omega^1(M) \tag{2.9}$$

(here $i(X)$ denotes the contraction of a differential form with the vector field X, as in (2.8), i.e. the insertion of X into the first 'slot' of a differential form).

In particular, therefore, the existence of ω allows us to associate a vector field X_f to every function $f \in C^\infty(M)$ via

$$i(X_f)\omega = -df \tag{2.10}$$

(the minus sign is for later convenience only). X_f, the 'symplectic gradient' of f, is known as the *Hamiltonian vector field* of f. It generates a flow on M which leaves ω invariant, as the Lie derivative of ω along X_f is zero,

$$L(X_f)\omega \equiv di(X_f)\omega + i(X_f)d\omega = -ddf = 0 \quad . \tag{2.11}$$

Via (2.10), the symplectic form provides an anti-symmetric pairing $\{f, g\}$ between functions f, g on M called the *Poisson bracket* of f and g. It is defined by

$$\{f, g\} := \omega(X_f, X_g) \in C^\infty(M) \quad , \tag{2.12}$$

and describes the change of g along X_f (or vice versa),

$$\{f, g\} = i(X_g)i(X_f)\omega = i(X_f)dg = L(X_f)g \quad . \tag{2.13}$$

In particular, f is constant (i.e. preserved) along the integral curves of X_f. The Poisson bracket satisfies the Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\} \quad . \tag{2.14}$$

This can be shown either by writing out explicitly $(d\omega)(X_f, X_g, X_h) = 0$ (ω is closed), or by using the tensoriality of the Lie derivative and $L(X_f)\omega = 0$ to deduce

$$L(X_f)\,(\omega(X_g, X_h)) = \omega(L(X_f)X_g, X_h) + \omega(X_g, L(X_f)X_h)$$

which is just a rewriting of (2.14). This gives $(C^\infty(M), \{.,.\})$ the structure of an infinite dimensional Lie algebra.

One further important identity we will need, which relates the Lie algebras of vector fields and functions on M, is

$$[X_f, X_g] = X_{\{f,g\}} \quad , \tag{2.15}$$

which shows that the Hamiltonian vector fields also form an infinite dimensional Lie algebra. Moreover, regarding the map (Lie algebra homomorphism) $f \to X_f$ as an assignment of differential operators to functions, the identity (2.15) is also an illustration of the quantization paradigm (Dirac's quantum condition)

$$\textit{Poisson Brackets} \rightarrow \textit{Commutators}$$

and will play an important role in the following. To prove (2.15) one again makes use of the tensoriality of the Lie derivative, this time in the form

$$i([X, Y]) = L(X)i(Y) - i(Y)L(X) \quad ,$$

to show that $i([X_f, X_g])\omega = i(X_{\{f,g\}})\omega$.

Lastly, we will need to consider certain submanifolds of symplectic manifolds. A subspace $(V, \omega|_V)$ of a symplectic vector space (W, ω) (i.e. a vector space W equipped with a non-degenerate antisymmetric two-tensor ω) is called *isotropic* if $\omega|_V = 0$. By linear algebra, an isotropic subspace of W has dimension at most $\frac{1}{2}\dim(W)$, and in that case V is called a *Lagrangian subspace*[3] of W. Likewise, we now define a Lagrangian submanifold of (M, ω) to be an n-dimensional submanifold $N \subset M$ such that $\omega|_{TN} = 0$. For example, it is evident from (2.5) that Q (defined by $p_k = 0$) is a Lagrangian submanifold of $M = T^*Q$, as is the fibre T^*_qQ of the cotangent bundle at $q \in Q$. Locally, any Lagrangian submanifold N is given by the vanishing of n functions F_k on M which are *in involution*, i.e. which satisfy

$$\{F_k, F_l\} = 0 \qquad \forall k, l \quad . \tag{2.16}$$

In fact, it follows from this condition that the Hamiltonian vector fields X_{F_k} are tangent to $\bigcap_l \{F_l = 0\}$ so that they locally span the tangent bundle TN. Reading (2.16) as $\omega(X_{F_k}, X_{F_l}) = 0$ then says that $\omega|_{TN} = 0$.

This concludes our crash-course on symplectic geometry. The second half of this century has witnessed a great deal of activity in this field, which has established itself as an independent mathematical descipline fertilized by the relation with classical mechanics. I have not mentioned any results of modern symplectic geometry which can be obtained within this framework and the adventurous reader is referred to [10] for a detailed account.

[3] The terminology arises from the relation between such subspaces and the Hamilton-Jacobi theory of Lagrangian mechanics, see [10, 13].

2.2 Relation with Classical Mechanics

Now, what has all this got to do with classical mechanics? In the simplest mechanical systems the areana for classical mechanics in the Hamiltonian (or first order) formalism is the *phase space*, a $2n$-dimensional real vector space $\sim \mathbf{R}^{2n}$ with coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ describing the position and the momentum (velocity) of the particles involved. The dynamics (time evolution) of the system is governed by *Hamilton's equations*

$$
\begin{aligned}
\tfrac{d}{dt} q^k &= \frac{\partial H}{\partial p_k} \\
\tfrac{d}{dt} p_k &= -\frac{\partial H}{\partial q^k} \ ,
\end{aligned}
\tag{2.17}
$$

where $H(q^k, p_k)$, the *Hamiltonian*, is a function on phase space describing the energy of the system.

Typically, H is of the form $H = T + V$ where $T \sim p^2$ is the kinetic energy and $V = V(q^k)$ is the potential energy whose gradient descibes the forces acting on the particles. For example, a harmonic oscillator in one dimension is described by the Hamiltonian $H = (p^2 + q^2)/2$, the equations of motion $\dot{q} = p, \;\; \dot{p} = -q$ leading to the characteristic oscillating behaviour $q(t) = q(0)\cos t + p(0)\sin t$.

If H does not depend on time explicitly, the equations of motion (2.17) imply that H is conserved along any trajectory in phase space,

$$
\begin{aligned}
\tfrac{d}{dt} H &= \frac{\partial H}{\partial q^k}\dot{q}^k + \frac{\partial H}{\partial p_k}\dot{p}_k \\
&= \frac{\partial H}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q^k}\frac{\partial H}{\partial p_k} = 0 \ ,
\end{aligned}
\tag{2.18}
$$

(summation over repeated indices being understood) while the evolution of any other function f on phase space (*observable*) is given by

$$
\tfrac{d}{dt} f = \frac{\partial f}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q^k} \ . \tag{2.19}
$$

In our simple one-dimensional example above, (2.18) already determines the phase space trajectories uniquely to be the circles $p^2 + q^2 = const.$, in agreement with the explicit solution of the equations of motion.

In general, any constant of motion, i.e. any function f on phase space in involution with the Hamiltonian, $\{H, f\} = 0$, can be used to reduce the dynamical system to a lower dimensional one on the common level surfaces of the functions H and f. It follows from the Jacobi identity (2.14) that the Poisson bracket of any two constants of motion is also a constant of motion. If it is possible to find n constants of motion in involution (and independent in the sense that their Hamiltonian vector fields are linearly independent) the system is called *integrable* and there are then standard methods available for solving the system completely (Hamilton-Jacobi theory, action-angle variables, ...). Most text-book examples of classical mechanics are integrable, but integrability is by no means prototypical and only occurs in systems with a high degree of symmetry. In the general case one has to resort to more qualitative (instead of quantitative) methods of investigation. For a detailed exposition with numerous applications (e.g. to the rigid body and celestial mechanics) see [10].

The equations (2.17-2.19), characterising Hamiltonian mechanics, arise naturally if we think of $\mathbf{R}^{2n}$ as the cotangent bundle $T^*\mathbf{R}^n$ of the configuration space $\mathbf{R}^n$, with the canonical symplectic form (2.5). Namely, in that case the Hamiltonian vector field X_f of a function $f(q^k, p_k)$ is

$$X_f = \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p_k} \quad , \tag{2.20}$$

as it is easily verified that $i(X_f)dp_k \wedge dq^k = -df$ (cf. (2.10)). Therefore the Poisson bracket is

$$\{f,g\} = \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} \quad , \tag{2.21}$$

and, in particular, the canonical Poisson brackets (classical canonical commutation relations) between the coordinates and momenta are

$$\begin{aligned} \{q^k, q^l\} = \{p_k, p_l\} &= 0 \\ \{p_k, q^l\} &= \delta_k^{\,l} \quad . \end{aligned} \tag{2.22}$$

The functions q^k and p_l form a *complete set* of observables in the sense that any function which Poisson commutes (has vanishing Poisson brackets with) all of them is a constant.

The equations (2.17-2.19) can now be written succinctly as

$$(2.17) \quad \Leftrightarrow \quad \tfrac{d}{dt}q^k = \{H, q^k\} \ , \quad \tfrac{d}{dt}p_k = \{H, p_k\} \ , \tag{2.23}$$

$$(2.18) \quad \Leftrightarrow \quad \{H, H\} = 0 \ , \tag{2.24}$$

$$(2.19) \quad \Leftrightarrow \quad \tfrac{d}{dt}f = \{H, f\} = X_H f \ , \tag{2.25}$$

so that time evolution in classical mechanics is determined by the Hamiltonian vector field X_H of the Hamiltonian H.

This formulation makes manifest the form-invariance of the equations of classical mechanics under canonical transformations or symplectomorphisms (diffeomorphisms leaving the symplectic form invariant). For instance, Liouville's theorem that the volume of any portion of phase space is invariant under time evolution (i.e. behaves like an incompressible fluid) is a trivial consequence of $L(X_H)\omega = 0$ and is thus built into the formalism from the outset.

It also has the added advantage of generalizing immediately to more complicated systems (e.g. with constraints) where the configuration space is some curved manifold Q, or even where the phase space is some compact symplectic manifold (and hence cannot possibly be a cotangent bundle). The necessity to consider such more exotic systems in physics has arisen in recent years in a number of different contexts, e.g. for the description of internal degrees of freedom and in topological and conformal field theory. In mathematics, quantization of compact symplectic manifolds plays a central role in representation theory (where the symplectic manifolds in question are coadjoint orbits).

3 What is Quantization?

In this section we take a first step away from the classical theory outlined above. It is, in a sense, a continuation of the introduction (why quantum theory?) and tries to give a general flavour of what quantization is about, without entering too far into the formalism and interpretation of quantum mechanics itself.

Classically, the space $C^\infty(M,\omega)$ of observables has, in addition to a Lie algebra structure provided by the Poisson bracket, the structure of a commutative algebra under pointwise multiplication,[4]

$$(fg)(x) = f(x)g(x) = (gf)(x) \quad . \tag{3.1}$$

It appears that it is this property which has to be sacrificed when moving from the classical to the quantum theory, the non-commutative nature of observables in the quantum theory being at the heart of the phenomena discussed in the introduction. More specifically, quantization usually refers to an assignment

$$\mathcal{Q}: \quad f \rightarrow \mathcal{Q}(f) \tag{3.2}$$

of operators $\mathcal{Q}(f)$ on some Hilbert space to classical observables f. This Hilbert space can be finite-dimensional (in which case one can think of the $\mathcal{Q}(f)$'s simply as finite-dimensional matrices) but will, in general, be infinite-dimensional. The scalar product in the Hilbert space is necessary for the probabilistic interpretation of the theory and is thus of fundamental importance. This assignment $\mathcal{Q}$ has to satisfy some more or less obvious requirements like

Q1: **R**-linearity,

$$\mathcal{Q}(rf+g) = r\mathcal{Q}(f) + \mathcal{Q}(g) \qquad \forall r \in \mathbf{R}, f,g \in C^\infty(M) \quad , \tag{3.3}$$

and the condition that

Q2: the constant function 1 is mapped into the identity operator or matrix **1**,

$$\mathcal{Q}(1) = \mathbf{1} \quad . \tag{3.4}$$

Furthermore, real functions should correspond to hermitian operators (as the eigenvalues of $\mathcal{Q}(f)$ are the possible results of measurements in the quantum theory and hence should be real),[5]

Q3:

$$\mathcal{Q}(f)^* = \mathcal{Q}(f) \quad . \tag{3.5}$$

But, of course, we need more guidelines than that to construct a quantum theory from a classical theory (even keeping in mind the limitations to this programme mentioned in the introduction). It is here where Dirac's observation enters that

[4]These are related by the Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$ and give $C^\infty(M,\omega)$ the structure of a *Poisson algebra*.

[5]Here and in the following I will gloss over functional analytic complications. This is, however, not meant to imply that they are not important.

it is the commutator of two operators which is the quantum counterpart of the classical Poisson bracket. More precisely, to the conditions (3.3-3.5) one adds

Q4: the quantum condition

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}(\{f, g\}) \quad . \tag{3.6}$$

Here $\hbar$ is Planck's constant, a constant of nature (dimension of an action) characteristic of quantum effects. It is a very small number, and for most macroscopic considerations the fact that it is not zero can be neglected. This is also reflected in the fact that for $\hbar \to 0$ (now treating $\hbar$ just as a parameter) one recovers from (3.6) the commutative structure of classical mechanics. At the microscopic level, however, order $\hbar$ effects can no longer be neglected and this is where classical mechanics needs to be replaced by quantum mechanics.

It would perhaps be more natural, if $\hbar$ appeared as a (free) parameter in the theory - see [4]. One could also contemplate the possibility of adding higher-order terms in $\hbar$ to the right hand side of (3.6); this leads to what is known as deformation quantization.

Experience has taught that there is yet one more condition to be imposed for the assignment (3.2) to produce a valid quantization (in those examples where one 'knows' what it should look like). This last requirement is some kind of irreducibility condition. A reasonably general and satisfactory way of phrasing it makes use of the concept of a complete set of observables introduced in section 2.2. In complete analogy, we define a complete set of operators to be one such that the only operators which commute with all the operators from that set are multiples of the identity. The condition then reads that

Q5: if $\{f_1, \ldots, f_k\}$ is a complete set of observables, $\{\mathcal{Q}(f_1), \ldots, \mathcal{Q}(f_k)\}$ is a complete set of operators.

Unfortunately, it is in general not possible to satisfy both Q4 (for all f and g) and Q5, and the best one can hope for is some 'optimal' compromise, e.g. demanding Q4 only for a complete set of observables and perhaps some additional observables which are of particular interest in the quantum theory. Of course, nothing tells us how to find a complete set of observables, or which one to choose. Nor is it ruled out that different choices of complete sets will lead to inequivalent quantum theories (i.e. to inequivalent predictions for the result of experiments). It is here, in what one means by 'optimal', that extraneous information and requirements enter the construction of a quantum theory, e.g. certain symmetries or geometric properties of the classical system which may make one complete set more 'natural' than another.

> Common sense must be used here to avoid embarking on an over-axiomatised, and hence misguided, piece of theoretical physics. We ... should not be trapped into axiomatising theoretical ideas out of existence. [4, p. 1155]

This discussion shows that it is very difficult to address the question of existence and classification of quantizations satsifying Q1-Q5 (in some sense) in general. By changing slightly the rules of the game, GQ nevertheless provides one method for doing precisely this. I will come back to this below.

First, it will be instructive to see how all this works in the simplest case $Q = \mathbf{R}^n$, $M = T^*Q$. In that case we have already encountered a complete set of observables in section 2.2, namely the coordinate functions q^k and p_l. According to the rule Q4 we demand the corresponding operators to satisfy the canonical commutation relations

$$\begin{aligned} [\mathcal{Q}(q^k), \mathcal{Q}(q^l)] &= [\mathcal{Q}(p_k), \mathcal{Q}(p_l)] = 0 \ , \\ [\mathcal{Q}(q^k), \mathcal{Q}(p_l)] &= i\hbar\delta^k_l \ . \end{aligned} \tag{3.7}$$

This is the so-called Heisenberg algebra and, by the Schur lemma, rule Q5 is now equivalent to finding an irreducible representation of the Heisenberg algebra (this is why I called Q5 'some kind of irreducibility condition' above). By the Stone - von Neumann theorem any such representation is unitarily equivalent to $L^2(Q) = L^2(\mathbf{R}^n)$ with q^k and p_l represented by

$$\mathcal{Q}(q^k)\psi(x) = x^k\psi(x) \ , \qquad \mathcal{Q}(p_l)\psi(x) = -\hbar i\frac{\partial\psi}{\partial x^l}(x) \tag{3.8}$$

(more precisely, for this uniqueness theorem to hold, one has to require that the representations exponentiate to representations of the Heisenberg group - there are inequivalent representations of the Heisenberg algebra). The spectrum (range of eigenvalues) of these operators is $(-\infty, +\infty)$. This is the standard Schrödinger picture of quantum mechanics. It is important to keep in mind that the fact that in this case 'wave functions' can be represented by functions on the configuration space is a consequence of our quantization rules Q1-Q5 and the Stone - von Neumann theorem, and not some fundamental dogma of quantization (as which it is often presented).

Now, the coordinates and momenta are certainly not the only observables of interest. Can we quantize any others as well in accordance with the rule Q4? Indeed we can, albeit not many more. One important observable is the kinetic energy operator $p^2 \equiv p^k p^l \delta_{kl}$ and evidently it should be represented by the Laplacian,

$$\begin{aligned} \mathcal{Q}(p_k p_l) &= -\hbar^2\frac{\partial^2}{\partial x^k \partial x^l} \ , \\ \mathcal{Q}(p^2) &= -\hbar^2\delta_{kl}\frac{\partial^2}{\partial x^k \partial x^l} \equiv -\hbar^2\Delta \ . \end{aligned} \tag{3.9}$$

Likewise, we have little choice but to represent observables quadratic in the coordinates by multiplication operators. Classically, the Poisson bracket between these two quadratic operators is proportional to $p_k q^l$ and we thus need to assign an operator to this third class of quadratic observables as well. Imposing either the hermiticity condition Q3 or the quantum condition Q4 one finds

$$\mathcal{Q}(p_k q^l) = \tfrac{1}{2}\left(\mathcal{Q}(p_k)\mathcal{Q}(q^l) + \mathcal{Q}(q^l)\mathcal{Q}(p_k)\right) \ . \tag{3.10}$$

This can be interpreted as a particular *operator ordering* of $\mathcal{Q}(p_k q^l) \sim \mathcal{Q}(p_k)\mathcal{Q}(q^l)$ (but note that a priori there is no logical necessity for the assignment $\mathcal{Q}$ to satisfy some condition like $\mathcal{Q}(fg) \sim \mathcal{Q}(f)\mathcal{Q}(g)$ in general).

The quadratic observables form a closed Lie algebra under Poisson brackets,

$$\begin{aligned}
\{p_i p_j, p_k p_l\} = \{q^i q^j, q^k q^l\} &= 0 \\
\{p_i q^j, p_k p_l\} &= -\delta^j_k p_i p_l - \delta^j_l p_i p_k \\
\{p_i q^j, p_k q^l\} &= \delta^i_l q^j p_k - \delta^j_k p_i q^l \\
\{p_i q^j, q^k q^l\} &= \delta^k_i q^j q^l + \delta^l_i q^j q^k \\
\{p_i p_j, q^k q^l\} &= \delta^k_j p_i q^l + \delta^l_j p_i q^k + \delta^k_i p_j q^l + \delta^l_i p_j q^k \quad , \quad (3.11)
\end{aligned}$$

the *symplectic Lie algebra* $sp(n)$ (in the non-compact form, $sp(1) \sim sl(2, \mathbf{R})$). Thus, what the above means is that when we quantize a symplectic vector space we can always obtain a representation of the symplectic Lie algebra on the quantum Hilbert space which reflects the classical symplectic invariance of the theory (and which exponentiates to a projective representation of the symplectic group).

If we now try to extend this quantization to cubic observables we run into conflict with the quantum condition Q4. That this is not due to some particularly unfortunate choices we have made but rather an inevitable consequence of the rules Q1-Q5 is the content of the *Groenewald - van Hove theorem* (for a careful exposition see [10]). Thus, even in the simplest case of a symplectic vector space no full (in the sense that Q4 holds for all observables) quantization exists. This is not a severe set-back, however, since there is no reason to expect any arbitrarily crazy classical 'observable' to be quantizable and to qualify as a true observable of the quantum system. The choice of classical functions which are to be promoted to quantum operators depends on the system under consideration and should feed its way back into e.g. the choice of complete set of observables entering the condition Q5.

Let us now look at the case when $Q \neq \mathbf{R}^n$, $M \neq \mathbf{R}^{2n}$. Even if M is a cotangent bundle, $M = T^*Q$, canonical coordinates (q^k, p_l) will in general not exist globally. It thus makes little sense to choose these as a complete set of observables and to impose the canonical commutation relations (3.7) at the quantum level. If one does that one is ignoring completely the geometry of the phase space and is thus performing something that could be regarded as only a tangent space approximation to the true quantum theory.

Take, for instance, the example $M = T^*S^1$ (a cylinder) with angular coordinate φ, angular momentum p and symplectic form $dp \wedge d\varphi$. If one required the canonical commutation relations $[\mathcal{Q}(\varphi), \mathcal{Q}(p)] = i\hbar$, this would imply that the spectrum of both operators is $(-\infty, +\infty)$, but this is wrong! In fact, it is known that in quantum mechanics angular momentum is quantized in units of $\hbar$, $\mathrm{spec}\,\mathcal{Q}(p) = \hbar\mathbf{Z}$ (while the spectrum of φ ought to be $[0, 2\pi)$). The source of the problem is, of course, the fact that φ is not a globally defined coordinate on the circle and that one is really dealing with the real line when one pretends that it is.

One way to get around this problem is to replace φ by a globally defined function on the circle, like $\sin\varphi$. If one does that, one is also forced to include $\cos\varphi$ to obtain a complete set of functions closed under Poisson brackets. One then arrives at the following globally well defined canonical Poisson bracket algebra of the cylinder,

$$\{p, \sin\varphi\} = \cos\varphi \ , \quad \{p, \cos\varphi\} = -\sin\varphi \ , \quad \{\sin\varphi, \cos\varphi\} = 0 \ . \tag{3.12}$$

Quantization of the cylinder then amounts to finding a representation of (3.12) and this yields the expected result.

Is there a more systematic way of arriving at (3.12)? And what is going to replace the canonical commutation relations (3.7) in general? A clue to this comes from the following observation. The fact that the canonical coordinates are globally defined for $M = T^*\mathbf{R}^n$ implies that (via their Hamiltonian vector fields) translations act on the phase space, completeness corresponding to the fact that these translations act transitively. The canonical Poisson bracket algebra (2.22) can thus be regarded as a central extension of the translation algebra, and the quantization conditions instruct one to find its irreducible representations.

This suggests a general strategy whenever the phase space is a homogenous space (i.e. one with a transitive action of some group G). Assuming that this action is generated by Hamiltonian vector fields, the corresponding functions form (roughly speaking) a complete set of observables. One then looks for irreducible representations of their Poisson bracket algebra. Thus the canonical commutation relations of the Heisenberg algebra are replaced by those of the 'canonical' group G and the problem of quantization is again reduced to one of representation theory. In general, there will be no Stone - von Neumann theorem so that quantization will not be unique. And, even if $M = T^*Q$, the Hilbert space of the quantum theory will not necessarily turn out to be $L^2(Q)$.

Applied to the above example one finds that quantization of the cylinder amounts to finding representations of the Euclidean group $E(2)$ which in turn can be expressed via the Poisson bracket relations (3.12), as had been anticipated above. Note that the first guess, that the canonical group could be chosen to be $S^1 \times \mathbf{R}$ itself, acting on the cylinder by rotations and translations, fails, because the generator $\partial/\partial p$ of translations is not globally Hamiltonian. This is the same problem in disguise we encountered above with regard to the 'naive' canonical commutation relations. For a detailed explanation of this programme, with many other finite and infinite dimensional examples, see [4].

More generally, one may say that whenever there is a preferred complete set of observables (in some sense) there is a preferred class of quantizations, and in this form Isham's programme has been applied successfully to gauge theories and quantum gravity in the Ashtekar variables by the Syracuse group.

In order to avoid these questions (which require more of a case by case analysis) and to geometrize the question of existence and classification of quantizations, GQ focusses on a slightly different way of looking at quantum mechanics on $\mathbf{R}^n$. Essentially, the concept of a complete set of observables (like the q^k and

p_l) is replaced by that of a maximally (Poisson) commuting set of observables (like the q^k). Thus, wave functions are considered to be functions (or sections of some complex line bundle) on the classical or quantum spectrum of a maximally commuting set of observables (which are diagonal in this representation).

Alternatively, still for $Q = \mathbf{R}^n$, wave functions can be characterized as functions on phase space which are annihilated by the Hamiltonian vector fields of a maximally commuting set of observables. It is this way of looking at Schrödinger quantization that generalizes most readily to other symplectic manifolds M. In section 2.2, we had already seen that such a maximally commuting set $\{F_k\}$ defines a Lagrangian submanifold of M. By varying the constants c_k in the equations $F_k = c_k$ one then obtains a *foliation* of M by Lagrangian submanifolds. This is also called a *real polarization* of M. The quantum Hilbert space is then constructed from those functions (sections of a line bundle) on M which are (covariantly) constant along the leaves of this foliation. It is, roughly speaking, this condition that replaces the quantization condition Q5 (emphasizing the role of a complete set of observables).

At first, this approach to quantization appears to be rather restrictive. In the finite dimensional case, however, there is considerable overlap among the results arising from this, Isham's, and other quantization schemes. One of the reasons for this is that the concept of a polarization is more flexible than it perhaps seems.

First of all, it is possible to replace real by complex polarizations (integrable Lagrangian subbundles of the complexified tangent bundle of M - see section 5.1). This is also familiar from ordinary quantum mechanics on $\mathbf{R}^n$ in the form of the *Bargmann representation* in which wave functions are represented by holomorphic functions of $z_k \sim q^k + ip_k$. In many cases covered by Isham's scheme there are more or less natural polarizations which are compatible with (i.e. invariant under) the canonical group. GQ can then be used to construct representations of this canonical group. In this context it can thus be regarded as investigating the question to which extent representations of the canonical group, featuring in Isham's approach, can be constructed from symplectic geometry.

Moreover, it can be seen in examples that, with due care, it is also possible to apply GQ when the $\{F_k\}$ or the real polarization are not globally defined but are singular somewhere. Such singular polarizations are more likely to exist and thus extend the range of applicability of GQ.

A final word of warning: it is possible that GQ is overambitious in attempting to make quantization 'work' for (almost, see section 4.1) arbitrary symplectic manifolds. Since it is really quantum theory that should be regarded as fundamental, there is no a a priori reason to believe that every classical theory has a quantum counterpart.

4 Geometric Quantization I: Prequantization

As mentioned a couple of times above, GQ accomplishes the quantization of a symplectic manifold in a two- (or more-) step procedure. The first is the construction of a faithful representation of the Poisson algebra of functions by linear operators on a Hilbert space. This is known as prequantization and satisfies the quantization conditions Q1-Q4. In a second step, a variant of Q5 is imposed in terms of a polarization, at the inevitable expense of sacrificing part of the quantum condition Q4.

In section 4.1 the construction of the prequantum Hilbert space from a complex line bundle over phase space is explained, as well as the classification of line bundles with connections. Examples are discussed in section 4.2.

4.1 The Prequantum Hilbert Space

In order to geometrize the notion of quantization, it is natural to attempt to construct the quantum phase space (= Hilbert space) from the space of functions on the classical phase space M instead of regarding the two as completely separate entities. An important role is played by the identity (2.15),

$$[X_f, X_g] = X_{\{f,g\}} \quad ,$$

which shows that Hamiltonian vector fields give a representation of the classical Poisson bracket algebra by first order differential operators on M. In fact, the assignment

$$f \to -i\hbar X_f \tag{4.1}$$

satisfies the conditions Q1 (obviously), Q3 (with respect to the Liouville measure, because X_f leaves ω invariant), and Q4 (by (2.15)). However, since the zero vector field is assigned to any constant function, (4.1) fails to satisfy Q2. One may try to remedy this by replacing $-i\hbar X_f$ by $-i\hbar X_f + f$, but this is also not quite right, now violating Q4. A little further experimenting shows that if $M = T^*Q$ (where the symplectic form ω is globally the differential of the canonical one-form θ), the assignment

$$\begin{aligned} \mathcal{P} \quad : \quad & f \to \mathcal{P}(f) \\ & \mathcal{P}(f) = -i\hbar X_f - \theta(X_f) + f \end{aligned} \tag{4.2}$$

indeed satisfies $Q1 - Q4$ and thus gives a faithful representation of the Poisson algebra by first order differential operators on $L^2(M, \omega)$. For $Q = \mathbf{R}^n$ one has (denoting the multiplication operator simply by q^k)

$$\mathcal{P}(q^k) = i\hbar \tfrac{\partial}{\partial p_k} + q^k \quad , \qquad \mathcal{P}(p_l) = -i\hbar \tfrac{\partial}{\partial q^l} \quad . \tag{4.3}$$

This evidently only reduces to the Schrödinger representation (3.8) when acting on functions of the coordinates alone. (4.3) also shows that $\mathcal{P}$ fails to satisfy the irreducibility condition Q5 as e.g. the operator $\partial/\partial p_j$ commutes with all the $\mathcal{P}(q^k)$ and $\mathcal{P}(p_l)$. Another problem with prequantization is, that it certainly fails to reproduce the second order differential operators (3.9) associated to observables quadratic in the momenta. To reobtain these from GQ requires much more work. The crucial point is, that the operator $\mathcal{P}(p^2)$ (being linear in the momenta) does not even act on the space of functions of the coordinates alone. It thus changes the representation space or, in the language of GQ, the polarization. Thus, to associate an operator to p^2 one has to compensate for the change in the polarization caused by the flow of p^2. This is an analytically rather involved procedure based on the so-called *Blattner - Kostant - Sternberg (BKS) kernels* which is very incompletely understood in general. We will only come back to it briefly in section 6.1.

There is still one minor difficulty with the above construction. Namely, instead of θ one could have chosen a symplectic potential of the form $\theta + df$ for some function f on M. This can be compensated for by multiplying the functions of $L^2(M,\omega)$ by the phase factor $\exp(if/\hbar)$ (showing that the resulting prequantizations are unitarily equivalent). f is, however, only determined by df up to a constant, resulting in an overall phase ambiguity of the prequantum wave functions.[6] This suggests that it is more convenient to regard the operators $\mathcal{P}(f)$ as acting on the space of sections of a trivial complex line bundle L over M equipped with a connection D which in a particular trivialization takes the form

$$D = d - (i/\hbar)\theta \quad . \tag{4.4}$$

As we will need the construction later on when switching from real to complex polarizations, I will briefly explain the reation between trivializations, sections, and connection forms in the case at hand. The same arguments apply to local trivializations (whose existence is guaranteed by the definition of a fiber bundle) in the general case of non-trivializable bundles.

If there is a global nowhere vanishing section s of the complex line bundle L, this section gives us an identification $L \sim M \times \mathbf{C}$. Conversely, beginning with $M \times \mathbf{C}$ (as we did above before starting to worry about line bundles) we can think of it as a line bundle L with the natural trivializing section $s_0(m) = (m, 1)$. Given a connection (covariant derivative) D on L, and a trivializing section s, the corresponding connection one-form β_s is defined by

$$Ds = -i\beta_s s \quad . \tag{4.5}$$

Any other section of L is of the form ψs for some complex valued function ψ, and one has

$$D(\psi s) = (d\psi)s + \psi(Ds) = (d\psi - i\beta_s\psi)s \quad , \tag{4.6}$$

which can be read as

$$D\psi = d\psi - i\beta_s\psi \tag{4.7}$$

'in local coordinates'. If one chooses a different trivializing section, say $s' = \exp(-if)s$ then the connection one-form will change according to

$$Ds' = D(\exp(-if)s) = -i(\beta_s + df)s' \equiv -i\beta_{s'}s' \quad . \tag{4.8}$$

[6] The more exotic possibility of replacing θ by $\theta + \alpha$, where α is a closed but non-exact one-form on M, will be dealt with below. It leads to a unitarily inequivalent theory.

With respect to s' the section ψs will be represented by $\exp(+if)\psi$ and this recovers the argument given above. The upshot of this is that we should think of (4.4) as being valid in the trivializtion $s_0(m) = (m, 1)$ and that we now know how to relate changes in the symplectic potential (which it is occasionally convenient to perform) to changes of local trivializations. This point of view actually becomes mandatory when one is dealing with general symplectic manifolds (M, ω) where ω is not necessarily exact. In that case expressions like (4.2) and (4.4) only make sense locally, with $\theta \to \theta_\alpha$, say, on a coordinate patch U_α while on overlaps $U_\alpha \cap U_\beta$ of a (good) cover one has

$$\theta_\alpha - \theta_\beta = df_{\alpha\beta} \ , \tag{4.9}$$

where $f_{\alpha\beta}$ is related to the *transition function* connecting the local trivializing sections s_α and s_β over $U_\alpha \cap U_\beta$.

In terms of D, the prequantum operator $\mathcal{P}(f)$ can be written as

$$\mathcal{P}(f) = -i\hbar D(X_f) + f \ . \tag{4.10}$$

There is another way of looking at $\mathcal{P}(f)$ which sheds some light on its definition. Via its Hamiltonian vector field X_f the function f generates a flow

$$\Phi_t^f : m \mapsto \Phi_t^f(m) \tag{4.11}$$

of canonical transformations of M. Up to an overall phase (related to the ambiguity $f \to f + c$, c a constant) there is a unique way of lifting this flow to an automorphism of L preserving the Hermitian structure and the compatible connection. This, in turn, induces a 'pull-back' action

$$\widehat{\Phi}_t^f : \psi \mapsto \widehat{\Phi}_t^f \psi \tag{4.12}$$

on sections of L and their local representatives ψ. Introducing the quantity

$$\begin{aligned} \mathcal{L}_f &= \theta(X_f) - f \\ &= p_k \frac{\partial f}{\partial p_k} - f \end{aligned} \tag{4.13}$$

the *Lagrangian* of f, one finds that (4.11) is given explicitly by

$$\left(\widehat{\Phi}_t^f \psi\right)(m) = \psi(\Phi_t^f(m)) \exp\left(-\tfrac{i}{\hbar} \int_0^t \mathcal{L}_f(\Phi_{t'}^f(m)) dt'\right) \ . \tag{4.14}$$

Thus 'time evolution' is given by the exponential of the classical action, something that is highly reminiscent of the path integral. There are numerous other connections between GQ and path integrals, see [13, 14]. Anyway, as $\mathcal{P}(f)$ can be expressed in terms of $\mathcal{L}_f$ as

$$\mathcal{P}(f) = -i\hbar X_f - \mathcal{L}_f \ , \tag{4.15}$$

it follows that $\mathcal{P}(f)\psi$ is nothing but the derivative of (4.14) at $t = 0$,

$$\mathcal{P}(f)\psi = -i\hbar \tfrac{d}{dt} \left(\widehat{\Phi}_t^f \psi\right)|_{t=0} \ . \tag{4.16}$$

We can thus interpret $\mathcal{P}(f)$ intrinsically as the generator of a connection preserving automorphism of L lifting the action of the Hamiltonian vector field X_f on M.

Let us now retrun to more down-to-earth matters. It follows from (4.4) that the curvature Ω of L, defined by

$$\Omega(X,Y) = i\left([D(X),D(Y)] - D([X,Y])\right) \quad , \tag{4.17}$$

is

$$\Omega = iD^2 = (1/\hbar)d\theta = (1/\hbar)\omega \quad . \tag{4.18}$$

The definition (4.10) still makes sense for non-trivial line bundles and

$$[\mathcal{P}(f),\mathcal{P}(g)] = -i\hbar\mathcal{P}(\{f,g\}) \tag{4.19}$$

is satisfied for all f and g provided that L is a line bundle with connection D whose curvature two-form is $(1/\hbar)\omega$. Moreover, $\mathcal{P}(f)$ can still be understood as the generator of a connection preserving automorphism of (L,D).

As ω is real, there always exists a compatible Hermitian structure on L and we thus arrive at the following

Definition: A prequantization of a symplectic manifold (M,ω) is a pair (L,D) where L is a complex Hermitian line bundle over M and D a compatible connection with curvature $(1/\hbar)\omega$. The prequantum Hilbert space $\mathcal{H}$ is the completion of the space of smooth sections of L, square-integrable with respect to the Liouville measure on M (and the Hermitian structure on the fibers).

Topologically, line bundles are classified by their first Chern class $c_1(L) \in H^2(M,2\pi\mathbf{Z})$. In de Rham cohomology, $c_1(L)$ can be represented by the curvature form of any connection on L (the cohomology class of the curvature form is independent of the choice of connection). Thus a necessary (and, in fact, sufficient) condition for a prequantization (L,D) of a symplectic manifold to exist is that $(1/2\pi\hbar)\omega$ represent an integral cohomology class or, in other words, that the integral of $(1/2\pi\hbar)\omega$ over every closed, orientable two-surface in M be an integer. Such symplectic manifolds are called quantizable (although prequantizable would be more accurate).

Cotangent bundles T^*Q, equipped with the canonical symplectic structure $\omega = d\theta$, are always quantizable as ω is exact. Moreover, as the cohomology class of ω is trivial, so is any prequantum line bundle L over (T^*Q,ω) (as we had already noted above). In certain cases, however, the quantizability condition imposes a quantization condition on parameters appearing in the classical system. For instance, Dirac's famous quantization condition on the electric charge e of a particle moving in the field of a magnetic monopole can be understood in this way. This is a consequence of the fact that the coupling of particles to an Abelian gauge field (connection) A with field strength (curvature) $F = dA$ can be accomplished by replacing the original symplectic structure $\omega = d\theta$ on T^*Q by the 'charged' symplectic structure

$$\omega_F = \omega + eF$$

(which is still non-degenerate and closed). This is equivalent to using the standard minimal coupling prescription $p_l \to p_l - eA_l(q^k)$ with the unmodified symplectic structure ω. Quantizability of (T^*Q, ω_F) is now equivalent to integrality of $(e/2\pi\hbar)F$. If F represents a non-trivial cohomology class (so that A is only defiined locally), this gives a restriction on the possible values of e and the prequantum line bundle L will be non-trivial as well. (As an aside: the coupling constant quantization conditions appearing in certain field theories like the Wess-Zumino-Witten model and topologically massive gauge theory can be understood in the same way.)

Likewise, if $M = S^2_r$, the two-sphere with radius r, and ω is the volume form $\omega = r \sin\vartheta d\vartheta d\phi$, then ω is only integral for certain discrete values of r, namely $r = n\hbar/2,\ n \in \mathbf{Z}$. It is, by the way, no coincidence that this looks like the quantization rule for angular momentum or like representation theory of $SU(2)$. S^2 is a homogeneous space for $SU(2)$, $S^2 \sim SU(2)/U(1)$ (in fact, a coadjoint orbit), and quantization of S^2 hence leads to representations of $SU(2)$. The prequantum Hilbert space is, of course, infinite dimensional, but by considering only holomorphic sections of the prequantum line bundle (which corresponds to a particular choice of complex polarization) one obtains finite dimensional Hilbert spaces which are irreducible representation modules of $SU(2)$. This relation between transitive group actions (homogeneous symplectic spaces) and irreducible representations is one of the origins of GQ. It is appropriate to regard GQ as a generalization of the Borel-Weil-Bott theorem and Kirilov's method of orbits [9] to the non-homogeneous case.

As the above examples may have given rise to the impression that all symplectic manifolds (M, ω) can be made quantizable by a rescaling of the symplectic form ω, I will mention a simple counterexample: the product of two two-spheres $M = S^2_r \times S^2_s$ with incommensurate radii r and s. Attempts have been made to generalize GQ to such spaces but I will have nothing to say about this here.

After having discussed this necessary condition for a prequantization to exist, we now turn to a brief discussion of the classification of prequantizations of a quantizable symplectic manifold (M, ω). As a key role is played by the connection D in the defintion of prequantization, the topological classification of line bundles (by their Chern class) is too coarse to provide a classification of prequantizations or prequantum line bundles on (M, ω). What one needs is a refinement in which two prequantizations (L, D) and (L', D') are regarded as equivalent if there is a bundle isomorphism $f : L \to L'$ such that $f^*D' = D$.

To address this problem in a somewhat more pedestrian manner, we will need the following terminology: a connection is called *flat* if its curvature vanishes; a *flat line bundle* is a line bundle with a flat connection. Furthermore we will need the fact that one can form the tensor product $E \otimes F$ of two vector bundles E and F (this is simply done fiberwise) and that the tensor product of two complex line bundles L and L' is again a complex line bundle (because $\mathbf{C} \otimes \mathbf{C} \sim \mathbf{C}$). These tensor product bundles inherit naturally a tensor product connection $D_{L\otimes L'} = D \otimes D'$ from connections on L and L'. In a local trivialization, if $D = d - i\alpha$ and $D' = d - i\alpha'$, then $D \otimes D' = d - i(\alpha + \alpha')$. Thus the curvature

(4.18) of $D \otimes D'$ is simply the sum of the curvatures of D and D'. Furthermore, the curvature of the complex conjugate line bundle (L^*, D^*) is minus that of (L, D) as $D^* = d + i\alpha$ and $(L \otimes L^*, D \otimes D^* = d)$ is the trivial flat line bundle with trivial connection d.

This implies that, given a prequantization (L, D) and a flat line bundle (L_0, D_0), the tensor product $(L \otimes L_0, D \otimes D_0)$ is again a prequantization. Conversely, given two prequantizations (L, D) and (L', D'), they differ by a flat line bundle because

$$\begin{aligned} (L', D') &\sim (L', D') \otimes ((L, D) \otimes (L^*, D^*)) \\ &\sim (L, D) \otimes ((L', D') \otimes (L^*, D^*)) \end{aligned} \tag{4.20}$$

and the second factor in the second line is flat. Thus the classification of prequantizations of (M, ω) amounts to the classification of flat line bundles on M (and is, in particular, independent of ω). This is quite standard and can be accomplished in a number of different ways. An argument using Cech cohomology and exact sequences can be found in [13] and leads to the result that isomorphism classes of flat line bundles (and prequantizations) are in one-to-one correspondence with the elements of

$$H^1(M, U(1)) \ ,$$

the first cohomology group of M with coefficients in $U(1)$. Alternatively, one can determine directly the (moduli) space of flat $U(1)$ connections on M modulo gauge transformations which is well known to be

$$\mathrm{Hom}(\pi_1(M), U(1)) \ ,$$

where $\pi_1(M)$ is the fundamental group of M. This result follows from the fact that the holonomy of a flat connection along a loop is invariant under deformations of the loop so that a flat connection is uniquely determined, modulo gauge transformations, by its holonomies along homotopy classes of non-contractible loops. By the universal coefficient theorem [16] the above two expressions are equal.

There are two possible sources of non-equivalent flat line bundles on M, and thus two kinds of contributions to $H^1(M, U(1))$. One is the possibility of having non-equivalent flat connections on a given line bundle. It is of the form[7]

$$H^1(M, \mathbf{R}) / H^1(M, \mathbf{Z}) \sim U(1)^{b_1(M)}$$

with $b_1(M) = \dim H^1(M, \mathbf{R})$ the first Betti number of M. This can be read as saying that, given a flat connection D_0 on the line bundle L_0, so that any other flat connection on L_0 is of the form $D_0 + \alpha$ with α a closed one-form, $D_0 + \alpha$ is inequivalent to D_0 provided that α is neither integral nor (a fortiori) exact. We had already seen above that symplectic potentials differing by exact one-forms (infinitesimal gauge transformations) lead to equivalent prequantizations.

[7] More precisely, $H^1(M, \mathbf{Z})$ should be replaced by its image $i_* H^1(M, \mathbf{Z})$ in $H^1(M, \mathbf{R})$ in this expression, where i is the inclusion $i : \mathbf{Z} \hookrightarrow \mathbf{R}$.

Connection forms differing by non-trivial integral one-forms, on the other hand, are related by 'large' gauge transformations. We will see an example of this torus' worth of prequantizations below. It has the interpretation of vacuum angles or Aharonov-Bohm phases.

The second contribution comes from topologically inequivalent flat line bundles. As line bundles are topologically classified by their Chern class $(c_1(L)/2\pi) \in H^2(M, \mathbf{Z})$ and the curvature form represents the image of this class in $H^2(M, \mathbf{R})$, these correspond to the kernel

$$\mathrm{Ker}\ i_* : \ H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{R})$$

(i_* kills the torsion in $H^2(M, \mathbf{Z})$). I will not give an example where such a possibility occurs but want to just mention that the choice of isomorphism class of flat line bundles can in certain cases be interpreted as a choice of statistics (Fermi versus Bose, for instance).

4.2 Examples

In this section we will take a brief look at some two-dimensional examples, the cylinder T^*S^1 and the two-sphere S^2. Each has its own characteristic features which serves to illustrate one or the other of the issues encountered above in a rather more abstract manner.

Example 1 $\underline{M = T^*S^1}$

As M is a cotangent bundle, the symplectic form $\omega = dp \wedge d\varphi$ is globally exact,

$$\omega = d\theta \ , \qquad \theta = p d\varphi \ , \tag{4.21}$$

and the prequantum line bundle L is trivial. A prequantization of M is given by the connection $D = d - (i/\hbar)\theta$. As M is not simply connected,

$$\pi_1(M) = \mathbf{Z} \ , \qquad H^1(M, U(1)) = U(1) \ , \tag{4.22}$$

we expect, however, to find not a unique but a $U(1)$'s worth of prequantiztions. This expectation is indeed borne out. $d\varphi$ is a non-exact closed one-form, the generator of $H^1(M, \mathbf{R})$, and we can thus modify the prequantum connection to

$$D^\lambda := d - (i/\hbar)\theta + i\lambda d\varphi \ . \tag{4.23}$$

One way of seeing that for $\lambda \in [0, 1)$ these are all mutually inequivalent is the following. The prequantum operator $\mathcal{P}^\lambda(p)$ of p with respect to the connection D^λ is

$$\mathcal{P}^\lambda(p) = -i\hbar \tfrac{\partial}{\partial\varphi} + \hbar\lambda \ . \tag{4.24}$$

As L is trivial, we can identify the prequantum Hilbert space with the space of functions on M which are, in particular, periodic in φ. Likewise, the Hilbert space in the Schrödinger representation, on which (4.24) is a well defined operator, is $L^2(S^1)$. Thus the spectrum of $(-i\partial/\partial\varphi)$ is the integers and that of $\mathcal{P}^\lambda(p)$ is also discrete (as expected) and is

$$\mathrm{spec}\left(\mathcal{P}^\lambda(p)\right) = \{(n + \lambda)\hbar, \ \ n \in \mathbf{Z}\} \ . \tag{4.25}$$

As these are only equal when λ is an integer, this shows that for all $\lambda \in [0,1)$ the quantum theories obtained from the prequantization (L, D^λ) are inequivalent.

The parameter λ leads to an additional contribution to the holonomy picked up by a state upon parallel transport around the circle. It can thus be regarded as a simple toy-model of the Aharonov-Bohm effect, $\lambda d\varphi$ representing a magnetic field running through the interior of the circle. Alternatively, the above example can be regarded as an embryonic illustration of the field theoretic phenomenon of vacuum angles (or theta vacua). Such topological quantization ambiguities (superselection sectors) occur (almost) always when the configuration space is not simply connected, most prominently in four-dimensional gauge theories where they are related to the strong CP problem.

Example 2 $\underline{M = S^2}$

Above, we have already discussed the conditions for M to be quantizable. Here we will fix the volume form ω by $\int_M \omega = 2\pi\hbar$ so that $(M, k\omega)$ is quantizable iff $k \in \mathbf{Z}$. As $H^2(M, \mathbf{Z}) = \mathbf{Z}$ and $\pi_1(M) = 0$ there is a unique prequantum line bundle (L_k, D_k) with $i(D_k)^2 = (k/\hbar)\omega$ in every topological (monopole) sector. Moreover, from the arguments of the previous section we can deduce that L_k is the k'th tensor power of L_1 and D_k the corresponding tensor product connection. Thus all we need to determine is the 'generator' (L_1, D_1). I will give three different descriptions of this bundle.

- The first is in terms of the *Hopf fibration*. The three-sphere is itself a $U(1)$ bundle over S^2 with monopole (Chern) number 1. By letting $U(1)$ act on the complex plane $\mathbf{C}$ in the standard fashion, one can associate to this $U(1)$ bundle over S^2 a complex line bundle over S^2 which is just L_1. This description is useful because it shows that element of the prequantum Hilbert space $\mathcal{H}$, sections of the non-trivial bundle L_1, can be represented by complex valued functions on S^3 transforming equivariantly under the action of $U(1)$ on S^3.

- The second makes use of the identification of S^2 with the complex projective plane $\mathbf{CP}^1$, the space of complex lines in $\mathbf{C}^2$. Over $\mathbf{CP}^1$ there is a natural complex line bundle obtained by attaching to each point of $\mathbf{CP}^1$ the complex line it represents. For obvious reasons this bundle is called the *tautological line bundle* and it again represents L_1. This description is useful because it makes it evident that L_1 can be regarded as a holomorphic line bundle.

- Finally, the last description is in terms of local coordinates. Think of S^2 as being given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbf{R}^3$. Let $x_\pm$ be the north and south poles of S^2 determined by $x_3 = \pm 1$. Then on the coordinate neighbourhoods $U_\pm = S^2 \backslash \{x_\pm\}$ one can introduce the complex coordinates
$$z_\pm = \frac{x_1 \pm i x_2}{1 \mp x_3}$$
related by $z_+ z_- = 1$ on the overlaps of the two regions. As the $U_\pm$ are topologically trivial, any line bundle is trivial when restricted to one of

these. Thus, all we have to do is to give a prescription for glueing these trivial bundles together over, say, the equator. If one does this with the transition function $(z_+)^k = (z_-)^{-k}$ one obtains the line bundle L_k. The advantage of this description is that it provides us with explicit expressions for the symplectic potentials (and hence for the prequantum connection D_k). Namely, on $U_\pm$ the symplectic form ω can be written as

$$\omega = -i\hbar \frac{d\bar{z}_\pm dz_\pm}{(1+|z_\pm|^2)^2} \quad , \tag{4.26}$$

and the symplectic potentials $\theta_\pm$ can be chosen to be

$$\theta_\pm = -i\hbar \frac{\bar{z}_\pm dz_\pm}{1+|z_\pm|^2} \quad . \tag{4.27}$$

This explicit description will also allow us to read off immediately the dimension of the space of holomorphic sections of L_k, which is $k+1$, the dimension of the spin $k/2$ representation of $SU(2)$, see section 6.3.

5 Geometric Quantization II: Polarizations

Up to now, GQ has been quite straightforward and elegant. Unfortunately, prequantization is not the end of the story and some additional structures have to be introduced to obtain a quantization of a symplectic manifold (in the sense of section 3) from this. In GQ, one of these is a polarization, and this leads to rather severe technical complications in general. Most of them are related to the fact that there is no natural measure on the space of quantum states and that, even when there is, GQ is still not completely 'correct'. One is then forced to modify the quantization scheme to what is known as half-form or metaplectic quantization. And although at this stage GQ becomes quite successful, it simultanously becomes rather complicated and unwieldy.

In section 5.1 I will show that the concept of a polarization arises rather naturally in GQ when one tries to 'cut down' the prequantum Hilbert space. The theory of real and complex polarizations and of Lagrangian submanifolds of symplectic manifolds is very rich and rewarding but I will not attempt to go far beyond the formal definition of a polarization.

In practice, there are two classes of symplectic manifolds for which GQ is fairly well understood and works with comparative ease, cotangent bundles and Kähler manifolds. These have natural and well-behaved polarizations which we will take a look at in sections 5.2 and 5.3. Although there are compact symplectic manifolds which are not Kähler and symplectic manifolds which admit no polarization whatsoever, an understanding of these two cases is usually sufficient for specific applications.

The construction of the quantum Hilbert space and of operators acting on it is then the subject of section 6.

5.1 Polarizations

In section 3 we have already seen that a possible generalization of Schrödinger quantum mechanics on $T^*Q = \mathbf{R}^{2n}$ is based not on the concept of a complete set of observables (as in Q5) but rather on that of a maximal commuting set. We have also seen that from that point of view it is possible to regard the Hilbert space $L^2(Q)$ as the space of functions on the phase space constant along the leaves of a polarization.

As this may appear to be a rather contrived and unnecessarily complicated way of arriving at the Hilbert space, I will now show that the concept of a polarization arises quite naturally if one attempts to construct the quantum Hilbert space from the prequantum Hilbert space $\mathcal{H}$. I want to point out, however, that the physical justification for this procedure

> ... is not based on general mathematical results (such as the Borel-Weil theorem), but on the way in which the construction works in particular examples. It generalizes and unifies a number of quantization techniques that, in the past, have not appeared to have any obvious connection with each other and that have sometimes seemed overspecialized with applications only to particular physical systems. [13, p. 171]

Roughly speaking, the problem with the prequantum Hilbert space $\mathcal{H}$ is that it is too large, consisting of functions ψ which depend on all the $2n$ coordinates of the symplectic manifold (M, ω). A way of eliminating 'half' of these is to demand that the wave functions are constant along n vector fields on M. As ordinary differentiation has no invariant meaning for sections of a bundle, one must take this to mean that they are covariantly constant. Thus, one way to proceed is to choose some n-dimensional subbundle P of the tangent bundle TM of M and to consider only those wave functions that satisfy

$$D(X)\psi = 0 \qquad \forall X \in P \tag{5.1}$$

(where '$X \in P$' is short, and sloppy, for 'X is a section of P'). Now there could be non-trivial integrability conditions for those equations which would form an obstruction to finding any (or a sufficient number of) solutions to (5.1). From (5.1) it follows that $[D(X), D(Y)]\psi = 0$ for all $X, Y \in P$. Combined with (4.17) this leads to the integrability condition

$$D([X,Y])\psi - (i/\hbar)\omega(X,Y)\psi = 0 \qquad \forall X, Y \in P \ . \tag{5.2}$$

We see that this condition is automatically satisfied provided that

$$X \in P, Y \in P \Rightarrow [X,Y] \in P \tag{5.3}$$

and

$$X \in P, Y \in P \Rightarrow \omega(X,Y) = 0 \ . \tag{5.4}$$

The first condition means that P is integrable, so that locally there exist integral manifolds in M through P. As these are n-dimensional the second condition means that these integral manifolds are Lagrangian. We have thus arrived precisely at the definition of a real polarization given in section 3. We see that there are no local integrability conditions[8] if we demand the wave functions to be covariantly constant along the leaves (integral manifolds) of a polarization P, i.e. of a Lagrangian subbundle of TM. This approach, which is a natural generalization of that based on a maximal commuting set of observables, thus arises quite naturally from prequantization and is the one adopted in GQ.

Life is, of course, not as simple as that. The problem with the above definition of a polarization is that it is far too restrictive. For instance, on a two-dimensional surface a polarization corresponds to a nowhere vanishing vector field. S^2 has none and among the closed two-dimensional surfaces the torus is the only one which has. The way to get around this problem is to complexify the tangent bundle of M, $TM \to TM^c$, and to consider integrable Lagrangian subbundles of TM^c. These are more likely to exist while the integrability condition (5.2) is still satisfied. We thus make the following

Definition: Let (M, ω) be a symplectic manifold. A polarization P of (M, ω) is an integrable maximally isotropic (Lagrangian) subbundle of the complexified tangent bundle TM^c of M.

Naively, one would now like to construct the quantum Hilbert space from the space $P(L)$ of *polarized sections*, i.e. sections of the prequantum line bundle L covariantly constant (parallel) along P. This is not as straightforward as one might have hoped it to be (e.g. because $\mathcal{H} \cap P(L)$ may be empty). We will come back to this problem in section 6, after having seen some examples of polarizations.

For technical reasons one imposes some additional conditions on P. The first, usually included in the definition of a polarization, is that the dimension of $P_m \cap \bar{P}_m \cap T_m M$ be constant. Here P_m denotes the fiber of P at $m \in M$ and $\bar{P}_m$ the complex conjugate of P_m. To state the other conditions we note that any complex subbundle F_c of TM^c satisfying $\bar{F}_c = F_c$ is the complexification of some real subbundle F of TM, $F_c = F^c$. Thus the complex subbundles $P \cap \bar{P}$ and $P + \bar{P}$ of TM^c are of the form

$$P \cap \bar{P} = D^c \quad , \qquad P + \bar{P} = E^c \quad , \tag{5.5}$$

where

$$D = P \cap \bar{P} \cap TM \quad , \qquad E = (P + \bar{P}) \cap TM \tag{5.6}$$

(this notation is standard, no confusion with the prequantum connection D should arise). As P is integrable, so is D. We assume that the integral manifolds of D are complete and we denote by M/D the space of all integral manifolds of D. A polarization is called *strongly admissible* if E is integrable and the spaces M/D and M/E are smooth Hausdorff manifolds.

In the following we will deal almost exclusively with polarizations which are either 'real',

$$P = \bar{P} \quad ,$$

[8]There can still be global integrability conditions related to the holonomy of D along the leaves of P. We will encounter these later on in the guise of Bohr-Sommerfeld conditions.

i.e. the complexification of a real polarization, or Kähler,

$$P \cap \bar{P} = \{0\} \ .$$

In the former case, $D = E$ so that $P = D^c$ is strongly admissible if the space of leaves of the underlying real polarization D is smooth and Hausdorff. In the latter, $D = \{0\}$ and hence $E = TM$ so that any Kähler polarization is strongly admissible. Other properties of polarizations will be mentioned below, in the context of either real or Kähler polarizations.

5.2 Real Polarizations

As noted above, real polarizations are characterized by the property $P = \bar{P}$ which implies that $P = D^c$. The prime example of a real polarization is the *vertical polarization* of a cotangent bundle $M = T^*Q$. In local coordinates it is spanned by the vectors $(\partial/\partial p_k)$ tangent to the fibers of T^*Q. Thus D is the vertical tangent bundle, P its complexification, and the integral manifolds of D are the fibers T^*_qQ, isomorphic to $\mathbf{R}^n$. The space M/D of integral manifolds is just the configuration space Q itself and all our regularity conditions are obviously satisfied.

As this vertical polarization always exists for cotangent bundles, so does (once the question of the measure has been settled, see section 6) the Schrödinger representation of quantum mechanics on Q, based on the Hilbert space $L^2(Q)$. Whether this is good or bad may be a matter of debate (after all, in section 3 we had understood the emergence of $L^2(Q)$ for $Q = \mathbf{R}^n$ as a consequence of the Stone - von Neumann theorem which is not available for general Q), but this is what GQ predicts.

There are real polarizations that are not vertical polarizations of some cotangent bundle, but there are not many more possibilities satisfying our rather stringent regularity conditions. To see an example of such a polarization, let us go back to the cylinder $M = T^*S^1$ discussed as example 1 of section 4.2. Instead of choosing the vertical polarization, spanned by $(\partial/\partial p)$, we can also choose a 'horizontal' polarization spanned by $(\partial/\partial\varphi)$ (as far as being Lagrangian and integrable is concerned there is nothing to prove when $n = 1$). This leads to what is known (for $Q = \mathbf{R}^n$) as the momentum representation. In this case the integral manifolds of D are circles S^1 and $M/D = \mathbf{R}$.[9].

However, in general something like a horizontal polarization (momentum representation) will not exist at all. And even when it does, it will not necessarily be of the naive form 'ψ is a function of the momenta'. This is something that is often ignored and therefore good to keep in mind. Again, see [17]. In fact, although Q is a Lagrangian submanifold of T^*Q (via the zero section, say), it cannot be an integral manifold of some polarization unless Q has the rather special form $Q = T^k \times \mathbf{R}^{n-k}$. This is a consequence of the interesting result that

[9] There are some interesting subtleties arising in this representation, related to the emergence of vacuum angles and the discreteness of the spectrum of the momentum operator on the real line M/D of momenta. See section 6.4 and [17]

(under our regularity conditions) the integral manifold of a real polarization of any symplectic manifold (M, ω) is necessarily of this form. This can easily be proven by showing that the operator ∇ defined by

$$i(\nabla_X Y)\omega = i(X)di(Y)\omega \qquad \text{for } X, Y \in D \tag{5.7}$$

satisfies all the conditions of a (partial) connection and restricts to a torsion free and flat affine connection on each leaf of D. ∇ is called the *Weinstein connection*. For instance,

$$X, Y \in D \Rightarrow \nabla_X Y \in D \tag{5.8}$$

is equivalent to

$$\omega(\nabla_X Y, Z) = 0 \qquad \forall Z \in D$$

(by the maximality of P) and follows from the formulae of section 2:

$$\begin{aligned} \omega(\nabla_X Y, Z) &= (i(\nabla_X Y)\omega)(Z) \\ &= i(Z)i(X)di(Y)\omega \\ &= L(X)i(Z)i(Y)\omega - i([X, Z])i(Y)\omega = 0 \ . \end{aligned} \tag{5.9}$$

Likewise the property that ∇ is torsion free,

$$\nabla_X Y - \nabla_Y X = [X, Y] \ , \tag{5.10}$$

can be established by calculating

$$\begin{aligned} i(\nabla_X Y - \nabla_Y X)\omega &= i(X)di(Y)\omega - i(Y)di(X)\omega \\ &= i(X)L(Y)\omega - L(Y)i(X)\omega = i([X, Y])\omega \ , \end{aligned} \tag{5.11}$$

etc.. . . Thus the most general possibility is indeed an integral manifold of the form $T^k \times R^{n-k}$. The two above examples are of the type $k = 0$ and $k = n$ respectively. And if the leaves are simply connected ($k = 0$) then essentially the only possibility is the vertical polarization.[10]

This concludes our discussion of real polarizations. Polarizations with $k \neq 0$ have their subtleties and generally force one to consider either distributional or cohomological wave functions, see section 6.4.

5.3 Kähler Polarizations

Kähler polarizations are characterized by the condition $P \cap \bar{P} = \{0\}$. They have this name because every Kähler manifold (complex manifold with a compatible symplectic structure) has a natural Kähler polarization and conversely the existence of such a polarization implies that (M, ω) is (pseudo) Kähler (the 'pseudo' referring to the possible indefiniteness of the Kähler metric).

[10] The precise statement is [13] that if $k = 0$ and if there is some Lagrangian submanifold N of (M, ω) intersecting each leaf nicely (transversally) in exactly one point, then there is a natural identification of M with T^*N.

We shall first take a look at what this means and how it works in the case of a $2n$-dimensional symplectic vector space (V,ω) (where, for concreteness, we can think of $V = T^*\mathbf{R}^n$). We begin with some linear algebra. V is a real vector space. A complex structure on V is a linear transformation $J : V \to V$ with $J^2 = -1$. Such a J gives V the structure of a complex vector space where multiplication by the complex number $a + ib$ is defined by

$$(a+ib)v := av + bJv \quad . \tag{5.12}$$

The complex structure J is called compatible with the symplectic structure ω if

$$\omega(Jv, Jw) = \omega(v,w) \qquad \forall v,w \in V \quad . \tag{5.13}$$

In that case, $\omega(v, Jw)$ is symmetric in v and w and defines a non-degenerate symmetric bilinear form $g(.,.)$ and a Hermitian metric $h(.,.)$ on V via

$$\begin{aligned} g(v,w) &:= \omega(v,Jw) \ , \\ h(v,w) &:= g(v,w) + i\omega(v,w) \ . \end{aligned} \tag{5.14}$$

h is antilinear in the first entry and linear in the second so that e.g.

$$h(Jv,w) = -ih(v,w) \quad . \tag{5.15}$$

J is called positive if g is positive definite. A symplectic structure with a compatible complex structure is called a pseudo Kähler structure and a Kähler structure if J is positive.

V can be complexified in the obvious (J-independent) way, $V \to V^c$, and V^c is a complex $2n$-dimensional vector space. If J is a complex structure on V then J can be diagonalized in V^c. The $\pm i$ eigenspaces of J are denoted by $V^{(1,0)}$ and $V^{(0,1)}$ and are spanned by vectors of the form $v \mp iJv$. Obviously, $V^{(1,0)}$ and $V^{(0,1)}$ are complex n-dimensional complex conjugates of each other and satisfy

$$V^{(1,0)} \cap V^{(0,1)} = \{0\} \quad .$$

If J is compatible with ω, then $V^{(1,0)}$ and $V^{(0,1)}$ are Lagrangian subspaces of V^c. Conversely, a Lagrangian subspace P of V^c satisfying $\bar{P} \cap P = \{0\}$ defines a compatible complex structure on V such that P is its $+i$ or $-i$ eigenspace in V^c (we will, by a slightly misleading usage of terms, refer to the latter as a *holomorphic polarization* because the corresponding polarized states can be represented by holomorphic functions).

Comparing with our above definition of a Kähler polarization we see that a Kähler polarization equips each tangent space of M with a compatible complex structure. A smoothly varying complex structure on the tangent bundle of a manifold M is called an almost complex structure. If this almost complex structure is integrable in the sense that the $\pm i$ eigenbundles are integrable, J is called a complex structure on M and gives M the structure of a complex manifold (i.e. there are local holomorphic coordinates with holomorphic transition functions). Thus, because polarizations are integrable, a Kähler polarization gives (M,ω) the structure of a complex manifold with a compatible symplectic structure. Such manifolds are called (pseudo) Kähler manifolds. And conversely every Kähler structure (J,ω) on M defines a positive holomorphic Kähler polarization of (M,ω) via $P=T^{(0,1)}M$, the $-i$ eigenspace subbundle of TM^c.

In local holomorphic coordinates z^k one has

$$\omega = i\omega_{jk}dz^j d\bar{z}^k \quad , \qquad \bar{\omega}_{jk}=\omega_{kj} \tag{5.16}$$

(it would be better to introduce barred and unbarred indices at this point, but I will refrain from doing so). Locally, any Kähler form can be written as

$$\omega = i\partial\bar{\partial}K \tag{5.17}$$

for some real valued function K, the Kähler potential, where

$$\begin{aligned}
&\partial = dz^k \wedge \tfrac{\partial}{\partial z^k} \quad , \qquad \bar{\partial} = d\bar{z}^k \wedge \tfrac{\partial}{\partial \bar{z}^k} \quad , \\
&d = \partial + \bar{\partial} \quad , \\
&\partial^2 = \bar{\partial}^2 = \partial\bar{\partial}+\bar{\partial}\partial = 0 \quad .
\end{aligned} \tag{5.18}$$

Thus natural local symplectic potentials on a Kähler manifold are $i\bar{\partial}K$ and $-i\partial K$.

All this is best illustrated in the case of a flat phase space $M=T^*\mathbf{R}^n$ with coordinates (q^k,p_l) and the canonical symplectic form. We will give it the structure of a (flat) Kähler manifold, $\mathbf{R}^{2n}\sim\mathbf{C}^n$, by introducing the complex coordinates

$$z^k = \tfrac{1}{\sqrt{2}}(p_k + iq^k) \tag{5.19}$$

corresponding to the complex structure defined by

$$J(\partial/\partial p_k) = (\partial/\partial q^k) \quad , \qquad J(\partial/\partial q_k) = -(\partial/\partial p_k) \quad . \tag{5.20}$$

The symplectic form can be written as

$$\begin{aligned}
\omega &= i\delta_{kl}dz^k d\bar{z}^l = i\partial\bar{\partial}K \quad , \\
K &= \delta_{kl}z^k\bar{z}^l = |z|^2 \quad .
\end{aligned} \tag{5.21}$$

The holomorphic polarization P is spanned by the vectors $\partial/\partial\bar{z}^k$ or, equivalently, by the Hamiltonian vector fields of the coordinate functions z^k. Later on we will use the symplectic potential $\theta_K=-i\partial K$ which vanishes on P so that the covariant derivative along directions in P takes the particularly simple form $D(\partial/\partial\bar{z}^k)=(\partial/\partial\bar{z}^k)$. This has the advantage that P-polarized sections can be identified directly with holomorphic functions in the corresponding trivialization. Generally, a connection potential vanishing on a given polarization P, $\beta|_P=0$, is called *adapted to* P. Under our regularity conditions local adapted potentials always exist.

Another example of a Kähler manifold is the two-sphere which we investigated from the point of view of prequantiztion in section 4.2. For the third description of its prequantum line bundle we introduced complex coordinates on $S^2\sim\mathbf{CP}^1$.

We now recognize the symplectic form (4.26) as a Kähler form with Kähler potential

$$K = \hbar \log(1 + |z|^2) \tag{5.22}$$

The local symplectic potentials given in (4.27) are also adapted to the holomorphic polarization spanned locally by $(\partial/\partial\bar{z}_{\pm})$.

6 Geometric Quantization III: Quantization

Now finally, after having accumulated all these bits and pieces of information, we come to the quantization of symplectic manifolds. This involves the determination of the quantum Hilbert space $\mathcal{H}_P$ corresponding to a polarization P, and the construction of operators acting on $\mathcal{H}_P$.

After some general remarks we will look at the question how to construct operators on the quantum Hilbert space. This turns out to be straightforward for observables preserving the polarization, but a rather drastic modification of that procedure is required to associate operators to observables whose flow moves the polarization. This will lead us to the pairing construction of Blattner, Kostant, and Sternberg and to BKS kernels whose construction I will sketch in the simplest of cases (a family of positive Kähler polarizations).

We will then deal seperately with the three examples of polarizations we have discussed above: vertical polarizatons (section 6.2), Kähler polarizations (section 6.3), and real polarizations with non-simply connected leaves (section 6.4). They all have their particular complications and pitfalls. Initially one is likely to expect the example of a vertical polarizations to be the least problematic of the lot, coresponding just to the familiar Schrödinger representation generalized to curved configuartion spaces. However, it turns out that there is no natural measure on the space of polarized states and although certain more or less ad hoc resolutions of this problem are conceivable one is eventually confronted with the necessity of modifying the entire quantization scheme. I will present a version of the half-density quantization scheme which is fairly easy to understand. Eventually this would have to be replaced by the significantly less transparent half-form quantization scheme, but this will only make a brief appearance in the following.

In the case of a general real polarization the situation is even worse because there may be no polarized sections at all. One is then forced to permit distributional wave functions to appear whose support is concentrated on the so-called Bohr-Sommerfeld varieties in M/D. In section 6.4 we will see how these arise in the case of the cylinder and the harmonic oscillator in the energy representation.

Given all these difficulties it may thus come as a surprise that in the case of Kähler polarizations there is a natural measure on the space of polarized states and no obstruction to constructing $\mathcal{H}_P$. In fact, a positive Kähler polarization just picks out a particular subspace of the prequantum Hilbert space $\mathcal{H}$. Unfortunately, this construction fails to give correct results in even the simplest of quantum mechanical examples, the harmonic oscillator. The same (wrong) spectrum of the harmonic oscillator is also predicted if one tries to quantize the system in a real polarization. The required modification is again that which works in the case of vertical polarizations, namely half-form or metaplectic quantization. This scheme also appears to account correctly for changes in the polarization and for the quantization of certain operators which do not preserve a given polarization, but the general theory is far from completely understood at the moment.

6.1 Polarized States and the Construction of Quantum Operators

We begin with some general remarks. Let us fix a prequantization (L, D) of a symplectic manifold (M, ω) and a strongly admissible polarization P. The basic idea is, as mentioned repeatedly above, to construct the quantum Hilbert space from the linear space $P(L)$ of P-polarized sections of (L, D), i.e. of (smooth) sections ψ of L satisfying

$$D(X)\psi = 0 \qquad \forall X \in P \ . \tag{6.1}$$

Ideally, one would like to go ahead and define the quantum Hilbert space $\mathcal{H}_P$ as $\mathcal{H}_P := \mathcal{H} \cap P(L)$, i.e. as the space of P-polarized sections of L square integrable with respect to the Liouville measure on (M, ω). This, however, usually does not work, either because polarized sections are not square integrable (e.g. the Schrödinger wave functions, which depend only on the coordinates so that the momentum integral will diverge), or because there are no smooth polarized sections of L at all. These difficulties are best illustrated by concrete examples and we will do this below. First, however, we will come to the issue of quantum operators acting on polarized states, which can be stated and addressed in more generality.

In section 4, to every function on M we were able to associatexd a prequantum operator $\mathcal{P}(f)$,

$$\mathcal{P}(f) = -i\hbar D(X_f) + f \ \ ,$$

acting on the sections of L and satisfying the quantization conditions Q1-Q4; in particular,

$$Q4: \ \ [\mathcal{P}(f), \mathcal{P}(g)] = -i\hbar\mathcal{P}(\{f, g\}) \qquad \forall f, g \in C^\infty(M) \ . \tag{6.2}$$

On the basis of our experience with flat space quantum mechanics and keeping in mind the Groenewald - van Hove theorem we expect to have to sacrifice at least parts of (6.2) when moving from prequantization to quantization. We also expect to have to modify the assignment $\mathcal{P}$ in general because we don't expect nor want all quantum operators to be at most first order differential operators. In fact, we know from quantum mechanics (3.9) that the usual kinetic energy term quadratic in the momenta should come out as proportional to the Laplacian, at least if $Q = \mathbf{R}^n$.

The first step is to check which of the prequantum operators $\mathcal{P}(f)$ can be promoted directly to operators on $P(L)$ (the necessary modifications due to half-density quantization are irrelevant for our present purposes and will be given in the next section). The requirement is obviously that $\mathcal{P}(f)$ map polarized states to polarized states, i.e.

$$D(X)\psi = 0 \ \ \forall X \in P \Rightarrow D(X)\mathcal{P}(f)\psi = 0 \ \ \forall X \in P \ \ . \tag{6.3}$$

The obstruction to this comes from the term $D([X_f, X])\psi$ so that (6.3) is equivalent to

$$\begin{aligned} (6.3) \quad &\Leftrightarrow \quad [X_f, X] \in P \qquad \forall X \in P \\ &\Leftrightarrow \quad [X_f, P] \subset P \ \ . \end{aligned} \tag{6.4}$$

This is a very intuitive result because it says that a classical observable f defines an operator on the space of P-polarized states via the prequantum assignment $f \to \mathcal{P}(f)$ provided that its flow leaves the polarization P invariant. This follows also from the argument given in (4.11-4.16): if Φ_t^f leaves P invariant, one can use (4.16) directly to define the operator $\mathcal{Q}(f)$ on polarized states. Let us call the space of these functions, which is not particularly large, $C_P^\infty(M)$. It is closed under Poisson brackets and contains, in particular, the functions whose Hamiltonian vector fields span the polarization. The latter are diagonal on polarized states in the sense that

$$\mathcal{Q}(f)\psi = f\psi \ \ . \tag{6.5}$$

For example, in the case of the vertical polarization of a cotangent bundle one finds

$$\begin{aligned} f \in C_P^\infty(T^*Q) \quad &\Leftrightarrow \quad [X_f, \tfrac{\partial}{\partial p_k}] \in P \ \ \forall k \\ &\Leftrightarrow \quad \left(\tfrac{\partial^2}{\partial p_k \partial q^l} f\right) \tfrac{\partial}{\partial p_l} - \left(\tfrac{\partial^2}{\partial p_k \partial p_l} f\right) \tfrac{\partial}{\partial q^l} \in P \ \ \forall k \\ &\Leftrightarrow \quad \tfrac{\partial^2}{\partial p_k \partial p_l} f = 0 \ \ , \end{aligned} \tag{6.6}$$

so that $f \in C_P^\infty(T^*Q)$ iff it is at most linear in the momenta,

$$f \in C_P^\infty(T^*(Q)) \Leftrightarrow f(q,p) = f_0(q) + f^k(q)p_k \ \ . \tag{6.7}$$

For such f the quantum operator is

$$(\mathcal{Q}(f)\psi)(q) = f_0(q)\psi(q) - i\hbar f^k \left(\tfrac{\partial}{\partial q^k}\psi\right)(q) \ \ . \tag{6.8}$$

This expression will still have to be modified by a correction term coming from the measure, see (6.25) below. By the same reasoning as above one finds that the only real valued observables preserving the holomorphic polarization on $T^*\mathbf{R}^n \sim \mathbf{C}^n$ (see section 5.3) are of the form

$$f(z,\bar{z}) = f_0 + f_k z^k + \bar{f}_k \bar{z}^k + f_{kl} z^k \bar{z}^l \ \ , \tag{6.9}$$

where $f_0 \in \mathbf{R}$ and f_k and $f_{kl} = \bar{f}_{lk}$ are complex constants. This makes the holomorphic representation particularly suitable for the quantization of the harmonic oscillator whose Hamiltonian is proportional to $|z|^2$, see section 6.3.

For the time being, this is all I have to say about polarization preserving observables and we are now confronted with the question what to do with functions which move the given polarization P. Another way of stating this, in which the polarization plays a more passive role, is that via its canonical lift $\widehat{\Phi}_t^f$ (4.14) to (L, D) the Hamiltonian flow on M generated by such a function f moves a P-polarized state ψ out of $P(L)$. The evolved state

$$\psi_t \equiv \widehat{\Phi}_t^f \psi \tag{6.10}$$

is now polarized with respect to the pulled-back polarization

$$P_t \equiv \left(\Phi_t^f\right)^* P \quad , \tag{6.11}$$

$$\psi \in P(L) \Rightarrow \psi_t \in P_t(L) \quad . \tag{6.12}$$

Thus evidently what we need is a way of relating states to each other which are polarized with respect to different polarizations. To proceed, let us make the simplifying assumption that the quantum Hilbert spaces $\mathcal{H}_t \equiv \mathcal{H}_{P_t}$ constructed from the family of polarizations P_t can all be regarded as subspaces of the prequantum Hilbert space $\mathcal{H}$. This assumption holds, for instance, when P_t is a family of positive Kähler polarizations. In that case we have the orthogonal (wrt the scalar product on $\mathcal{H}$) projections

$$\begin{aligned} \Pi_{t't} \quad &: \quad \mathcal{H}_t \rightarrow \mathcal{H}_{t'} \quad , \\ \Pi_{0t} \equiv \Pi_t \quad &: \quad \mathcal{H}_t \rightarrow \mathcal{H}_0 \equiv \mathcal{H}_P \end{aligned} \tag{6.13}$$

available to project the state ψ_t back to $\mathcal{H}_P$. Thus, in analogy with (4.16) we can now attempt to define the quantum operator $\mathcal{Q}(f)$ on $\mathcal{H}_P$ by

$$\begin{aligned} \mathcal{Q}(f)\psi \quad &:= \quad -i\hbar \tfrac{d}{dt} \left(\Pi_t \psi_t\right) |_{t=0} \\ &= \quad -i\hbar \tfrac{d}{dt} \left(\Pi_t \widehat{\Phi}_t^f \psi\right) |_{t=0} \quad . \end{aligned} \tag{6.14}$$

This is the basic idea of the *Blattner - Kostant - Sternberg construction.* Even in this situation, determining when (6.14) exists, when it exists as a self-adjoint operator and when the projections are unitary is a highly non-trivial problem.

In the general case, when the quantum Hilbert spaces cannot be regarded as subspaces of $\mathcal{H}$, the orthogonal projection operators have to be replaced by some other, less natural, linear maps from one Hilbert space $\mathcal{H}_P$ to the other, $\mathcal{H}_{P'}$, and the problem becomes correspondingly more difficult.

One case which is tractable is the following: Consider a symplectic vector space, regarded as the flat symplectic manifold $M = T^*\mathbf{R}^n$, and let P, P' and P'' be the vertical, horizontal, and holomorphic polarization respectively. In all three cases we have an irreducible representation of the Heisenberg group on the corresponding quantum Hilbert space. Thus, by the Stone - von Neumann theorem (section 3), the existence of unitary(!) linear operators from $\mathcal{H}_P$ to $\mathcal{H}_{P'}$ and $\mathcal{H}_P$ to $\mathcal{H}_{P''}$ is guaranteed. The former is just the Fourier transform from the coordinate to the momentum representation, and the latter is the

Bargmann transform realizing the unitary equivalence of the configuration and holomorphic representations.

It can, moreover, be shown that in the case of a vertical polarization the quantum operator associated to the kinetic energy function in flat space comes out correctly to be the Laplacian (plus scalar curvature terms in the case of a curved configuration space). The calculation is, unfortunately, too lengthy to be reproduced here, see [14, 13].

By such considerations one is also naturally led to the important and subtle question to which extent the resulting quantum theory depends on the choice of polarization and which polarizations give rise to unitarily equivalent theories. Some interesting progress has been made on this question recently [5] motivated by topological and conformal field theory. Unfortunately, I will not be able to go into this here. For an explanation of half-form quantization from this point of view see [13].

6.2 The Vertical Polarization and Half-Densities

We now discuss the construction of the quantum Hilbert space in the most familiar looking case of the vertical polarization. Recall that this is the polarization $P = D^c$ spanned by the tangents to the fibers of a cotangent bundle $M = T^*Q$. The prequantum line bundle (L, D) is trivializable and in terms of the trivializing section $s_0(m) = (m, 1)$ the Hermitian structure on the fibers and the compatible connection potential are

$$< s_0, s_0 > (m) = 1 \quad , \qquad \beta = (1/\hbar)\theta = (1/\hbar)p_k dq^k \quad . \tag{6.15}$$

β is adapted to the vertical polarization so that the covariant derivative along the fibers of P is simply the ordinary derivative acting on functions on T^*Q,

$$D(\partial/\partial p_k) = \partial/\partial p_k \quad ,$$

Thus polarized sections corespond to functions which are independent of the momenta p_k, i.e. to functions on Q. We now need to turn this into a Hilbert space. The first guess would be to use the Hilbert space structure on the prequantum Hilbert space $\mathcal{H}$ and to define the quantum Hilbert space $\mathcal{H}_P$ as $\mathcal{H} \cap P(L)$, i.e. as the space of sqare integrable P-polarized sections of L. Unfortunately, this space is empty as p_k-independent wave functions are certainly not square integrable with respect to the Liouville measure - the integral over the fibers diverges.

In this particular example a (partial) remedy to the problem immediately comes to mind: one should integrate polarized sections not over M but over Q (which is, more invariantly, to be regarded as the space M/D of leaves of the polarization). However, there is no natural measure on Q. If a metric on Q is given, perhaps implicitly via a Hamiltonian of the form $H = g^{kl}p_k p_l/2 + \ldots$, then one can construct the density $\sqrt{g}d^n q$ ($g = \det g_{kl}$) which can be used to define a scalar product on $P(L)$. Alternatively, and more generally, one can try to work from the outset with a bundle whose (polarized) sections are square-roots of densities (n-forms) on Q so that the scalar product of two such objects is automatically well defined. This leads to the *half-density quantization* scheme.

Under the assumption that Q is oriented we can, following [13], proceed as follows (for the full fledged half-density quantization scheme see [14] or the first edition of [13]). The material will be presented in such a way that the extension to other real polarizatons (with simply connected leaves) should be self-evident. Let us introduce the (determinant) line bundle

$$\mathrm{Det}(Q) := \Lambda^n \left((T^*Q)^c\right) \tag{6.16}$$

whose sections are complex valued volume forms on Q. As we assumed Q to be orientable (and oriented) we can form the square root $\mathrm{Det}^{1/2}(Q)$, e.g. by choosing real and positive transition functions for $\mathrm{Det}(Q)$ and using their positive square roots to define $\mathrm{Det}^{1/2}(Q)$. Via the projection $\pi : T^*Q \to Q$ we can pull these line bundles back to T^*Q where we denote them by

$$\begin{aligned} \pi^*(\mathrm{Det}(Q)) &=: K_D \\ \pi^*(\mathrm{Det}^{1/2}(Q)) &=: \delta_D = (K_D)^{1/2} \end{aligned} \tag{6.17}$$

It should be kept in mind that, as bundles over T^*Q, their spaces of sections are now $C^\infty(T^*Q)$-modules (i.e. sections can be multiplied by functions on T^*Q). Thus sections of K_D are not necessarily pull-backs of volume forms on Q. We would now like to replace the prequantum line bundle L by $L_D = L \otimes \delta_D$. In order to define P-polarized sections we need the notion of a covariant derivative of sections of δ_D along P. We define the covariant derivative of a section μ of K_D along P by

$$D(X)\mu = i(X)d\mu \ \ . \tag{6.18}$$

This (partial) connection is flat,

$$\left([D(X), D(Y)] - D([X,Y]\right)\mu = 0 \tag{6.19}$$

(because $d\mu$ can have at most one 'vertical' direction) and μ is the pull-back of an n-form on Q iff it is covariantly constant along P. (6.18) gives rise to a covariant derivative on sections ν of δ_D via the obvious definition

$$D(X)\nu^2 = 2\nu D(X)\nu \qquad \text{or} \qquad D(X)\mu^{1/2} = \tfrac{1}{2}\mu^{-1/2}D(X)\mu \ \ . \tag{6.20}$$

For vector fields preserving the vertical polarization one can also define the Lie derivative of sections of K_D via the usual formula $L(.) = di(.) + i(.)d$ for differential forms. Obviously, $L(X)\mu = D(X)\mu$ if $X \in P$. $L(.)$ extends to δ_D in the same way as $D(.)$ in (6.20).

Sections of L_D are of the form $s\nu$ where s is a section of L and ν a section of δ_D and we call $s\nu$ a P *wave function* if

$$D(X)(s\nu) \equiv (D(X)s)\nu + s\,D(X)\nu = 0 \qquad \forall X \in P \ \ . \tag{6.21}$$

As the product of two sections of δ_D is a section of K_D and the scalar product on the fibers,

$$< s_1\nu_1, s_2\nu_2 > := < s_1, s_2 > \bar{\nu}_1\nu_2 \ \ , \tag{6.22}$$

is parallel along P by (6.21), we can identify it with an n-form on Q. We have thus arrived at our goal of defining a natural scalar product on the space of polarized sections, namely

$$<< s_1\nu_1, s_2\nu_2 >>:= \int_Q < s_1, s_2 > \bar{\nu}_1\nu_2 \quad . \tag{6.23}$$

The quantum Hilbert space $\mathcal{H}_P$ is now defined to be the L^2-completion of the space of smooth P wave functions with resepct to this scalar product.

The construction of quantum operators acting on $\mathcal{H}_P$ now proceeds exactly as in section 6.1. For a polarization preserving function f we saw that the prequantum operator $\mathcal{P}(f)$ was well defined on the space $P(L)$ of P-polarized sections of L. It thus remains to define its action on sections of δ_P. Keeping in mind that $\mathcal{P}(f)$ is nothing but the generator of the canonical flow of X_f on sections of L (4.16) and that its generator on differential forms is the Lie derivative, we set

$$\mathcal{Q}(f)(s\nu) = (\mathcal{P}(f)s)\nu - i\hbar s L(X_f)\nu \quad , \tag{6.24}$$

In particular, if we fix a volume element μ on Q then P wave functions are of the form $s_0\psi(q)\mu^{1/2}$ with scalar product

$$\int_Q \bar{\psi}_1\psi_2\mu$$

and the required modification to (6.8) is

$$(\mathcal{Q}(f)\psi)(q) = \left(f_0(q) - \tfrac{1}{2}i\hbar\, \mathrm{div}_\mu(f^k)(q)\right)\psi(q) - i\hbar f^k\left(\tfrac{\partial}{\partial q^k}\psi\right)(q) \quad . \tag{6.25}$$

Here the divergence of a vector field Y on Q with respect to μ is the function on Q defined by

$$\mathrm{div}_\mu(Y)\mu := L(Y)\mu = di(Y)\mu \quad . \tag{6.26}$$

This correction term can be regarded as arising from a particular symmetric 'operator ordering' of the classical expression $f^k p_k$, an issue which, as such, is not present in GQ. It vanishes, e.g., when $\mu = \sqrt{g}d^nq$ and Y is a Killing vector of the metric g_{kl}.

The construction of operators corresponding to observables not preserving the vertical polarization proceeds via BKS kernels whose naive construction I indicated in section 6.1. Matters are complicated by the fact that one has to take into account the variation in δ_{D_t}. In order to keep track of relative phase factors in the corresponding Hilbert spaces one eventually has to modify the definition of δ_D in such a way that it does not depend on the orientation of M/D. The resulting quantization is half-form or metaplectic quantization. Virtually the same correction term as above appears as the metaplectic correction to a polarization preserving operator in other polarizations. In that form it will turn out to be responsible for the ground state energy of the harmonic oscillator (see the discussion in sections 6.3 and 6.4).

6.3 Kähler Quantization and Metaplectic Correction

We now deal with the simpler case of a positive Kähler polarization. We thus consider a Kähler manifold (M,ω,J), a prequantum line bundle (L,D), and choose $P=T^{(0,1)}M$ (see section 5.3) to be the polarization spanned by the $-i$ eigenspaces of J. Locally, P is spanned by the vector fields $(\partial/\partial\bar{z}^k)$ where the z^k are holomorphic coordinates on M. The space of holomorphic (P-polarized) sections of (L,D) can be shown to be a closed subspace of the prequantum Hilbert space $\mathcal{H}$. It is thus a Hilbert space in its own right which we take to be the quantum Hilbert space $\mathcal{H}_P$ of the system.

To obtain a more explicit description of $\mathcal{H}_P$ it is useful to work with local connection potentials adapted to P. We noted in section 5.3 that a convenient choice is $\theta_K=-i\partial K$ where K is the Kähler potential. Let us see how this works in the case of a Kähler vector space. According to (5.21), K is given by

$$K=\delta_{kl}z^k\bar{z}^l \quad ,$$

so that

$$\theta_K=-i\delta_{kl}\bar{z}^l dz^k \quad . \tag{6.27}$$

To account for this change from θ to θ_K, we have to change the local trivializing section. Tracing through the formulae of section 4 one finds that s_0 is to be replaced by

$$s_K=\exp\left(-\tfrac{1}{4\hbar}(\delta_{kl}q^kq^l+\delta^{kl}p_kp_l-2ip_kq^k)\right)s_0 \quad . \tag{6.28}$$

Polarized sections of the (trivial) prequantum line bundle are thus of the form

$$s(z,\bar{z})=s_K(z,\bar{z})\psi(z) \tag{6.29}$$

where $\psi(z)$ is a holomorphic function on $\mathbf{C}^n$. It follows from $<s_0,s_0>=1$ and (6.28) that the Hermitian structure in this trivialization is given by

$$<s_K,s_K>(z,\bar{z})=\exp(-|z|^2/\hbar)=\exp(-K/\hbar) \quad . \tag{6.30}$$

From this we can also read off that with respect to the canonical symplectic potential and the section s_0 polarized wave functions are of the form

$$\phi(z,\bar{z})=\psi(z)\exp\left(-|z|^2/2\hbar\right) \quad ,$$

as could of course also have been deduced directly from solving $D(\partial/\partial\bar{z}^k)(s_0\phi)=0$ in this trivialization. Either way $\mathcal{H}_P$ can be identified with the space of holomorphic functions on $\mathbf{C}^n$ with scalar product

$$<<\psi_1,\psi_2>>=\int_{\mathbf{C}^n}d^np\,d^nq\,\overline{\psi_1(z)}\psi_2(z)\exp(-K/\hbar) \quad . \tag{6.31}$$

It is clear from this expression that the Hilbert space would have been empty if we had chosen a non-positive Kähler polarization (with K corresponding to an indefinite quadratic form) because there would not have been any normalizable holomorphic functions.

As we already noted above, this holomorphic representation with

$$\mathcal{Q}(z^k)\psi(z) = z^k\psi(z) \quad , \qquad \mathcal{Q}(\bar{z}^k)\psi = \hbar\tfrac{\partial}{\partial z^k}\psi(z) \quad , \tag{6.32}$$

is unitarily equivalent to the Schrödinger representation. It is also known as the *oscillator representation*, with $\mathcal{Q}(z^k)$ and $\mathcal{Q}(\bar{z}^k)$ interpreted as creation and annihilation operators respectively. As one has a direct particle (occupation number) interpretation in this representation, it is the conventional starting point in canonical quantum field theory.

This representation is particularly convenient for quantizing the harmonic oscillator as its Hamiltonian (we take $n=1$ for notational convenience)

$$H(q,p) = \tfrac{1}{2}(p^2+q^2) = z\bar{z} \tag{6.33}$$

preserves the holomorphic polarization, c.f. (6.9). Acting on holomorphic functions, the corresponding (pre-) quantum operator is

$$(\mathcal{Q}(H)\psi)(z) = \hbar z\tfrac{\partial}{\partial z}\psi(z) \quad . \tag{6.34}$$

which spells doom because its eigenfunctions are the monomials z^n with eigenvalues $n\hbar$. It is, of course, well known from quantum mechanics that this is incorrect and that the spectrum should be shifted by the ground state energy $\frac{1}{2}\hbar$. In the standard treatment this term arises from symmetrizing

$$H = z\bar{z} \to \tfrac{1}{2}(z\bar{z}+\bar{z}z)$$

before substituting the operators (6.32) for z and $\bar{z}$ so that one obtains the quantum operator

$$\widehat{H} = \hbar(z\tfrac{\partial}{\partial z}+\tfrac{1}{2}) \quad . \tag{6.35}$$

This shows that, in spite of the fact that everything has run so smoothly so far in the case of positive Kähler polarizations, the necessity arises to modify the quantization procedure for Kähler manifolds as well. Interestingly, it turns out that the half-form quantization scheme which solves a number of problems arising in the context of real polarizations also takes care of the present shortcoming. Namely, one of the consequences of the metaplectic correction is that it gives rise to an additional term in the expression for the quantum operator of a polarization preserving observable (similar to the one encountered in (6.25)). A recipe for constructing the corresponding quantum operator will be given below. For the harmonic oscillator the effect of this will be precisely to replace (6.34) by (6.35). This is quite remarkable because a priori it is not at all clear what half-forms have to do with operator ordering.

A general rule of thumb for including the metaplectic correction to the operator corresponding to a polarization preserving obervable is the following (see [14]). Let the polarization P be spanned by the n complex vector fields X_k, $k=$

$1, \ldots, n$. If f preserves the polarization P, there is a matrix $\mathbf{a}(f) \equiv (a_{kl}(f))$ of functions on M satisfying

$$[X_f, X_k] = a_k^l(f) X_l \quad . \tag{6.36}$$

In terms of this matrix, the half-form corrected quantum operator is

$$\mathcal{Q}(f) = \mathcal{P}(f) - \tfrac{1}{2} i\hbar \, \mathrm{tr}(\mathbf{a}(f)) \quad . \tag{6.37}$$

In the case of the harmonic oscillator, P is spanned by $\frac{\partial}{\partial \bar{z}}$ and the Hamiltonian vector field is

$$X_H = i(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) \quad .$$

Thus,

$$[X_H, \frac{\partial}{\partial \bar{z}}] = i\frac{\partial}{\partial \bar{z}} \quad ,$$

leading to the correct(ed) expression

$$\mathcal{Q}(H) = \hbar(z\frac{\partial}{\partial z} + \tfrac{1}{2}) \tag{6.38}$$

and the energy spectrum

$$\mathrm{spec}(\mathcal{Q}(H)) = \{(n + \tfrac{1}{2})\hbar, \;\; n \geq 0\} \quad . \tag{6.39}$$

In the general case of a Kähler manifold (M, ω, J) everything works as above. In particular, expressions like (6.27) and (6.29) are still valid locally and allow one to represent polarized sections locally by holomorphic functions on M. Certain interesting and new features arise when M is compact so that the only globally defined holomorphic functions on M are the constants. To explain these, I will make use of some algebraic geometry (see e.g. [18]).

First of all, we can use local non-vanishing polarized sections of L as trivializing sections. Then the transition functions will be holomorphic and hence give L the structure of a holomorphic Hermitian line bundle over M. The space of holomorphic sections of L can be identified with the zero'th sheaf cohomology group $H^0(M, \underline{L})$ of M with values in the sheaf $\underline{L}$ of germs of holomorphic sections of L. By general theorems, this is finite dimensional so that the quantum Hilbert space will be finite dimensional as well,

$$\dim \mathcal{H}_P = \dim H^0(M, \underline{L}) < \infty \quad . \tag{6.40}$$

It is for this reason that compact symplectic manifolds are usually used to introduce internal degrees of freedom. For instance, if one quantizes $T^*\mathbf{R}^n \times M$ using the vertical polarization in the first and the holomorphic representation in the second factor, the resulting Hilbert space is a tensor product of $L^2(Q)$ with the finite dimensional Hilbert space $\mathcal{H}_P(M)$. If M is e.g. a coadjoint orbit of a group G such that $\mathcal{H}_P(M)$ carries an irreducible representation of G, then the resulting tensor product wave functions are usually interpreted as wave functions taking values in this representation or carrying a representation of G.

I will illustrate this in the case of the two-sphere whose prequantization we had discussed in Example 2 of section 4.2. Recall that we introduced two coordinate patches $U_\pm$ with local complex coordinates $z_\pm$ and that the transition functions for the line bundle L_k were given by $(z_+)^k = (z_-)^{-k}$. Furthermore, for $k=1$ the Kähler potential, symplectic form and adapted symplectic potential were

$$\begin{aligned} K_\pm &= \hbar \log(1+|z_\pm|^2) \ , \\ \omega &= -i\hbar \frac{d\bar{z}_\pm dz_\pm}{(1+|z_\pm|^2)^2} \ , \\ \theta_{K_\pm} &= -i\hbar \frac{\bar{z}_\pm dz_\pm}{1+|z_\pm|^2} \end{aligned}$$

(see (4.26,4.27,5.22)). To determine the global holomorphic sections of L_k in this trivialization, we have to check which local holomorphic functions on $U_\pm$ can be patched together via the transition functions. This is easy. A basis for holomorphic functions on U_+ is given by the monomials $(z_+)^l$ and on U_- by $(z_-)^m$ for l and m non-negative integers. We thus have to find the solutions to the equation

$$(z_+)^l = (z_+)^k (z_-)^m \ .$$

This gives

$$l = k - m$$

which has non-negative integer solutions for $l \leq k$ and $m \leq k$. Thus the dimension of the space of holomorphic sections is $k+1$, precisely the dimension of the spin $k/2$ representation of $SU(2)$. As the patch U_+ covers everything but one point on S^2 we can define the scalar product by integration over U_+ alone. As in (6.30) the Hermitian structure on the fibers of L_k gives an extra contribution to the measure of the form (we now call z_+ simply z)

$$\exp(-kK/\hbar) = (1+|z|^2)^{-k} \tag{6.41}$$

so that overall the scalar product (with the standard normalization) is

$$<< \psi_1, \psi_2 >>= \tfrac{1}{2\pi} \int \frac{i\overline{\psi_1(z)}\psi_2(z) dz \wedge d\bar{z}}{(1+|z|^2)^{k+2}} \tag{6.42}$$

(the additional power of two coming from the symplectic form).

There are two other things worth noting about (6.40). On the one hand, if the Kähler polarization is not positive, then $\dim H^0(M, \underline{L}) = 0$ and the Hilbert space is empty (as in the non-compact case). On the other hand, if L is 'sufficiently positive' then the dimension of $\mathcal{H}_P$ can be computed from the Riemann-Roch theorem. More precisely, the RR theorem expresses the Euler characteristic

$$\chi(M, \underline{L}) := \sum_i (-1)^i \dim H^i(M, \underline{L}) \tag{6.43}$$

in terms of characteristic classes,

$$\chi(M, \underline{L}) = \int_M \mathrm{ch}(L) \mathcal{T}(M) \ . \tag{6.44}$$

Here $\mathrm{ch}(L)$ is the Chern character of L and can be represented by $\exp(\omega/2\pi\hbar)$ while $\mathcal{T}(M)$ is the Todd class of M whose precise form will not interest us. If one replaces L by L^k (and hence ω by $k\omega$) for some positive integer k, then for some sufficiently large value of k the higher cohomology groups in (6.43) will vanish and the right hand side of (6.44) calculates directly the dimension of $\mathcal{H}_P$. In particular, $\mathcal{T}(M)$ does not contribute for $k \to \infty$ and one finds (see (2.6))

$$\dim H^0(M, L^k) \to \frac{k^n}{(2\pi\hbar)^n n!} \int_M \omega^n = (\tfrac{k}{2\pi\hbar})^n \mathrm{Vol}_\omega(M) \ . \tag{6.45}$$

The limit $k \to \infty$ can be interpreted as the semi-classical limit $\hbar \to 0$ and one thus recovers the folklore wisdom that in the semi-classical limit the number of quantum states is equal to the number of cells in phase space (measured in units of $\hbar$). Equation (6.45) has been used recently by Witten to calculate the symplectic volume of certain moduli spaces of flat connections from quantum field theory [19].

6.4 Real Polarizations and Bohr-Sommerfeld Varieties

In this final section we will, following [14], take a brief look at the complications which can arise when the leaves of a real polarization are not simply connected. In that case, there can be global integrability conditions to the equation (5.1) defining polarized states. These lead to the necessity of permitting distributional wave 'functions' whose support is restricted to lower dimensional subvarieties of M.

Let $P = D^c$ be a real polarization and (L, D) a prequantization of (M, ω). We denote by Λ a leaf (integral manifold) of D and, more specifically, by Λ_m the leaf passing through $m \in M$. The operator D, restricted to covariant differentiation along P, induces a flat connection D_Λ on $L|_\Lambda$. If Λ is not simply connected, then it is possible for D_Λ to have non-trivial holonomy along the non-contractible loops in Λ. On the other hand, the condition (5.1) implies that ψ_Λ is a covariantly constant section of L_Λ. It is thus invariant under parallel transport and, in particular, cannot pick up a phase from the non-trivial holonomy of D_Λ. Therefore either $\psi_\Lambda = 0$ or the holonomy group of D_Λ is trivial (i.e. D_Λ is the trivial flat connection). Call $S \subset M$ the union of points in M such that D_{Λ_m} is trivial. S is known as the *Bohr - Sommerfeld variety* and $S = M$ if all the Λ_m are simply connected. From the above it follows that polarized sections of (L, D) vanish in the complement of S,

$$\psi \in P(L) \Rightarrow \mathrm{supp}(\psi) \subset S \ . \tag{6.46}$$

Instead of working with such distributional wave functions, it is possible to work with so-called cohomological wave functions (i.e. one trades singularities for cohomology as is familiar from algebraic geometry), see [14].

The relation with the usual Bohr - Sommerfeld quantization conditions is that

$$m \in S \Leftrightarrow \exp(i/\hbar) \oint_\gamma \theta = 1 \tag{6.47}$$

for all loops γ in Λ_m, where θ is a local symplectic potential. In terms of local canonical coordinates (q^k, p_l) one can write θ as $p_k dq^k$ and thus (6.47) becomes the quantization condition

$$\oint_\gamma p_k dq^k = 2\pi\hbar n_\gamma \ , \qquad n_\gamma \in \mathbf{Z} \ . \tag{6.48}$$

Taking into account the contribution $\exp(-2\pi i d_\gamma)$ to the holonomy from the flat connection on the bundle of half-forms (d_γ defined up to an integer), one obtains the *modified Bohr - Sommerfeld conditions*

$$\oint_\gamma p_k dq^k = 2\pi\hbar(n_\gamma + d_\gamma) \ . \tag{6.49}$$

As the above discussion was rather abstract, let us now take a look at two simple examples where these distributional wave functions and the corresponding Bohr - Sommerfeld conditions arise quite naturally (and turn out to be important). The first of these is the cylinder, whose prequantization and quantization in the vertical polarization we have already dealt with in Example 1 of section 4.2. Here we shall look at the same model in the momentum representation [17]. This is the representation defined by the horizontal polarization spanned by $\partial/\partial\varphi$. Consequently, polarized sections have to satisfy

$$D^\lambda(\tfrac{\partial}{\partial\varphi})\psi = 0$$

(recall equation (4.23)) or

$$\tfrac{\partial}{\partial\varphi}\psi(\varphi,p) = \tfrac{i}{\hbar}(p - \hbar\lambda)\psi(\varphi,p) \ . \tag{6.50}$$

Polarized sections are thus of the form

$$\psi(\varphi,p) = \exp(\tfrac{i}{\hbar}(p-\hbar\lambda)\varphi)\phi(p) \tag{6.51}$$

where $\phi(p)$ is some function of the momentum. The first thing to note is that this is not what one would naively have expected the momentum representation to look like. The phase factor appears because of the choice of symplectic potential, $\theta = pd\varphi$. If we had been able to choose $-\varphi dp$ as a symplectic potential (adapted to the horizontal polarization), polarized states would have been of the expected form $\phi(p)$. However, this potential is not globally defined, and the fact that we are dealing with quantum mechnics on the circle and not on the real line is reflected in the peculiar form of the wave functions. It is this which guarantees that the vacuum angle λ is equally visible in the momentum representation although the space of momenta itself is topologically trivial.

Now we have to remember that the prequantum Hilbert space is the space of L^2-functions on the cylinder so that, in particular, ψ has to be periodic in φ. From (6.51) it follows that this is only possible if the support of $\phi(p)$ is restricted to those p which satisfy

$$p = (n + \lambda)\hbar$$

for some integer n. This is nothing but the Bohr - Sommerfeld condition (6.48) and obviously leads inevitably to distributional wave functions with support on

$S \sim \mathbf{Z}$. Along the way we have also recovered the discrete and shifted spectrum (4.25) of the momentum operator (diagonal in this representation).

As a second example let us take a brief look at the one-dimensional harmonic oscillator in the real energy representation. This corresponds to the polarization defined by the Hamiltonian vector field X_H of the Hamiltonian H. In polar coordinates (r, φ) on the plane we have

$$\begin{aligned} H(r,\varphi) &= \tfrac{1}{2}r^2 \ , \\ \omega &= r dr \wedge d\varphi = d(H d\varphi) \ , \\ X_H &= \tfrac{\partial}{\partial \varphi} \ . \end{aligned} \tag{6.52}$$

To avoid having to deal with singular polarizations, we remove the origin from the plane and thus the phase space is $\mathbf{R}^2 \backslash \{0\}$ with the above symplectic form. Polarized wave functions are of the form

$$D(X_H)\psi = 0 \Rightarrow \psi(r,\varphi) = \exp(\tfrac{i}{2\hbar}r^2)\phi(r) \ , \tag{6.53}$$

so that single valuedness of ψ imposes the (Bohr - Sommerfeld) condition

$$\tfrac{1}{2}r^2 = n\hbar \ .$$

Unfortunately, this is only almost correct as it leads to the same wrong energy spectrum we initally found in the previous section. As is also apparent from the form of the wave function, topologically this example is the same as the cylinder we discussed above, so that we could obtain the correct spectrum by fine-tuning the value of the vacuum angle to $\lambda = \frac{1}{2}$. However, this is rather ad hoc. A more satisfactory way of obtaining the result is to take into account the contribution from the bundle of half-forms. In the previous section, this changed the form of the energy operator. Here, it gives rise to the modified Bohr - Sommerfeld condition (6.49). The flat connection on the bundle of half-forms is non-trivial, with $d_\gamma = \frac{1}{2}$. This leads to the same shift in the spectrum as the choice $\lambda = \frac{1}{2}$ and to the correct result (6.39).

References

[1] P.A.M. Dirac, *The Principles of Quantum Mechanics* (Fourth Edition), Clarendon Press, Oxford, 1958.

[2] J.-M. Souriau, *Quantification Géométrique*, Commun. Math. Phys. 1 (1966) 374; *Structure des Systèmes Dynamiques*, Dunod, Paris, 1970.

[3] B. Kostant, *Quantization and Unitary Representations*, in *Lecture Notes in Mathematics* Vol. 170, Springer, Berlin, 1970.

[4] C.J. Isham, *Topological and Global Aspects of Quantum Theory*, in *Relativity, Groups and Topology II*, North Holland, Amsterdam, 1984.

[5] S. Axelrod, S. della Pietra and E. Witten, *Geometric Quantization of Chern-Simons Gauge Theory*, J. Diff. Geom. 33 (1991) 787.

[6] M.J. Gotay, *Constraints, Reduction, and Quantization*, J. Math. Phys. 27 (1986) 2051.

[7] A. Ashtekar and M. Stillerman, *Geometric Quantization and Constrained Systems*, J. Math. Phys. 27 (1986) 1319.

[8] M. Blau, *Constraints and Polarizations*, Phys. Lett. B205 (1988) 525; *On the Geometric Quantization of Constrained Systems*, Class. Quantum Grav. 5 (1988) 1033.

[9] A.A. Kirilov, *Elelements of the Theory of Representations*, Springer, Berlin, 1976.

[10] R. Abraham and J.E. Marsden, *Foundations of Mechanics* (Second Edition), Benjamin, Reading MA, 1978.

[11] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer, New York, 1978.

[12] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[13] N.M.J. Woodhouse, *Geometric Quantization* (Second Edition), Clarendon Press, Oxford, 1992.

[14] J. Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer, New York, 1980.

[15] N.E. Hurt, *Geometric Quantization in Action*, Reidel, Dordrecht, 1982.

[16] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Springer, New York, 1986.

[17] M. Blau, *What happens to the Vacuum Angle in the Momentum Representation?*, Mod. Phys. Lett. A4 (1989) 927; *On the Representation Independence of Topological Effects in Quantum Field Theory*, Int. J. Mod. Phys. A17 (1989) 4627.

[18] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer, New York, 1966.

[19] E. Witten, *Quantum Gauge Theories in Two Dimensions*, Commun. Math. Phys. 141 (1991) 153.